P139 - Relativity
Problem Set 12: GRAVITATIONAL WAVES 2003 April

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1 GRAVITATIONAL PLANE WAVES

1.1 Introduction to Gravitational Waves

By analogy to electromagnetism we expect that there should be gravitational waves – moving disturbances in the metric. These should have the following properties:
(a) Gravitational Waves should exist and travel with speed c which is the only Lorentz invariant velocity and is the relation between space and time.
(b) Every from of matter and energy with which we are familiar has the same gravitational effect – positive attraction. i.e. like having all one sign electric charge Thus no dipole radiation only quadrupole and higher order.
(c) Gravitational Waves are very weak. Consider two neutron stars of one solar mass ($M_{NS} = \mathrm{M}_\odot$) and radius $r_{NS} = 6$ km orbiting each other essentially in contact. (This is for example, clearly tidal forces will have a very disruptive effect on the two neutron stars.) The characteristic acceleration at the neutron stars is

$$a = \frac{2G\mathrm{M}_\odot}{(2r_{NS})^2} \simeq 9 \times 10^{11} m/s^2 \simeq 10^{11} g$$

The orbital period for half a rotation (symmetric pattern with identical neutron stars and thus the period of the graviational radiation) is $T/2 = 2\pi/\sqrt{r^3/G\mathrm{M}_\odot} \simeq 10^{-4}$ sec, so that the frequency of the gravitational radiation is about 10 kHz.

Consider a test mass at the Earth one parsec away (1 pc = 3.26 light years). This distance is the distance to the nearest known stars, excepting the Sun, and the test mass will be in the next section our detector. The acceleration $A_\oplus$ from the gravitaional radiation, i.e. proportional to $1/r$ rather than $1/r^2$, is

$$A = \frac{G\,(2m_\odot)\,a}{c^2 r} \simeq 0.09 m/s^2 \simeq 10^{-2} g$$

(d) Detection of gravitational waves is difficult. Consider a resonant frequency pendulum tuned to the gravitational waves. Using the formula for a gravitational pendulum $\omega = \sqrt{g/\ell}$, one finds that to get $f = 10^4$ Hz requires $\ell = 20\mathring{A}$. (24.8A)

However, the wavelength of the gravitational wave is $\lambda_{GW} = c/f \simeq 3x10^5 km/s/10^4/s = 30$ km. Thus a quarter-wavelength long detector would need to be about 7.5 km. Thus the single pendulum is not a good match to the gravitational wave.

The velocity produced by the gravitational wave on the test mass on Earth is of order $V = A/\omega = A/(2\pi f) \simeq 3 \times 10^{-6}$ m/s. This is a very low velocity and difficult to dectect even with the Mössbauer effect. The amplitude of oscillation is $A/(2\pi f)^2 \simeq 0.5\mathring{A}$.

(e) The generation of gravitational waves is difficult. Just as it is difficult to detect gravitational waves; it is difficult to generate them. We will find that for astrophysical systems the dimensionaless strain h is roughly

$$h \sim \frac{r_{Schwarschild}}{r}\left(\frac{v}{c}\right)^2$$

The power in gravitational waves is proportional to the square of the internal power of the radiating object. The constant of proportionality is equivalent to dividing by a very large power.

$$P = \frac{P_{internal}^2}{P_{cGW}}, \quad \text{where} \quad P_{cGW} = \frac{c^5}{G} = 3.63 \ \times 10^{59}\text{erg/s} = 2 \times 10^5 \text{M}_\odot/s$$

1.2 Properties of Gravitational Waves

There are many similarities between gravitation and electromagnetism. In particular they have similar Poisson Equations relating the field to its sources. For electromagnetism

$$\Box^2 A^\nu = 4\pi J^\nu, \text{ where } \Box^2 A^\nu \equiv \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{1}{c^2}\frac{\partial^2 A^\nu}{\partial t^2} - \frac{\partial^2 A^\nu}{\partial x^2} - \frac{\partial^2 A^\nu}{\partial y^2} - \frac{\partial^2 A^\nu}{\partial z^2} \quad (1)$$

where J^ν is the 4-D electric current and the last equality holds for the gauge condition $\partial_\mu A^\mu = 0$ and the usual rectilinear coordinates. For gravity one has (in the weak field limit)

$$\Box^2 \phi^{\mu\nu} = -16\pi G T^{\mu\nu} \quad \text{or} \quad \partial_\lambda \partial^\lambda (h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h) = -16\pi G T^{\mu\nu} \quad (2)$$

where $\kappa = \sqrt{16\pi G} \cong 2.04 \times 10^{-24}$ s (g-cm)$^{-1/2}$ and $T^{\mu\nu}$ the stress-energy tensor. Note the relation between the linearized part of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric tensor and the field variable $\phi^{\mu\nu}$

$$\phi^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h \quad \rightarrow \quad h^{\mu\nu} = \phi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\phi \quad (3)$$

Such a field is convenient for the gauge condition

$$\partial_\mu \phi^{\mu\nu} = 0 \quad (4)$$

It is no surprise that both Maxwell's and the linearized Einstein equations should have radiation solutions and that both solutions waves move with the speed of light ($c = 1$ in our notation and is left out of the equation above). It is clear that adding any solution of the form

$$\Box^2 \phi^{\mu\nu} = 0, \quad \text{or, if} \quad \partial_\mu \phi^{\mu\nu} = 0, \quad \partial_\lambda \partial^\lambda \phi^{\mu\nu} = 0 \tag{5}$$

will work.

Try the plane-wave solution

$$\phi^{\mu\nu} = \epsilon^{\mu\nu} cos k_\alpha x^\alpha \tag{6}$$

where $\epsilon^{\mu\nu}$ is a constant tensor, and k_α is a constant vector in the direction of propagation, which are named the polarization tensor and the wavevector, respectively.

(a) Show that this is a solution to equation 5, if

$$k^\alpha k_\alpha = 0, \quad \text{and} \quad \epsilon^{\mu\nu} k^\mu = 0 \tag{7}$$

meaning that $\omega = ck$ where $k = \sqrt{k_x^2 + k_y^2 + k_z^2} = |\mathbf{k}|$ and thus the speed of the gravity wave is the speed of light. Note that this means in the weak field limit there is no dispersion. (The c for your information later. Ignore it for this exercise.)

Now without loss of generality, we can restrict ourself to a plane wave moving in the z-direction $k^\alpha = (\omega, 0, 0, \omega)$ so that

$$\phi^{\mu\nu} = \epsilon^{\mu\nu} cos(\omega t - \omega z) = \epsilon^{\mu\nu} cos(kt - kz) \tag{8}$$

(b) You can either show or take my word for it that equations 7 and the requirement that $h^{\mu\nu}$ and $\phi^{\mu\nu}$ are symmetric means that there only only six possible independent polarizations. Note that there are 16 components to the tensor and symmetry reduces that to 10 independent components. The two relations of equation 7 cut the 10 to 6. (You heard that these correspond to the ± 0, ± 1, and ± 2 helicity modes in class.)

(c) Two of the independent polarization solutions can be

$$\epsilon_{\oplus}^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

and

$$\epsilon_{\otimes}^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

Show that these are traceless and transverse to the direction of propagation. The claim is that these two are the only one of their set of six that can actually carry energy and momentum. The others can be transformed away by gauge transformations.

We can return to general plane waves

$$\begin{aligned}\phi^{\mu\nu}_{\oplus} &= A_{\oplus}\epsilon^{\mu\nu}_{\oplus} cos k_\alpha x^\alpha \\ \phi^{\mu\nu}_{\otimes} &= A_{\otimes}\epsilon^{\mu\nu}_{\otimes} cos k_\alpha x^\alpha\end{aligned}$$

with energy-momentum tensors

$$\begin{aligned}t^{\mu\nu}_{\oplus} &= A^2_{\oplus} k^\mu k^\nu sin^2 k_\alpha x^\alpha \\ t^{\mu\nu}_{\otimes} &= A^2_{\otimes} k^\mu k^\nu sin^2 k_\alpha x^\alpha\end{aligned}$$

The average flux in the direction of propagation is

$$t = \frac{1}{2c}(A^2_{\oplus} + A^2_{\otimes})\omega^2 = \rho_{GW}/c \tag{11}$$

and the energy density is related to the flux by c. The stress-energy tensor for waves propagating in the z-direction has for its non-zero components

$$T^{00} = \frac{1}{c}T^{0z} = \frac{1}{c}T^{zz} = c^2 < A^2_{\oplus} + A^2_{\otimes} >= \frac{1}{16\pi}\frac{c^2}{G} < h^2_{+} + h^2_{\times} > \tag{12}$$

1.3 Relative Motion

In class I showed that such gravitational plane waves caused two test particles to oscillate relative to each other. I used the weak field geodesic equation

$$\frac{dp_\mu}{d\tau} = \frac{1}{2} m h_{\alpha\beta,\mu} u^\alpha u^\beta \tag{13}$$

or for a massive particle

$$\frac{du_\mu}{d\tau} = \frac{1}{2} h_{\alpha\beta,\mu} u^\alpha u^\beta \tag{14}$$

to show that a particle at rest $u^\alpha = (1,0,0,0)$ experiences no acceleration and remains at rest. However, two particles that were separated by a distance d in a direction transverse to the direction of propagation and aligned with the $\oplus$ polarization experience a periodic variation in their position to first order

$$\delta l \cong d \times (\frac{1}{2} A_{\oplus} cos \omega t) \tag{15}$$

If a typical strong ($A_{\oplus} \simeq 10^{-18}$) gravity plane wave passes through the earth-moon system ($d \cong 3.84 \times 10^{10}$ cm), what is the variation in the earth-moon distance? If laser ranging to the moon is accurate to 10 cm, can the motion be readily detected?

If the relative distance to a space craft at a distance of 5 astronomical units (7.5×10^{13} cm) can be measured to 1 mm by doppler tracking, can one detect this wave? How accurately must one measure the separation to be able to observe such a wave? Could it be done with laser interferometry?

1.4 Effective tidal Force

A gravitational plane wave causes and apparent deformation of objects. We discussed the appearance of a tube-shaped distribution of matter lying along the direction of propagation of the plane wave. For the two linear polarizations $\oplus$ and $\otimes$ the tube appears alternately squashed in the directions of the polarization axis. Specifically for a plane wave propagating along the z-axis, the $\oplus$ polarization has at a fixed time (for the right phase)

$$\begin{aligned} x(z) &= r(1 - \frac{1}{2}A_{\oplus}\cos\omega z) \\ y(z) &= r(1 - \frac{1}{2}A_{\oplus}\sin\omega z) \end{aligned}$$

The $\otimes$ polarization would look the same but rotated 45°.

For the circular polarizations

$$\begin{aligned} \epsilon^{\mu\nu}_{RH} &= (\epsilon^{\mu\nu}_{\oplus} - i\epsilon^{\mu\nu}_{\otimes})e^{ik_\alpha x^\alpha} \\ \epsilon^{\mu\nu}_{LH} &= (\epsilon^{\mu\nu}_{\oplus} + i\epsilon^{\mu\nu}_{\otimes})e^{ik_\alpha x^\alpha} \end{aligned}$$

the shape is a twisted squashed tube. Waves with the first circular polarization $\epsilon^{\mu\nu}_{RH}$ have positive helicity meaning the angular momentum is parallel to the direction of energy flux. Waves with the second circular polarization $\epsilon^{\mu\nu}_{LH}$ have negative helicity meaning the angular momentum is antiparallel to the direction of energy flux. The angular momentum in the wave is just $2/\omega$ times the energy in the wave.

What is the effective tidal force on the particles in a tubular shape for a gravity wave of amplitude $A_{\oplus} \simeq 10^{-18}$? Hint: Use the equivalence principle and calculate the effective differential acceleration.

$$\frac{d^2x}{dt^2} = -\frac{1}{2}A_{\oplus}x_0\frac{d^2}{dt^2}cos\omega t \tag{16}$$

You can get a similar equation for y. Show that the radial component of the tidal force is

$$f_r = m\ \frac{1}{2}A_{\oplus}r_0\ \omega^2 cos\omega t\ cos2\phi \tag{17}$$

where $r_0^2 = x_0^2 + y_0^2$ is the radial coordinate and ϕ is the azimuthal angle to the polarization axis.

2 Generation of Gravitational Waves

We continue briefly with our analogy to electromagnetism. It is appropriate to note that the relative strength of gravity to electromagnetism for a simple effect such as and atomic transition of energy order $E \sim 1$ eV is $GE^2/e^2 \approx 3 \times 10^{-54}$. Thus we cannot expect to find gravity waves being significant in atomic or individual particle interactions but will be significantly only as a bulk effect for a large aggregate collection. This (equivalent to the small value of κ) justifies our weak field approximation. This keeps the waves linear and with no dispersion.

The general solution to Poisson equations (1) and (2) are

$$\Box^2 A^\nu = 4\pi J^\nu, \quad A^\nu(\mathbf{x},t) = \frac{1}{c}\int d^3x' \int dt' \frac{J^\nu(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta(t' + |\mathbf{x}-\mathbf{x}'|/c - t) \tag{18}$$

integration over t' replaces $J^\nu(\mathbf{x}',t')$ by $J^\nu(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c - t)$ and similarly

$$\Box^2 \phi^{\mu\nu} = -16\pi G T^{\mu\nu}, \quad \phi^{\mu\nu} = -\frac{16\pi G}{4\pi}\int d^3x' \frac{T^{\mu\nu}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c - t)}{|\mathbf{x}-\mathbf{x}'|} \tag{19}$$

(These were derived from the relation $\nabla^2(1/|\mathbf{x}-\mathbf{x}'|) = -4\pi\delta(\mathbf{x}-\mathbf{x}')$.)

In considering radiation one usually divides the situation into various zones when the size of the system radiating is smaller than a wavelength.

Zone	*Conditions*	
Near or Static	$d << r << \lambda$	
Intermediate	$d << r \sim \lambda$	(20)
Far or Radiation Zone	$d << \lambda << r$	

where d is the characteristic size of the radiating system, λ is the wavelength, and r is the distance to the observer.

In the far field region the field scales as $1/r$ and the field is perpendicular to $\mathbf{r}$ as can be seen from the following argument. In the far field zone $kr >> 1$ so that $|\mathbf{x}-\mathbf{x}'| \cong r - \hat{n}\cdot\vec{\mathbf{x}'}$ so that

$$\lim_{kr\to\infty} \vec{\mathbf{A}} \to \frac{e^{ikr}}{cr}\int e^{-ik\hat{n}\cdot\mathbf{x}'}\ \vec{\mathbf{J}}\ d^3x' \cong \frac{e^{ikr}}{cr}\sum_n \frac{(-ik)^n}{n!}\int (\hat{n}\cdot\vec{\mathbf{x}'})^n\ \vec{\mathbf{J}}\ d^3x' \tag{21}$$

where the second part comes from expanding $exp(-ik\hat{n}\cdot\vec{\mathbf{x}'})$ in a Taylor series since we have assumed that $kd << 1$ which implies that $k|x| << 1$.

We expect that quadrupole radiation is the lowest order radiation that gravity can produce since all gravitational charges carry the same sign and the amplitude of the field will be proportional to k^3 and the energy radiated proportional to k^6.

Aside: The general solution

$$\vec{\mathbf{A}} = \frac{4\pi ik}{c}\sum_{l,m} h_l^{(1)}(kr) Y_{lm}(\theta,\phi)\int j_l(kr') Y^*_{lm}(\theta',\phi')\ \vec{\mathbf{J}}\ d^3x' \tag{22}$$

where $h_l^{(1)}(kr)$ are the Hankel functions, $Y_{lm}(\theta,\phi)$ are the spherical harmonics, and $j_l(kr)$ are the Bessel functions.

Making similar approximation for the distance $|\mathbf{x}-\vec{\mathbf{x}'}| \cong r - \hat{n}\cdot\vec{\mathbf{x}'}$ for the gravitational case, one has

$$\phi^{\mu\nu} \cong \frac{16\pi G}{4\pi r}\int d^3x' T^{\mu\nu}(t-r,\mathbf{x}') \tag{23}$$

In the linear approximation, $T^{\mu\nu}$ satisfies the gauge condition

$$\partial_\nu T^{\mu\nu} = 0 \tag{24}$$

which implies

$$\frac{\partial}{\partial t} T^{k0} = -\partial_l T^{kl} \quad \text{and} \quad \frac{\partial}{\partial t} T^{00} = -\partial_l T^{0l}. \tag{25}$$

We can use the first relationship to get

$$\int T^{kl} d^3x = \frac{1}{2}\frac{\partial}{\partial t}\int (T^{k0}x^l + T^{l0}x^k) d^3x \tag{26}$$

and the second to get

$$\int (T^{k0}x^l + T^{l0}x^k) d^3x = \frac{\partial}{\partial t}\int T^{00}x^k x^l d^3x \tag{27}$$

This can be proven by integrating the right hand side by parts. Combined these give us

$$\int T^{kl} d^3x = \frac{1}{2}\frac{\partial^2}{\partial t^2}\int T^{00}x^k x^l d^3x. \tag{28}$$

This indicates that the integral over the stress-energy can be expressed in terms of derivatives of T^{00}. This is simply conservation of energy and momentum.

Putting this back into equation (23) we have

$$\phi^{\mu\nu} \cong -\frac{16\pi G}{4\pi r}\int T^{\mu\nu}(t-r,\mathbf{x}') = -\frac{16\pi G}{8\pi r}\frac{\partial^2}{\partial t^2}\int T^{00}(t-r,\mathbf{x}') d^3x' \tag{29}$$

We will make two other approximations in addition to the weak field and the far zone in our treatment; (1) the gravitational effects are relatively small so that we can replace T^{00} by ρ and (2) we will neglect special relativistic corrections ($v/c << 1$). Note that these approximations mean that we will be in the weak field limit in that the dimensionless strain

$$h \sim \frac{r_{schwarzschild}}{r}\frac{v^2}{c^2} \tag{30}$$

Thus for non-relativistic matter one has

$$\phi^{kl} \cong -\frac{16\pi G}{8\pi r}\frac{\partial^2}{\partial t^2}\int \rho(t-r,\mathbf{x}') d^3x' \tag{31}$$

This integral can be expressed in terms of the quadrupole moment tensor.

$$Q^{kl} = \int (x^k x^l - \frac{1}{3}r^2\delta^l_k)\rho(t-r,\mathbf{x}') d^3x \tag{32}$$

so that

$$\phi^{kl} \cong -\frac{16\pi G}{8\pi r}\{\frac{\partial^2}{\partial t^2}Q^{kl} + \delta^l_k\frac{\partial^2}{\partial t^2}\int r'^3\rho(t-r,\mathbf{x}') d^3x'\} \tag{33}$$

For the calculation of energy flux, the term proportional to δ^l_k can be omitted because that part of the wave carries no energy as it has the wrong polarization.

$$\phi^{kl} \cong -\frac{16\pi G}{8\pi r}\frac{\partial^2}{\partial t^2}Q^{kl} = -\frac{16\pi G}{8\pi r}\ddot{Q}^{kl} \tag{34}$$

This is the lowest order term for gravitational radiation and is clearly quadrupolar as asserted earlier. There can be higher order terms; however, in general one expects them to be significantly smaller than the quadrupole contribution.

It can be shown that the energy radiated per unit solid angle is

$$-\frac{d^2E}{dtd\Omega} = \frac{dPower}{d\Omega} = (\frac{16\pi G}{8\pi r})^2[\frac{1}{2}\dddot{Q}^{kl}\dddot{Q}^{kl} - \dddot{Q}^{kl}\dddot{Q}^{km}n^l n^m + \frac{1}{4}\dddot{Q}^{kl}\dddot{Q}^{mr}n^k n^l n^m n^r] \tag{35}$$

This angular distribution is quite complicated except in very symmetric cases. However, we can integrate over all solid angles to find the total power radiated

$$-\frac{dE}{dt} = \frac{G}{5c^5}\dddot{Q}^{kl}\dddot{Q}^{kl} \tag{36}$$

Now we have a general set of equations we can use to treat the case of various cosmologically and astrophysically interesting objects. We expect gravitational radiation to fall into one of three classes:

Sources of Gravitational Radiation

Type	Source	Characteristic Frequency
Periodic	orbiting and oscillating objects	$10^{-7} - 10^2$ Hz and $10^2 - 10^5$ Hz
Burst	Collisions, Supernova & infall/collapse	$10^{-1} - 10^4$ Hz
Stochastic	CGR, Inflation, Phase Transitions	all

We treat these in the next subsections.

2.1 Order of Magnitude Estimates

We estimated the size of the dimensionless strain produced by a mass

$$h \sim \frac{R_{schwarzschild}}{R}\frac{v^2}{c^2} \tag{37}$$

Now we can estimate the quadrupole moment and its third derivative to estimate the power loss.

$$\dddot{Q}^{lk} \sim \frac{GMR^2}{t_c^3} \sim \frac{GMv^3}{R} \tag{38}$$

where M, R, t_c, and v are the characteristic mass, size, time scale, and velocity of the source. Using our formula for the power radiated

$$-\frac{dE}{dt} = \frac{G}{5c^5}\dddot{Q}^{kl}\dddot{Q}^{kl} \sim \frac{G}{c^5}(\frac{M}{R})^2 v^6 \sim L_{cGW}(\frac{R_{Schwarzschild}}{R})^2(\frac{v}{c})^6 \tag{39}$$

where

$$L_{cGW} \equiv \frac{c^5}{G} = 3.63 \times 10^{59} \text{ erg s}^{-1} = 2.03 \times 10^5 \mathrm{M}_{\odot} c^2 \mathrm{s}^{-1} \tag{40}$$

If a steel rod of mass 100 tons (10^8 grams) and length 20 meters rotates at its breakup speed (30 radians/sec), what is the approximate radiated power in gravitational waves?

Show that the power in gravity waves for a system with internal power in quadrupole motion $L_{internal} \sim Mv^2/t_c$ is roughly

$$L_{GW} = \frac{L^2_{internal}}{L_{cGW}} \tag{41}$$

Since astrophysical systems are generally gravitationally bound, the virial theorem tells us

$$\text{Kinetic Energy} \sim \frac{MR^2}{t_c^2} \sim |\text{Potential Energy}| \sim \frac{GM^2}{R}. \tag{42}$$

So the characteristic time scale is

$$t_c \sim \sqrt{\frac{R^3}{GM}} \tag{43}$$

so that

$$L_{GW} \sim L_{cGW}(\frac{R_{Schwarzschild}}{R})^5 \tag{44}$$

and the gravitational wave energy radiated by a nonspherical self-gravitating system in a characteristic time is roughly

$$\Delta E \sim L_{GW} t_c \sim Mc^2(\frac{R_{Schwarzschild}}{R})^{7/2}. \tag{45}$$

Thus the gravitational wave emission efficiency is then

$$\Delta E \equiv \epsilon M c^2, \quad \epsilon \sim (\frac{R_{Schwarzschild}}{R})^{7/2}. \tag{46}$$

Now we can estimate the order of magnitude for dimensionless strain of astrophysical objects at distance r

$$h \sim \epsilon^{2/7}\frac{R_{schwarzschild}}{r} \sim 3 \times 10^{-18}(\frac{\epsilon}{0.1})^{2/7}\frac{(M/\mathrm{M}_{\odot})}{(r/10kpc)}. \tag{47}$$

(10 kpc is roughly the distance to our Galactic center.)

3 Periodic Gravitational Waves

We consider here gravitational waves produced by sources whose density changes periodically.

3.1 Emission by a Vibrating Quadrupole

According to the equations above any system of masses with a time-dependent quadrupole moment whose third derivative $\dddot{Q}^{kl} \neq 0$ is not zero will radiate power in gravitational waves. Consider two equal masses connected by a spring - a linear oscillating quadrupole. This is a simplification of problems of interest but will give a scale for estimating situations of interest. We can use the result that a system of spherical masses has the same quadrupole moment as a system of point masses located at the centers of the spherical masses. We align the masses on the z-axis a distance b from the origin with oscillation amplitude a so that their positions are:

$$z = \pm(b + asin\omega t), \quad \text{and} \quad z^2 = b^2 + 2absin\omega t + a^2 sin^2\omega t \tag{48}$$

One can easily show that the quadrupole moment is

$$Q^{kl} = \frac{1}{3}\begin{bmatrix} -2mz^2 & 0 & 0 \\ 0 & -2mz^2 & 0 \\ 0 & 0 & 4mz^2 \end{bmatrix} = \frac{1}{3}(1+2\frac{a}{b}sin\omega t+(\frac{a}{b})^2 sin^2\omega t)\begin{bmatrix} -2mb^2 & 0 & 0 \\ 0 & -2mb^2 & 0 \\ 0 & 0 & 4mb^2 \end{bmatrix} \tag{49}$$

Thus to first order in a/b (so that we only get radiation at frequency ω) this gives

$$\phi^{kl} \cong \frac{16\pi G}{8\pi r}\frac{2a}{3b}\omega^2 sin\omega(t-r)Q(a=0)^{kl} \tag{50}$$

One can use the formula above to work out the angular distribution which comes to a simple pattern.

$$-\frac{d^2E}{dtd\Omega} = (\frac{16\pi G}{8\pi})^2[2mab\omega^3 cos\omega(t-r)]^2 \sin^4\theta \tag{51}$$

where θ is the usual polar angle. Note that this is similar to EM dipole radiation which depends on $\sin^2\theta$. The total emitted power is

$$-\frac{dE}{dt} = \frac{G}{5c^5}\dddot{Q}^{kl}\dddot{Q}^{kl} = \frac{32G}{15c^5}[mab\omega^3 \cos\omega(t-r)]^2 \tag{52}$$

and averaging over a cycle yields

$$-\frac{dE}{dt} = \frac{G}{5c^5}\dddot{Q}^{kl}\dddot{Q}^{kl} = \frac{16G}{15c^5}[mab]^2\omega^6. \tag{53}$$

The damping rate γ_{GW} of the oscillator due to gravity waves is defined as

$$\gamma_{GW} \equiv -\frac{1}{E}\frac{dE}{dt}. \tag{54}$$

Since each mass has kinetic energy of $\frac{1}{2}m\omega^2 a^2$, the damping rate for this oscillator is

$$\gamma_{GW} \equiv -\frac{1}{E}\frac{dE}{dt} = \frac{32G}{15c^5}mb^2\omega^4 \tag{55}$$

and the time for the energy to be damped to $1/e$ is $1/\gamma_{GW}$. Now this is a simple case but it allows us to make order of magnitude estimates for some more realistic astrophysical systems.

3.1.1 Oscillating Neutron Star

A neutron star could be formed asymmetrically or it might be impacted by infalling material such as an asteroid. Suppose that the neutron star has a mass of $m_{ns} = 0.7\mathrm{M}_{\odot}$, and radius $R_{ns} = 10$ km and that it is initially deformed on one axis so that $\delta R/R \sim 0.1$. It will oscillate on that axis as a nearly incompressible fluid. These parameters give the neutron star an approximate density of $\rho \sim 10^{14}$ gm/cm^{-3} . (Note, I have not checked to see that these are fully consistent with our formula - the Oppenheimer-Volkoff equation or the stiffer equation of state. If I or one of you have time, then we can fix it in the solution hand out.)

How much power is radiated in gravitational waves and what is the damping time? Is this likely to be the most efficient damping mechanism?

Hint: It is easy to show that the oscillation frequency is roughly $\omega \sim \sqrt{G\rho} \sim 3 \times 10^3$ Hz for an incompressible fluid. Use the relation that $v = \sqrt{\partial P/\partial \rho} \sim \sqrt{P/\rho}$ and $P = Gm^2/R^4$ and that $\omega \approx v/R$. Define the vibration energy as

$$E_{vib} = \frac{1}{2}m_{ns}(\delta R)^2\omega^2 \sim (\frac{\delta R}{R})^2 \times 10^{53} \text{ ergs} \tag{56}$$

3.1.2 White Dwarf, Sun, & Earth Oscillations

Gravitational radiation emitted by quadrupole vibrations might be important in the case of novas. Nova outbursts occur on white-dwarf stars in binary systems, where accretion from a companion star gradually accumulates nuclear fuel on the white dwarf surface until the fuel reaches a critical mass and explodes. Among other things, the explosion initiates vibrations in the body of the white dwarf, at characteristic frequencies of 0.01 to 1 Hz. The energy released in a nova explosion is typically 10^{45} ergs, of which as much as 10% goes into the vibrational motion of the star and is subsequently radiated in the form of gravitational waves.

What if the neutron star considered above was a white dwarf instead? How much power would be radiated from a 10% deformation and what would be the characteristic decay time through the emission of gravitational radiation? Does this follow the order of magnitude scaling laws from earlier and the numbers in the paragraph above? Are gravity waves likely to be important to the dynamics of solar oscillations and earth oscillations? Make a table of system (neutron star, white dwarf, sun, earth) of the form

Table 1
Vibration GW Damping

Object	Mass	R	R_{BH}/R	GW Power	Damping Time
Neutron Star	0.7 $M_\odot$	10 km			
White Dwarf	0.7 $M_\odot$	10^4 km			
Sun ($M_\odot$, $R_\odot$)	1.99×10^{33} g	6.96×10^{10} cm	4.48×10^{-6}		
Earth	5.98×10^{27} g	6.38×10^{8} cm	1.4×10^{-8}		

Note that one might consider using the excitation of vibration modes in the earth as a gravitational wave detector. The confusion background comes from earthquakes, volcanos and mankind.

3.2 Emission by a Rotating Quadrupole

Another periodic source of gravitational waves, which occurs very commonly in astrophysics, is the rotating quadrupole consisting of two masses in orbits around their common center of mass.

As the damping effects of gravitational waves circularizes orbits and it is simpler to calculate, we will consider circular orbits. Since the quadrupole moment repeats when the masses interchange sides in the orbit, the frequency of the emitted gravitational waves is twice the orbital frequency. The angular distribution will be relatively complicated and we skip over it but note that in the direction perpendicular to the orbital plane the gravitational radiation will be circularly polarized while in the plane it will be linearly polarized as in the case of the vibrating quadrupole.

The radiated power of the system is

$$-\frac{dE}{dt} = \frac{32G}{5c^5}\left[\frac{m_1 m_2}{m_1 + m_2}\right]^2 R^4\omega^6 = \frac{32G}{5c^5}\mu^2 R^4\omega^6 \tag{57}$$

where R is the separation between the masses m_1 and m_2 and ω is the orbital frequency.

For an astrophysical system, such as a binary star system with circular orbits, gravity holds the masses together and Kepler's third law gives a relationship between the separation R and the orbital period and thus angular frequency:

$$\omega^2 = \frac{G(m_1 + m_2)}{R^3} \quad \text{or} \quad P = 0.545\left(\frac{R}{1000\ \text{km}}\right)^{3/2}\left(\frac{M_\odot}{m_1 + m_2}\right)^{1/2}\ \text{sec} \tag{58}$$

so that the radiated power is

$$-\frac{dE}{dt} = \frac{32G^4}{5c^5R^5}(m_1m_2)^2(m_1+m_2). \tag{59}$$

As the binary loses energy by gravitational waves, the distance between the masses decreases at a rate given by

$$\frac{dR}{dt} = -\frac{64G^3}{5c^5R^3}m_1m_2(m_1+m_2) \tag{60}$$

at the same time the orbital frequency increases at a rate given by

$$\frac{d\omega}{dt} = -\frac{3\omega}{2R}\frac{dR}{dt} = \frac{96}{5}\left[\frac{G(m_1+m_2)}{c^2R^3}\right]^{3/2}\frac{Gm_1m_2}{c^2R} = \frac{96}{5}\frac{G}{c^5}\omega^3\frac{Gm_1m_2}{R} \tag{61}$$

where we have used the relationship that

$$E = -\frac{Gm_1m_2}{2R} \;\rightarrow\; \frac{dE}{dt} = \frac{Gm_1m_2}{2R^2}\frac{dR}{dt} \tag{62}$$

One can rearrange this to find the change in period $P_b = 2\pi/\omega$ (frequency) of the binary pair.

$$\frac{1}{P_b}\frac{dP_b}{dt} = \frac{3}{2}\frac{1}{R}\frac{dR}{dt} = -\frac{3}{2}\frac{1}{E}\frac{dE}{dt} = -\frac{96}{5}\frac{G^3(m_1+m_2)^2\mu}{c^5R^4} \tag{63}$$

$$= \frac{1}{2.61\times10^7\text{yr}}\left(\frac{m_1+m_2}{\mathrm{M}_\odot}\right)^{2/3}\frac{\mu}{\mathrm{M}_\odot}\left(\frac{1\text{hr}}{P_b}\right)^{8/3} \tag{64}$$

Assuming that the expressions above remain valid as $R\rightarrow0$, show that the time $\Delta t_{R\rightarrow0}$ until $R\rightarrow0$

$$\Delta t_{R\rightarrow0} = \frac{5}{256}\frac{c^5}{G^3}\frac{R^4_{now}}{(m_1+m_2)m_1m_2} \cong 1.84\left(\frac{R}{100\text{ km}}\right)^4\left(\frac{m_1+m_2}{\mathrm{M}_\odot}\right)^{-2}\left(\frac{\mathrm{M}_\odot}{\mu}\right) \tag{65}$$

and that

$$\frac{\Delta t_{R\rightarrow0}}{P_b} \sim 10^5\left(\frac{P}{1s}\right)^{5/3} \quad\text{or}\quad \frac{1}{P_b}\frac{dP_b}{dt} \sim \left(10^5\left(\frac{P}{1s}\right)^{5/3}\right)^{-1} \tag{66}$$

Now we are in a position to calculate the gravitational radiation power, orbital radius change, and relaxation time for some astrophysical systems.

3.2.1 Jupiter Power

As by far the largest component of the solar system quadrupole, estimate the power emitted by Jupiter in gravitational radiation and the rate of change of its orbit (in cm/year). (Useful numbers for this problem: orbit radius 5.2 AU = 7.8×10^8 km, period of orbit 11.9 years implying $\omega = 1.68\times10^{-8}$ sec^{-1}, mass $= 1.9\times10^{30}$ g $\approx 10^{-3}\mathrm{M}_\odot$.

While you are at it, you might want to calculate the rate of change of the earth's orbital radius due to gravity wave emission and compare it to that of other effects you calculated in problem set 5.

3.3 SuperMassive Black Holes

Our current view is that in the center of most galaxies there are supermassive ($10^5 M_\odot$ to $10^8 M_\odot$) black holes. Occassionaly, smaller black holes or supermassive blach holes from the merging of galaxies find themselves in (co-)orbit around the central supermassive black hole. This happens at first because the black holes are so massive and thus sink towards the central of the potential. Later the gravitational radiation they emit causes them to sprial into closer and closer orbits until they eventually merge. One can show that the quadrupole radiation from two inspiraling black holes has the strain of

$$h_\times = 3.6 \times 10^{-22} \frac{1Gpc}{r} \left(\frac{M_1}{10 M_\odot}\right) \left(\frac{M_2}{10^6 M_\odot}\right)^{2/3} \left(\frac{f}{0.01 Hz}\right)^{2/3}$$

The frequency determines the orbit separation. The time at frequency f until inspiral is

$$T = 1.41 \times 10^6 sec \left(\frac{0.01Hz}{f}\right)^{8/3} \left(\frac{10M_\odot}{M_1}\right) \left(\frac{10^6 M_\odot}{M_2}\right)^{2/3}$$

$$h_{pred} = 3.4 \times 10^{-22} \left(\frac{f}{10^{-2} Hz}\right)^{2/3} \left(\frac{M_{BH}}{10^4 M_\odot}\right)^{5/3} \frac{x}{(1+x)^{1/3}} \frac{1}{1-\sqrt{1+z}} \qquad f \sim 10^4 \left(\frac{M_\odot}{M_{BH}}\right) \frac{Hz}{1+z}$$

What is the strain expected at 3 Gpc from two supermassive black holes each with mass about $10^5 M_\odot$ with a frequency $f = 0.01 Hz$? Roughly how long before they inspiral to a frequency of $f = 0.03 Hz$? Make a rough plot of strain versus frequency with a few characteristic times called out -e.g. 1 year, 1 month, 1 week, 1 day, etc. Use the characteristic times for the proposed LISA (laser interferometer space array) of arond $10^{-4}\ Hz \le f \le 10^{-2}\ Hz$.

3.3.1 Rapidly Rotating Deformed Neutron Star

For a slightly deformed, homogeneous neutron star with moment of inertia $I = 2MR^2/5$, mass M, radius R, rotation period P_{ns}, and ellipticity, ϵ, the gravitational wave luminosity is

$$L_{GW} = -\frac{dE}{dt} = \frac{32G}{5c^5} I^2 \epsilon^2 (\frac{2\pi}{P})^6 \qquad (67)$$

$$\approx 10^{38} \text{erg s}^{-1} (\frac{I}{4 \times 10^{44} \text{g cm}^2})^2 (\frac{P_{ns}}{0.033\text{sec}})^{-6} (\frac{\epsilon}{10^{-3}})^2 \qquad (68)$$

and at a time t after its birth since it will slow down

$$L_{GW} \approx 10^{45} \text{erg s}^{-1} (\frac{I}{4 \times 10^{44}})^{1/2} (\frac{10^{-3}}{\epsilon}) (\frac{10^6 \text{sec}}{t + 10^4 \text{sec}})^{3/2} \qquad (69)$$

3.3.2 Close Binary Stars

Table 2
Interesting Binary Sources of GW

Object	Masses $M_\odot$	Distance (pc)	Frequency 10^{-6} Hz	GW Power ergs/s	Damping Time years	Strain at Earth (10^{-22}
Eclipsing Binaries						
ι Boo	1.0, 0.5	11.7	86			51
μ Sco	12, 12	109	16			210
V Pup	16.5, 9.7	520	16			46
Cataclysmic Binaries						
Am CVn	1.0, 0.041	100	1900			5
WZ Sge	1.5, 0.12	75	410			8
SS Cyg	0.97, 0.83	30	84			30
Binary X-ray Sources and/or Pulsars						
Cyg X-1	30, 6	2500	4.1			4
PSR1913+16	1.4, 1.4	5000	70*			0.12
PSR1534+12	1.4, 1.4	0.5	54*			
PSR2127+11C	1.4, 1.3	10.6	70*			

* Pulsars with elliptical orbits emit significantly at harmonics.
The next section will deal with these three binary pulsars.

There is a whole class of interesting binary star pairs that can be calculated approximately. The most spectacular binary star pair known at the moment is AM CVn (AM Canem Venticorum). Am CVn consists of a blue white dwarf and a low-mass white dwarf in an exceptionally small orbit around each other. Their period is 17.5 minutes! Do you know which star has the bigger radius? (Table 2 tells us that their masses are about 1 and 0.041 $M_\odot$, respectively.) What do you estimate for the ratio of the radii of the lower to higher mass star? Fill out the following table. (See formula in next section to help with the numerical factors.)

3.3.3 Binary Pulsars

There are three known binary pulsars and these and some of their parameters are listed in table 3.

Table 3
Galactic Binary Pulsars

Parameter	PSR 1534+12	PSR 1913+16	PSR 2127+11C
Reference	Wolszczan et al. 1991	Hulse & Taylor 1974	Prince et al. 1991
Flux (400 MHz)	36 mJy	5 mJy	0.6 mJy
Distance	1.1 ± 0.2 kpc	7.3 kpc	10.6 kpc (in M15)
Period P	10.098 hr	7.7519 hr	8.047 hr
eccentricity	0.273677	0.61713	0.68141
$M_1 + M_2$	2.6784 $M_\odot$	2.82837 $M_\odot$	2.71 $M_\odot$
$q = M_1(pulsar)/M_2$	0.97 ± 0.03	1.04	1.0 ± 0.2
$P/(2\dot{P})$	2.5×10^8 yr	1.1×10^8 yr	1.0×10^8 yr
$t_{merge-theory}$	27.3×10^8 yr	3.01×10^8 yr	2.20×10^8 yr
t_{meas}	30×10^8 yr	4.1×10^8 yr	3.20×10^8 yr

Table adapted from Phinney 1991 Ap J 380 L17.

As one can see the eccentricity is significant for these systems. This leads to significant emission at the orbital frequency (as opposed to twice) and in harmonics. The modified energy loss equation according to Peters and Mathews is

$$-\frac{dE}{dt} = L_{GW} = \frac{32G^4}{5c^5a^5}\mu^2(m_1+m_2)^3\ f(\epsilon) \tag{70}$$

$$\approx 3.0 \times 10^{33} \text{erg s}^{-1}(\frac{\mu}{M_\odot})^2(\frac{m_1+m_2}{M_\odot})^{4/3}(\frac{P_b}{1\text{hour}})^{-10/3}\ f(\epsilon) \tag{71}$$

where

$$f(\epsilon) = \frac{1 + \frac{73}{24}\epsilon^2 + \frac{37}{96}\epsilon^4}{(1-\epsilon^2)^{7/2}} \tag{72}$$

and a is the semi-major axis.

If the two masses are in an elliptical orbit with eccentricity ϵ then the energy emission in gravity waves rate (to a sign) is

$$L_{GW} = \frac{dE}{dt} = \frac{dE}{dt}|_{\epsilon=0}\ f(\epsilon) \tag{73}$$

and the angular momentum loss rate is

$$\frac{dJ}{dt} = \frac{dJ}{dt}|_{\epsilon=0}\ g(\epsilon) = -\frac{32}{5}\frac{G^{7/2}\mu^2(m_1+m_2)^{3/2}}{a^{7/2}}g(\epsilon) \tag{74}$$

where

$$g(\epsilon) = \frac{1+\frac{7}{8}\epsilon^2}{(1-\epsilon^2)^2} \tag{75}$$

where these are averaged over the orbit.

Using Kepler's laws we find

$$\frac{1}{P_b}\frac{dP_b}{dt} = \frac{1}{P_b}\frac{dP_b}{dt}|_{\epsilon=0}\ f(\epsilon) = -\frac{96}{5}\frac{G^3(m_1+m_2)\mu}{c^5 a^4}\ f(\epsilon) \tag{76}$$

For an elliptic orbit

$$\epsilon^2 = 1 + \frac{2EJ^2}{G^2\mu^3(m_1+m_2)^2} \tag{77}$$

One can use this relationship to derive an equation for $d\epsilon/dt$ and find that it is negative, so that gravitational radiation tends to circularize an elliptical orbit.

$$\frac{d\epsilon}{dt} = -\frac{304}{15}\frac{G^3\mu\ (m_1+m_2)^2\epsilon}{a^4\ (1-\epsilon^2)^{5/2}}\left(1+\frac{121}{304}\epsilon^2\right) \tag{78}$$

and the semi-major axis a decreases at the rate

$$\frac{da}{dt} = \frac{dR}{dt}|_{\epsilon=0} f(\epsilon) = -\frac{64}{5}\frac{G^3\mu\ (m_1+m_2)^2}{a^3} f(\epsilon) \tag{79}$$

In the Newtonian regime, if we orient the polarization axes along the major and minor axes of the projection of the orbital plane, then the dimensionless strain is

$$h_+ = 2(1+cos^2 i)\ \frac{\mu}{R}[\pi(m_1+m_2)f]^{2/3} cos(2\pi \int \frac{df}{dt}dt\ t) \tag{80}$$

$$h_\times = \pm 4cos^2 i\ \frac{\mu}{R}[\pi(m_1+m_2)f]^{2/3} sin(2\pi \int \frac{df}{dt}dt\ t) \tag{81}$$

3.4 Death Spiral

As the orbital pair loses energy through emitting gravitational waves the orbit becomes more circular, the major axis decreases in size, and the orbital frequency increases. The binary pair will begin a gradual spiral towards each other. The inward motion is slow at first but increases rapidly as the orbit gets smaller. AS the distance gets fairly small the rate gets large and results in a crescendo of gravitational radiation and a death spiral that ends in their coalescence. During the death spiral phase the gravitational radiation frequency will move higher and higher - rapidly in an up glissando. This is a reasonable term since one can easily calculate that the maximum frequency for a binary neutron star is about 1 kHz. It may be up to about 10 times that for optimal black holes.

Let us consider the steps leading up to the end. We will trust our results to the point that Newtonian gravitation is a reasonable approximation. We can use the circular orbit parameters.

One can integrate the rate of change of the orbit radius to find

$$\frac{1}{4}(R^4 - R_0^4) = -\frac{64G}{5c^5} m_1 m_2 (m_1 + m_2)(t - t_0) \quad (82)$$

where the t_0 indicates the starting time of the calculation.

Earlier we derived a formula for the time remaining $\Delta t_{R\to 0}$ until the radius would go to zero, if the equations remained valid. The time remaining went as

$$\Delta t_{R\to 0} = \frac{5}{256} \frac{c^5}{G^3} \frac{R_{now}^4}{(m_1 + m_2)^2 \mu} = [1 - (\frac{r}{r_0})^4] \frac{r_0}{4} \frac{1}{[-dR/dt|_{t_0}} \quad (83)$$

Thus it is clear that the system spends most of its time at the large distance and much less at small separations.

It is easy to show that the orbital frequency is

$$f_{orbit} = \frac{\omega}{2\pi} = \frac{1}{P_b} = \frac{1}{2\pi} [\frac{5}{256} \frac{c^5}{G^{5/3} \mu (m_1 + m_2)^{2/3}} \frac{1}{t_{R\to 0} - t}]^{3/8} \quad (84)$$

where $t_{R\to 0} = t_0 + \Delta t_{R\to 0}$. The gravitational wave frequency will be twice f_{orbit}.

Consider two neutron stars with masses 1.4 $M_\odot$orbiting at 100 km. The intial radiated power is about 6×10^{51} erg/s. The rate of decrease of the orbit radius is $-dR/dt|_{t_0} \sim 7 \times 10^6$cm/s = 70km/s. Clearly to get from 100 km to 20 km is going to take on the order of 1 second. The formula gives 0.36 seconds. Starting from 130 km separation would take about 1 second and 200 km would take about 6 seconds.

At 200 km the orbital frequency is about 50 Hz and the orbital velocity is about 0.1c. By 130 km these are up to 93 Hz and 0.13c and at 100 km this becomes 138 Hz and 0.289c. At 20 km the frequency is 1550 Hz and the orbital speed is 0.32c. Clearly a careful calculation will have to take into account special relativistic effects but our numbers should be accurate to better than 1% at 200 km and better than 10%. We would also have to take into account the spin-orbit coupling due to the frame-dragging of the rotating stars and orbit. This mostly causes a beat in the emitted radiation.

We can estimate the power radiated in gravitational waves in those last few moments.

$$\Delta E = \frac{G}{2} m_1 m_2 [\frac{1}{r_0} - \frac{1}{r}] \quad (85)$$

where the factor of two comes from the gravitational energy splitting between gravitational wave and kinetic energy. With $r_0 = 200$ km and $r = 20$ km one has

$$-\Delta E \simeq 10^{53} \text{ erg} \approx 0.04 M_\odot \quad (86)$$

which means that about 1.5% of the rest mass was radiated. Most of this energy is released in the last states of the coalescence. An extra pulse will be emitted when the neutrons stars collide and the resulting neutron star or black hole oscillations/vibrations is likely to produce even more. We consider this in the next section.

In the May 1995 Scientific American article by Tvsi Piran that I handed out in class, the author claims that two neutron stars 700 km apart take 15 minutes before collision and and their orbital period shrinks from a fifth of a second to a few milliseconds. Check these numbers and see what mass system one needs for this to be true and are these numbers consistent with the quoted parameters for the three know binary pulsars?

4 Gravitational Wave Bursts

Now we move from periodic (or quasiperiodic) case to the burst case. Any system with a quadrupole whose third derivative is non-zero will radiate.

4.1 Little hits very Big

For illustration we will first consider a small particle falling towards a very large mass. This is just so that we can neglect the motion of the large mass and its radiation. That should make the effects clearer to understand. In the next section we treat this in the fashion of reduced mass. The energy loss can be found by evaluating the quadrupole radiation loss formula and using the quadrupole matrix for a single mass at a distance z on the z axis we found for in the vibrating quadrupole section above.

$$Q^{kl} = \begin{bmatrix} -\frac{1}{3}mz^2 & 0 & 0 \\ 0 & -\frac{1}{3}mz^2 & 0 \\ 0 & 0 & \frac{2}{3}mz^2 \end{bmatrix} \tag{87}$$

Hence,

$$-\frac{dE}{dt} = \frac{G}{5c^5}\dddot{Q}^{kl}\dddot{Q}^{kl} = \frac{2Gm^2}{15c^5}(6\dot{z}\ddot{z} + 2z\dddot{z})^2 \tag{88}$$

Now we need to evaluate the various derivatives of z. Here we consider the case that the small test mass m is falling from rest at a great distance (i.e. infinity) as a result of gravitational attraction to the large stationary mass M. Later you can consider how to treat a more general case of interactions. Conservation of energy gives

$$\frac{1}{2}m\dot{z}^2 = \frac{GmM}{|z|} \tag{89}$$

and Newtonian law of universal interaction and the equivalence principle gives

$$\ddot{z} = \frac{GM}{z^2} \tag{90}$$

Taking the derivative with respect to time

$$\dddot{z} = -\frac{2GM}{z^3}\dot{z} = -\frac{2GM}{|z|^{7/2}}(2GM)^{1/2} \tag{91}$$

Now we can estimate the energy loss by integrating dE along the particle trajectory.

$$\Delta E = -\int_{\infty}^{R} \frac{1}{|z|^{9/2}} 2Gm^2\ 15c^5 (2GM)^{5/2} dz = -\frac{1}{R^{7/2}} \frac{4Gm^2}{105c^5} (2GM)^{5/2} \tag{92}$$

Clearly the closer the test particle gets the more radiation energy it emits. The question is where to cut off? Clearly, the Schwarzschild radius of the large mass is a hard cut off since none of the gravitational waves would escape.

$$\Delta E_{max} = -\frac{2}{105} mc^2 (\frac{m}{M}) \cong -0.019 mc^2 (\frac{m}{M}) \tag{93}$$

In the next section we will see that this may be as much as roughly a factor of two over more detailed calculations but does neglect the radiation from the oscillations of the system as the masses coalesce.

This is a moderately efficient gravitational wave generator. Consider a one solar mass object falling into a 10 $M_{\odot}$black hole.

$$\epsilon = \frac{\Delta E}{mc^2} = 0.019(\frac{m}{M}) = 0.002 \tag{94}$$

In the next section we handle the reduced mass case and get essentially same answer.

4.2 Collision of Two Objects

Consider two point masses m_1 and m_2 falling towards each other under the influence of gravity from rest at infinity. For this case we will assume that the collision is head on and that they move on the x axis. The center of mass is $x = 0$.

$$m_1 x_1 = -m_2 x_2 = \mu x \tag{95}$$

where

$$x \equiv x_1 - x_2, \qquad \mu \equiv \frac{m_1 m_2}{M}, \qquad M = m_1 + m_2 \tag{96}$$

Since

$$m_1 x_1^2 + m_2 x_2^2 = \mu x^2, \tag{97}$$

$$Q_{xx} = \frac{2}{3}\mu x^2, \qquad Q_{yy} = Q_{zz} = -\frac{1}{3}\mu x^2 \tag{98}$$

The equation of motion is

$$\ddot{x} = -\frac{GM}{x^2} \tag{99}$$

and

$$\frac{\dot{x}^2}{2} = \frac{GM}{x}, \tag{100}$$

yielding

$$\dddot{Q}_{xx} = -\frac{4}{3} G\mu M \frac{\dot{x}}{x^2}. \tag{101}$$

Using the formula for radiated power

$$-\frac{dE}{dt} = \frac{G}{5c^5}\left\langle \dddot{Q}^2_{xx} + \dddot{Q}^2_{yy} + \dddot{Q}^2_{zz} \right\rangle = \frac{8}{15}\frac{G}{c^5}\mu^2 M^2 \left\langle \frac{\dot{x}^2}{x^4} \right\rangle. \tag{102}$$

This integral will diverge as $x \to 0$. However, we know that this non-relativistic treatment will fail as we get to the Schwarzschild radius as the gravitational radiation will not escape. If we cut the integral off at $x_{min} = 2GM/c^2$

$$\Delta E = \frac{2}{105}\frac{\mu^2 c^2}{M}. \tag{103}$$

Now we should expect that this is an overestimate as the closer one gets to the Schwarzschild radius the less gravitational wave energy escapes but we also ignored other relativistic effects. The fall of a test particle into Schwarzschild black hole of mass M gives

$$\Delta E = 0.0104 \frac{\mu^2 c^2}{M}. \tag{104}$$

This is roughly half of our semi-classical approach. Epply and Smart have calculated the radiation form the head-on collision of two equal-mass Schwarzschild black holes to find

$$\Delta E = 0.001 M c^2 \tag{105}$$

however, there is uncertainty which is at the level of a factor of 2 so it might be in reasonable agreement with the formula above. We can presume that there is more radiation if the collision is not head-on or if there is significant angular momentum in the system.

How would you do this calculation for two particles colliding if in addition to the gravitational attraction, they had significant peculiar velocity? For example, gravitational bremsstrahlung between an electron and proton or two protons in a hot plasma.

The frequency spectrum from a burst of gravitational waves will clearly not be periodic or quasiperiodic. Instead it will be much like EM bremsstrahlung radiation. Modes will be excited from low frequencies right on up to the inverse of the fastest time in collision.

$$\Delta E = \int \frac{dE}{dt} dt = \frac{G}{5c^5}\int \dddot{Q}^{kl}\dddot{Q}^{kl} dt \tag{106}$$

The total energy radiated during the collision is

$$\Delta E = \int \frac{dE}{dt} dt = \frac{G}{5c^5}\int_{-\infty}^{\infty} \dddot{Q}^{kl}\dddot{Q}^{kl} dt \tag{107}$$

$$= \frac{1}{10\pi}\frac{G}{c^5}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \tilde{Q}^{kl}(\omega)\tilde{Q}^{kl}(\omega') e^{i(\omega-\omega')t} \tag{108}$$

where $\tilde{Q}^{kl}(\omega)$ is the fourier transform of $\dddot{Q}^{kl}$:

$$\tilde{Q}^{kl}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \dddot{Q}^{kl} e^{i\omega t} dt, \quad \text{and} \quad \dddot{Q}^{kl} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{Q}^{kl}(\omega) e^{-i\omega t} dt, \tag{109}$$

Parseval's theorem gives

$$\frac{dE}{d\omega} = \frac{2}{5}\frac{G}{c^5}|\tilde{Q}^{kl}(\omega)|^2 \tag{110}$$

For this case of two particles coming together under the influence of gravity in the low frequency regime

$$\frac{dE}{d\omega} \rightarrow \frac{2^{11/3}, (\frac{1}{3})^2}{5\pi 3^{7/3}} \frac{G}{c^5} (GM\omega)^{4/3}\mu^2 \quad \text{as} \quad \omega \rightarrow 0 \tag{111}$$

The high frequency cut off corresponds to the time scale at the Schwarzschild horizon. The figure handed out in class shows a representative spectrum of this type.

4.3 GW from Non-Spherical Collapse

One promising source of gravitational radiation is the non-spherical collapse of stellar cores (e.g. supernova). Detailed calculations are clearly very complicated. We consider a highly idealized symmetric case to get an order of magnitude estimate of the effect. Consider the Newtonian collapse of a homogeneous spheroid or ellipsoid.

$$-\frac{dE}{dt} = \frac{2}{375}\frac{GM^2}{c^5}\left\langle (\dddot{a}^2 - \dddot{c}^2)^2 \right\rangle \tag{112}$$

where a and c are the major and minor semi-axes. This collapsing configuration will give out radiation in proportion to the fourth power of the angular momentum.

$$\Delta E \sim (\frac{J}{GM^2/c})^4 Mc^2 \tag{113}$$

so with a lot of angular momentum the efficiency can be significant.

Estimate the efficiency $\epsilon = \Delta E/Mc^2$ in the formation of a remnant with the angular momentum of the Crab pulsar $J \sim 2 \times 10^{47}$ erg s^{-1}and for the pulsar PSR 1937+214 with $J \sim 4 \times 10^{48}$ erg s^{-1}.

5 Chaotic Gravitational Waves

The first set of chaotic gravitational waves that we encountered were the potential Cosmic Gravity Wave Thermal Background which you estimated would have an effective temperature of about 0.9 K. In a moment we will see how this could have come about; however, we should first consider how these might have been erased (stretched to very long wavelengths) and replaced by a new chaotic field by cosmic inflation.

5.1 Gravitons from Inflation

We saw in a previous lecture that inflation, in addition to producing density (scalar) fluctuations, would produce a random, chaotic field of gravitational (tensor) fluctuations. While these are outside the event (particle) horizon they are essentially frozen fluctuations in curvature

$$\delta ln(a) = \frac{\delta\rho}{\rho + P} \tag{114}$$

which Bardeen showed was a gauge invariant quantity. To first order we expect that these curvature fluctuations will be produced with roughly equal amplitude and thus potential fluctuations with dimensionless strain of order

$$(A_{\oplus,\otimes})_k = (h_{+,\times})_k \approx \frac{2}{\sqrt{\pi}} \frac{H_{inflation}}{M_{Planck}} \tag{115}$$

These fluctuations started their existence as part of the universe's wave function and as individual gravitons.

Once the universe is sufficiently old these curvature fluctuations were free to propagate as gravitational waves. That happens when a given graviton mode re-enters the horizon and the tensor metric fluctuations then propagate as gravitons. They also begin to redshift away due to the expansion of the universe.

The energy density of a chaotic field of gravitational waves has an energy density

$$\rho_{graviton} =< \dot{A}_{\oplus}^2 + \dot{A}_{\otimes}^2 >= \frac{\dot{h}_+^2 + \dot{h}_\times^2}{16\pi G} \tag{116}$$

or in power spectrum

$$\rho_{graviton} = \frac{1}{16\pi G} \int [(h_+)_k^2 + (h_\times)_k^2] k dk. \tag{117}$$

This leads us to the estimate of energy density power spectrum

$$k \frac{d\rho_{graviton}}{dk} = \frac{k^2[(h_+)_k^2 + (h_\times)_k^2]}{16\pi G} \tag{118}$$

Thus for the radiation-dominated phase of the universe the change in the energy density at each wavelength is proportional to a^{-2}. .....argument of why as energy per mode/wavelength is originally $\omega^2 A^2$ and it is redshifted away for a longer time inversely proportional to ω, one finds that for modes entering the horizon during radiation dominated phase all have equal energy density per mode (logarithmic interval) in λ .... Thus there is over the range of wavelengths $10^{10} to 10^{26}$ cm an essentially flat energy density or a dimensionless strain falling as $\propto \omega^{-2} \propto \lambda^2$. For longer wavelengths the mode entered during the matter-dominated phase and the energy density rises proportional to λ until one reaches the horizon. thus the dimensionless strain grows proportional to the wavelength $\propto \omega^{-1} \propto \lambda$. Outside the horizon the dimensionless strain is (constant) equal to the value set by the expansion rate of inflation divided by the Planck mass (to the factor $2/\sqrt{\pi}$. See Figure xx. for an example of the gravitational wave spectrum expected for a sample model of inflation.

We can estimate the present (after expansion) strain of inflation epoch gravitational waves as a function of wavelength

$$h_0 \sim \Omega_\gamma^{1/2} \frac{\lambda_0 H_0}{c} (\frac{H_{inf}}{m_{pl} c^2})^2 \tag{119}$$

Thus a detector operating at a wavelength of $\lambda_0 = 10^3$ km, $H_0 \lambda_0 / c \sim 10^{-20}$. The density in radiation $\Omega_\gamma \sim 10^{-4}$, so $h_0 \sim 10^{-20} (H_{inf}/m_{pl}c^2)^2$.

5.2 Gravitational Waves from Thermal Collisions

We know that gravitational radiation is only emitted when particles actually undergo accelerations. In thermal plasmas, which are common in astrophysics and cosmology, there are thermal collisions - usually coloumb and elastic scattering - frequently. The energy per unit frequency interval emitted as gravitational waves in a collision of particles is

$$\frac{dE}{d\omega} = \frac{G}{2\pi} \sum_{in,out} (-1)^{in/out} m_n m_n \frac{1+\beta_{nm}^2}{(1-\beta_{nm}^2)^{1/2}} ln(\frac{1+\beta_{nm}}{1-\beta_{nm}}) \tag{120}$$

where β_{nm} is the relative velocity between particles n and m. For non-relativistic two-body scattering this reduces to

$$\frac{dE}{d\omega} = \frac{8G}{5\pi} \mu^2 \, v^4 \, sin^2\theta \tag{121}$$

where μ is the usual reduced mass, v is the relative velocity, and θ is the scattering angle in the center-of-mass frame.

The gravitational radiation produced by the collisions in a plasma or gas can be obtained by summing the radiated energies per collision, provided the collisions are incoherent (i.e. there is more time between collisions than it takes to radiate so that they do not interfere with each other coherently). This condition is that that we are considering radiation with $\omega_{GW} >> \omega_c$ where ω_c is the collision frequency per particle. In the incoherent regime

$$\frac{dE}{d\omega} = \frac{8G}{5\pi} \sum_{a,b} \mu_{ab}^2 n_a n_b \left\langle v_{ab}^2 \int \frac{d\sigma_{ab}}{d\Omega} \, sin^2\theta \, d\Omega \right\rangle \tag{122}$$

where n_a and n_b are the number densities of particles of type a and b, $d\sigma_{ab}/d\Omega$ is the center-of-mass-system differential scattering cross-section. The sum runs over all paris of particles and the average is taken over all collisions.

5.2.1 GW from a Plasma

As an example we calculate the gravitational radiation emitted by Coulomb scattering in a plasma. The Rutherford scattering we can retcross-section is

$$\frac{d\sigma_{ab}}{d\Omega} = \frac{q_a^2 q_b^2}{4v_{ab}^4 \mu_{ab}^2 sin^4(\theta/2)} \tag{123}$$

The integral over θ must be cut off at a minimum angle determined by the Debye screening of the Coulomb force at large impact parameters;

$$\int \frac{d\sigma_{ab}}{d\Omega}\ sin^2\theta\ d\Omega = \frac{4\pi q_a^2 q_b^2 ln(1/\theta_{min})}{v_{ab}^4 \mu_{ab}^2} \tag{124}$$

To average over collisions we must find $< v_{ab} >$, as one power is left. For a Maxwell-Boltzmann distribution

$$\langle v_{ab} \rangle = 2(\frac{2kT}{\pi \mu_{ab}})^{1/2} \tag{125}$$

Typically $ln(1/\theta_{min}) \sim 10$. For a plasma of completely ionized hydrogen we must take into account electron-proton and electron-electron collisions. (Why don't we worry about proton-proton and electron-helium plus proton-helium collisions?)

$$\frac{dP}{d\omega} = \frac{64 G n_e^2 e^4}{5c^5}\ (\frac{2kT}{\pi \mu_{ab}})^{1/2}\ (1+\sqrt{2})\ ln(1/\theta_{min}) \tag{126}$$

The electron collision frequency can be estimated as

$$\omega_c \approx \frac{e^4 n_e < v >}{(kT)^2} \approx \frac{e^4 n_e}{(kT)^{3/2} m_e^{1/2}}. \tag{127}$$

5.2.2 GW from the Sun & Stars

Now let us apply this to the hydrogen plasma in the solar or a stellar core. Within a volume of roughly 2×10^{31} cc this plasma has $T \simeq 10^7$ K, $n_e \simeq 3 \times 10^{25}$ cc, and $ln(1/\theta_{min}) \sim 4$. The collision frequency is roughly 10^{15} s^{-1}, which is three orders of magnitude less than the thermal frequency $\omega_{thermal} = kT/\hbar \approx 10^{18}$ s^{-1}, so that an estimate of the total power produced in gravitational radiation can be found by multiplying by $VkT/\hbar$. This gives about 10^8 watts or 100 megawatts.

5.2.3 GW from the Early Universe Plasma

We could do this same calculation for the early universe. What would be power in gravitational radiation produced by thermal collisions of the plasma at the time of nucleosynthesis? (Take a time of 1 sec and $kT = 1$ MeV.) We neglected photons in the plasma calculation above, should we take them into account here? How far back do we have to go in the universe before there is good thermal interaction between the gravitons and other constituents of the universe?

5.2.4 GW from the ElectroWeak Phase Transition

Extra gravitational radiation could be produced during the ElectroWeak Phase Transition (as well as other phase transitions) over the simple plasma conditions due to the traveling domain wall of the phase transition as well as their collisions with each other. ...

5.2.5 GW from Cosmic Strings

Due to the high tension equal to the mass per unit length μ in cosmic strings, they both try to straighten out at the speed of light as well as have oscillations and traveling waves. These lead to the copious production of gravity waves by strings.

We can make a rough order of magnitude estimate of the gravitational radiation from cosmic strings. Most radiation will come from oscillating loops which are formed when a string crosses itself. A loop naturally forms. Loops surviving to epoch given by time t in the radiation era will have a typical size $l \sim G\mu t$ and will produce gravitational waves of frequency $\omega \sim 1/l$ and energy density

$$\rho_{GW-s} = (\dot{M}t)n(l) \simeq \ \mu \ ln(l) \tag{128}$$

where $n(l)$ is the density of loops larger than l. During the radiation dominated era

$$n(l) \sim t^{-3/2} l^{-3/2} \tag{129}$$

and thus

$$\rho_{GW-s} \simeq \frac{(G\mu)^{1/2}}{Gt^2} \tag{130}$$

Since the energy density in relativistic radiations scales as $a^{-4} \propto t^{-2}$ and is essentially the critical density during the radiation dominated error, we have $\rho_\gamma \approx 1/(30Gt^2)$,

$$\Omega_{GW-s} \simeq 30(G\mu)^{1/2}\Omega_\gamma = 6 \times 10^{-7} h^{-2} (G\mu/10^{-6})^{1/2} \tag{131}$$

A more precise calculation (Brandenberger et al. 1986) gives a slightly lower estimate

$$\Omega_{GW-s} = 4 \times 10^{-8} h^{-2} \Omega_0^{-1} (G\mu/10^{-6})^{1/2} \tag{132}$$

Calculate the limit that is set by precision pulsar timing, on the energy density of gravity waves with frequencies of 1/4 yr^{-1}. Five kpc away is PSR 1937+21 discovered by Don Backer which has had a steady 1.6 ms pulse for four years. The time between the main pulse and an interpulse is given as 744.9 plus or minus 1.3 microseconds over those four years. It is part of a group of pulsars whose relative timing is good to about 3 microseconds. Does this come close to the expected signal from GUT cosmic strings? Do more years of observation help?

Hint: Calculate the strain needed to change the timing by 1 microsecond (distance = 5 kpc). Next calculate the energy density for gravitational waves with wave length of 4 light years and compare that to ρ_c to get Ω_{GW}.

6 Anisotropies from Gravitational Waves

What is the typical CBR anisotropy expected from a long wavelength gravity wave in which we are immersed? (Calculate a numerical value that sets the scale for the mean expected rms fluctuation if the energy scale of inflation at current horizon crossing is the GUT scale, first for the quadrupolar term due to the gravity wave in which we currently find ourselves and then for the chaotic field filling the universe. What would we expect for the apparent random velocity dispersion of clusters of galaxies relative to each other due to this chaotic field of gravity waves?

We expect the frequency shift for entering photons to be given the difference in the dimensionless strain at the point of reception and the point of emission.

$$\frac{\Delta T}{T} = z(\theta, \phi) = \frac{1}{2}(h_r - h_e)\ (1 - cos\theta)\ cos2\phi \tag{133}$$

where h_r and h_e are the dimensionless strain at the receiver and emitter respectively and theta is the angle between the line of sight and the direction of propagation of the plane wave $(\hat{\mathbf{n}} \cdot \vec{\mathbf{k}})$ and ϕ is the azimuthal angle around the direction of motion relative to the polarization of the wave.

For a random chaotic field we can estimate the resulting mean square amplitude of temperature or velocity variations as

$$\left\langle z^2 \right\rangle = \left\langle \frac{\Delta T}{T} \right\rangle = \frac{1}{3\pi^3}\ \rho_{GW}\ \omega^{-2} \tag{134}$$

where 5/8 of this is produced by the quadrupole.

Consider the random chaotic field produced by inflation which will have $< A >= 0$ but will have an rms value

$$\left\langle (\frac{\Delta T}{T})^2 \right\rangle = \frac{1}{4} \left\langle \omega h H_0^{-1} \right\rangle^2 \simeq \frac{\omega^2 h^2}{4 H_0^2} \approx \frac{8\pi G \rho_{GW}}{H_0^2} \simeq 3\Omega_{GW} \tag{135}$$