Physics 139 - Relativity
Problem Set 10 Due Week April 12, 1999

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Show your development of the solution by equations and diagrams before the evaluation to indicate your understanding of the problem.

1 The Curvature of Space at the Earth's Surface

Calculate the curvature of space-time at the surface of the Earth using the equivalence principle. Extend Galileo's (Newton's) principle of equivalence of inertial and gravitational mass to the equivalence of the local gravitational acceleration to inertial acceleration. Assume that the curvature of space-time at the surface of the Earth is spherical. The radius of the Earth is 6.38×10^8 cm and the surface gravity is 9.81×10^2 cm sec^{-2}. Express the answer both in cm and in light years.

Suggestion: Calculate the path of a photon traveling between two horizontal points assuming that it undergoes acceleration according to the Earth's surface gravity. An alternate way to approach the problem is to assume that light in free fall travels a straight path and determine how the photon's path would appear to an observer in an elevator accelerating upward from the Earth's surface. Then calculate the radius of curvature that matches the photon's path. Would you get the same answer doing the calculation for a ball or a bullet going through the same two points?

2 Black Hole Radius

Calculate the black hole radius of the Earth. ($G = 6.67 \times 10^{-8}$ $\mathrm{cm^2 gm^{-1} sec^{-2}}$)

a) What is the radius for which the escape velocity would be the speed of light, if all the mass (5.98×10^{27} gm) associated directly with the Earth were concentrated in an infinitesimally small region (that is, smaller than the black hole radius)?

b) What is the radius for which the gravitational potential energy just matches the rest mass (mc^2) of a test particle?

c) What would be the speed at these radii of a test particle that came from rest at infinity? Calculate this using Newtonian physics. What would be the velocity in the case of a treatment of special relativity? Remember that $E = m(1 - (v/c)^2)^{-1/2}$.

d) What does the Schwarzschild metric give for the blackhole radius (see 3.3).

2.1 Surveying the Curvature

If Gauss's surveyors could have measured with ideal, instead of the $0.7''$, accuracy the sum of the angles in the approximately 100 km equilateral triangle, then what answer would they get in the following cases: i.e. How different would the sum of the angles be from 180 degrees?

a) The surface of the Earth was a perfect sphere of radius, $R_\oplus = 6.37 \times 10^8$ cm, and they put out measuring tape (or line) along the surface of the Earth?

b) They used light and so you can now use the radius of curvature of space at the surface of the Earth you calculated earlier in this problem.

What sign would the difference be? I.e. is space here positively or negatively curved.

Hint: For calculating the angle difference from 180 degrees: Consider a triangle where two vertices on the equator and the third on the north pole. The two angles at the vertices will be right angles adding to 180 degrees so that the included angle at the pole will be the excess angle over 180 degrees. If it is an equilateral triangle, then the angle at the pole will be 90 degrees and the triangle's area will be one eight of the spheres. If the angle at the top is as large as possible 180 degrees then the area of the triangle will be one fourth of the sphere. For this configuration there is a clear relation between the excess angle is proportional to the area of the triangle divided by the area of the sphere: Excess angle (in radians) = $4\pi A_{triangle}/A_{sphere}$ = solid angle of triangle Use this relation to estimate the difference in the sum of the angles from 180 degrees?

2.2 Sign of Effect

How would you make this calculation for a negatively curved space-time?

2.3 Extra Credit/Thought Provoking

How could Gauss and his surveyors (or we) tell if space was actually curved or was the angular difference from a plane simply caused by the curving of a light beam being accelerated by gravity? (the equivalence principle)

3 The Bending of Light by the Sun

The successful prediction of the bending of light by the Sun was the triumph that elevated Einstein's General Relativity to the point that it got the world's attention. In this problem you will calculate the bending of light by the Sun and see if you can get the correct answer by using Newtonian physics and the principle of equivalence - so that GR was not needed - and by dimensional analysis. The mass of the Sun is $M_\odot = 1.99 \times 10^{33}$ gm. The radius of the Sun is $R_\odot = 6.96 \times 10^{10}$ cm. You can use the impact approximation approach from a previous problem set.

3.1 Bending According to Equivalence Principle

Use the principle of equivalence - gravity = acceleration - and Newtonian physics to calculate the angle by which a light from a distant star (effectively infinity) nearly grazing the Sun's surface is bent by the time it passes us on Earth (also treat this as effectively infinity). If you need it in this approach, because of the spherical symmetry, angular momentum is conserved relative to the center of the Sun.

3.2 Dimensional Analysis

What two numbers do you have? Make two distances by calculating the black hole radius of the Sun and compute an angle by taking the ratio of two distances. How close is your answer to what you got in 3.1?

3.3 Use the Schwarzschild metric

Use the Schwarzschild metric, i.e for a spherically symmetric mass outside of the mass:

$$ds^2 = (1 - 2GM/rc^2)c^2dt^2 - (1 - 2GM/rc^2)^{-1}dr^2 + r^2d\Omega^2$$

and calculate the angle for the bending of starlight grazing the Sun. Use the impact parameter approximation to determine the geodesic deviation from a straight line.

4 The Gravitational Redshift

This problem will give you some familiarity with the gravitational redshift.

4.1 Formula

Give a formula for the fractional change in frequency (energy) of photons when its gravitational potential changes by an amount $\delta\phi$. (Hint: Use the equivalence principle to calculate the fractional difference in detected frequencies or energies.)

4.2 Gravitational Redshift in a Tower

Suppose you have a source of 14.4 keV photons emitted by an ^{57}Fe source on a tower of height 22.6 meters (74 ft). You detect the frequency of the emitted photons with a detector on the same level as the emitter and with another detector at the base of the tower. What frequency or energy difference do you observe.

Pound and Rebka (1960) performed this experiment and detected the gravitational shift using the Mössbauer effect What speed would the detector would give the same effect as the gravitational redshift? (Give the answer in m/s.)

4.3 Redshift from Sun

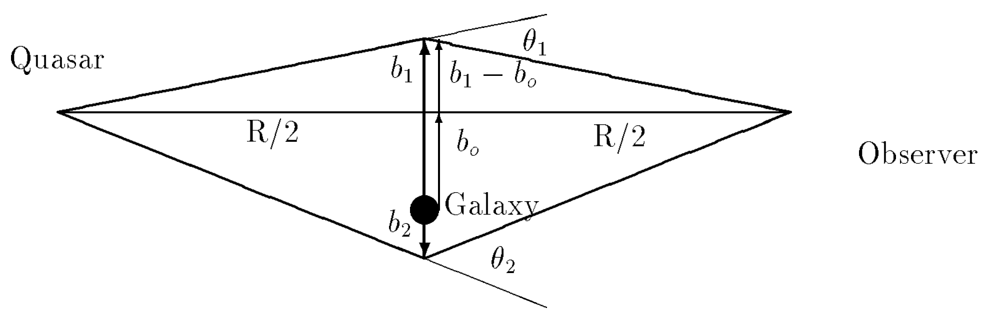
What is the fractional difference (redshift) for photons leaving the surface of the Sun?

4.4 Redshift from Center of the Earth

Suppose the Earth were transparent, so we could see the light emitted by the hot, glowing center of the Earth. What would be the redshift of this light received at the surface of the Earth? For this calculation, pretend that the Earth is a sphere of uniform density, and ignore the rotation of the Earth.

5 Gravitational Lenses and Time Delay

Suppose a galaxy acting as a "gravitational lens" produces two "images" of a distant quasar. Suppose that the galaxy is at a distance R/2, midway between the quasar and observer (as illustrated). Suppose that the impact parameters of the two light rays passing by the galaxy are b_1 and b_2. Treating the galaxy as a point-like mass M and using the linear approximation for the relativistic gravitational field, find an expression for the difference in travel times along the two light rays. Evaluate the time delay numerically for $b_1 = 5 \times 10^5$ light years, $b_2 = 1 \times 10^5$ light-years, and $M = 10^{11}$ solar masses.



Hint: Take into account the difference in path length between the two rays and the difference in the effective speed of light in the gravitational field; in lowest order of approximation, these two contributions can be combined additively. Ignore cosmological effects, that is, pretend that the background space-time is flat and static.