P139 - Relativity

Problem Set 13: Compact Objects/Black Holes 1998 April

G. F. SMOOT
Department of Physics,
University of California, Berkeley, USA 94720

1 Black Hole History Summary

1795: Laplace points out that Newtonian theories of gravitation and corpuscular light implies that some stars might be so massive and compact that no light could escape.
1915: Einstein publishes theory of General Relativity.
1915 December: four months later Schwarzschild publishes first solution.
1931 Chandrasekhar publishes upper limit on White Dwarf Mass
1932: Chadwick discovers neutron; Landau gives generic argument on upper limit for self-gravitating mass of degenerate baryons.
1935: A. Eddington understand the implication that Chandrasekhar (or Landau) argument implies the ultimate gravitational collapse of massive stars.
1939: J. R. Oppenheimer and H. Snyder treat the collapse of homogeneous sphere and show it gets cut off from communication with the rest of the Universe.
1968: John Wheeler coins the name Black Hole.
1965: Penrose and 1973: Penrose and Hawking theorems on gravitational collapse.

1.1 The End State of Stars

(a) Ordinary stars are self-gravitating and therefore must have hot interiors to sustain the thermal pressure that resists the inward pull of gravity.
(b) Space, outside of stars, is dark and cold. Thus heat flows continuously outward. (Stars shine at night!)
(c) As long as a star behaves as a classical gas, there is no true thermodynamic equilibrium. A star will continue to lose heat to the Universe. Thus a star will contract and get hotter and hotter internally gaining heat from the gravitational contraction.

It is a never ending struggle against gravity. The second law of thermodynamics, nuclear energy sources can provide only temporary respite. When they run out a star inevitably faces death, either convulsive, e.g. as a supernova or a lingering slide towards collapsed state.
(d) What endings are possible:

(i) Nothing
(ii) White Dwarf
(iii) Neutron Star
(iv) Black Hole

2 Schwarzschild Metric

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

The Schwarzschild Metric may be derived by assuming that the metric depends only on the radius for a static, spherically-symmetric mass. This means that when written in the form

$$ds^2 = e^{\lambda} dt^2 - e^{\nu} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

the metric can depend only upon the coordinate r. The Einstein equations imply that $\lambda + \nu = 0$ and that $e^{\lambda} = 1 - 2GM/c^2 r$ exterior to the spherical mass. (Note we are using $c = 1$ for these equations. Below, c is included when it is appropriate, i.e. when calculating numbers.) One could use the same Einstein equations to solve for the metric interior to the mass. However, the relativistic analog of the Newtonian theorem that says for the interior of a spherical shell, the potential is constant. This means that the metric at any location can be found by using the interior mass, i.e. $g_{00} = e^{\lambda} = 1 - 2GM(r)/c^2 r$ where the interior mass is the integral of the stress-energy tensor, T_0^0,

$$M(r) = \int_o^r T_0^0 4\pi R^2 dR.$$

This matches to the Schwarzschild metric if M is the total integrated $M(r)$ to the surface of the mass.

Make the same assumption as Einstein—that the energy density consists of only a uniform distribution of mass, ρ, and a cosmological constant energy density, $\Lambda/8\pi G c^2$. Find the interior solution for the metric. What is the metric for the region outside of the mass, total value M, if the cosmological energy density continues? Note that this metric does not become asymptotically flat as $r \to \infty$. However, as we discussed in class, $\Lambda/c^2 \leq 10^{-58}$ cm^{-2}, is very small, so there is a range of distances

$$\frac{c}{\sqrt{\Lambda}} >> r >> \frac{GM}{c^2}$$

in which the metric is nearly flat.

In the weak field limit $g_{00} \cong 1 + 2\Phi/c^2$. What is the effective potential Φ for this case? What is the force as a function of distance? What sign does it have relative to gravity? This is why the cosmological constant is sometimes referred to as a force that increases with distance.

Consider a cluster of galaxies extending as far as 10 Mpc (10 million parsecs). How big could the cosmological constant correction be? If there are about 2000 galaxies in the cluster with typical masses on the scale of $10^{11} M_{\odot}$, then what is its ratio relative to Newtonian gravity?

3 Maximum Field

In class I gave an argument based upon the Uncertainty Principle as to the maximum electric field that could exist in space. Reproduce that argument and provide a formula for the electric field and find its value in volts per cm.

4 Electrostatics: Reissner-Nördstrom Metric

An electrically charged mass will be surrounded by an electric field, which gives rise to a nonzero energy-momentum tensor through-out space. To find the metric requires solving the coupled Einstein and electromagnetic equations. For the stationary spherically symmetric solution, the metric will have the general form of the equations in Problem 3. However, the functions $\lambda(r)$ and $\nu(r)$ will have a different interpretation. The electric field will be in the radial direction and the electromagnetic field tensor, in spherical coordinates, is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where T^{00} is the well-known energy density for an electromagnetic field, and T^{0k} is the Poynting vector:

$$T^{00} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) \quad T^{0k} = \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})^k.$$

The trace is zero: $T^{\mu}_{\mu} = 0$. Setting $c = 1$ and using the same theorem for a spherically symmetric energy system, the results for the metric and the electric field are:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{r^2}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega^2$$

and

$$E(r) = \frac{Q}{r^2}$$

Is it clear that Gauss's law is valid?

Clearly there is a radius of a charged particle at which the metric will go flat. What is that radius?

What is that radius for an electron? Express your answer in terms of the classical radius of the electron. What is the field strength E there in volts per cm? How does this compare to the maximum field strength computed in the previous problem? Can you conclude that the real electric charge on the electron is actually greater than e because of vacuum polarization effects? What if the electron was really a point particle? Does the metric have a singularity if $Q^2 > GM^2$?

5 White Dwarfs

In 1932 Landau gave a rough derivation of the Chandrasekhar (1931) mass limit for an electron-degenerate star. His argument applies to both white dwarf and neutron stars.

Suppose that there are N fermions in a star of radius R, so that the number density of fermions is $n \sim N/R^3$. The volume per fermion is $\sim 1/n$ since the Pauli exclusion principle indicates that they cannot occupy the same phase space. Then, the Heisenberg uncertainty principle implies the momentum of the fermions is $\sim \hbar n^{1/3}$. Thus, the Fermi energy of a gas particle in the relativistic regime is

$$E_{\mathrm{F-rel}} \sim \hbar n^{1/3} c \sim \frac{\hbar c N^{1/3}}{R}.$$

For a non-relativistic particle $E \cong p^2/2m$, so that

$$E_{\mathrm{F-nonrel}} \sim \hbar n^{2/3} c \sim \frac{\hbar c N^{2/3}}{R^2}$$

The gravitational energy per fermion is

$$E_G \sim -\frac{GMm_b}{R}$$

where $M = Nm_b$ and we assume that most of the mass is in baryons (even if the pressure comes from electrons). Equilibrium is achieved at a minimum of the total energy, $E = E_{\mathrm{F}} + E_G$. In the relativistic case

$$E_{\mathrm{rel}} \sim \frac{\hbar c N^{1/3}}{R} - \frac{GMm_b}{R}$$

Both terms scale as $1/R$. When E is positive, decreasing R will increase both E and E_{F}; increasing r decreases both E and E_{F}. If E_{F} is decreased sufficiently the electrons (or neutrons) will become non-relativistic and $E_{\mathrm{F}} \sim \hbar c N^{2/3}/R^2$. Thus eventually E_G will become larger than E_{F}, and then E will be negative (eventually tending to zero as $R \to \infty$). There will therefore be a stable equilibrium at a finite value of R. (A stable equilibrium requires a minimum in the potential and the above argument is that the potential is positive at small R, zero at $R \to \infty$ and negative in between. This implies that there is a minimum in the potential at a finite value for R.)

If the sign of E is negative, E can be decreased without bound by decreasing R and no equilibrium exists so that gravitational collapse will occur.

The maximum baryon number for equilibrium is therefore determined by setting $E = 0$:

$$N_{\mathrm{max}} \sim \left(\frac{\hbar c}{Gm_b^2}\right)^{3/2} \approx 2 \times 10^{57}$$

$$M_{\max} \sim N_{\max} m_b \approx 1.5 M_\odot, \qquad M_\odot = 1.989 \times 10^{33} \text{ gm}$$

With the exception of composition-dependent numerical factors, the maximum mass of a degenerate star thus depends only on fundamental constants. What would you estimate for the ratio of the maximum masses of a star made of helium and one made of hydrogen?

The equilibrium radius associated with masses M approaching $M_{\max}$ is determined by the onset of relativistic degeneracy:

$$E_{\mathrm{F}} \geq mc^2$$

where m is the mass of the degenerate fermion (i.e. electrons or neutrons). The formula for $E_{\mathrm{F-rel}}$ gives an idea of the radius scale.

$$\begin{aligned} R &\sim \frac{\hbar}{mc}\left(\frac{\hbar c}{G m_b^2}\right)^{1/2} \\ &\sim 5 \times 10^8 cm, \qquad m = m_{\mathrm{e}} \\ &\sim 3 \times 10^5 cm, \qquad m = m_b. \end{aligned}$$

There are thus two distinct regimes of collapse: one for densities above white dwarf values and another for densities above nuclear densities.

Though these arguments give the correct order of magnitude, note that the rough radius for collapse of a neutron star is approximately equal to the Schwarzschild radius. Thus we expect that the equilibrium value for the radius of such a neutron star is not much greater than the Schwarzschild radius and a more careful treatment will be necessary.

We expect a star to be in hydrostatic equilibrium; the inward pull of gravity should be matched by the outward push of the pressure. For a spherically symmetric distribution of matter, the mass interior to a radius r is

$$M(r) = \int_0^r \rho \, 4\pi R^2 \, dR, \quad \text{or} \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho$$

(compare with the expression in Problem 3). To find the equation of hydrostatic equilibrium, consider an infinitesimal fluid element lying between r and $r + dr$ and having area dA perpendicular to the radial direction. The gravitational attraction between $M(r)$ and the mass $dm = \rho \, dA \, dr$ is $GM(r)dm/r^2$. The net outward pressure force on dm is $-[P(r+dr) - P(r)]dA$, so in equilibrium these are matched giving

$$-\frac{dP}{dr} \, dr \, dA = \frac{GM(r)}{r^2} \, dm$$

or

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}.$$

These two equations can be combined to give the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

This is useful, provided we know the equation of state (i.e., the relationship between the pressure P and density ρ) for matter. For many gases this is simply

$$P = \mathrm{K}\rho^{\gamma} \quad \text{where K and } \gamma \text{ are constants.}$$

This relation is true for a degenerate fermion gas. A volume V at pressure P has a pressure energy PV. A degenerate fermion gas with N particles in volume V has a number density $n = N/V$ and a Fermi energy E_{F} per particle given in the formulae above. Thus $P = NE_{\mathrm{F}}/V \sim nE_{\mathrm{F}} \sim n\hbar n^{1/3} = \hbar n^{4/3}$ for the relativistic case, and $\sim n\hbar n^{2/3} = \hbar n^{5/3}$ for the non-relativistic case. Since $n \propto \rho$, $\gamma = 4/3$ for the relativistic case and $\gamma = 5/3$ for the non-relativistic case. The last is familiar since

$$PV^{\gamma} = \text{constant} \quad \text{or} \quad P = K\rho^{\gamma}$$

with $\gamma = 5/3$ is the adiabatic relation for an ideal monoatomic gas (having only translational degrees of freedom). Thus, this relation holds for the thermal phase of a star.

With the equation of state in hand, this differential equation can be solved (although usually numerically).

The central pressure required to support a self-gravitating sphere of radius R and mass M, where $P \propto \rho^{5/3}$ is

$$P_{\mathrm{c}} = 0.770\,\frac{GM^2}{R^4},$$

and the corresponding central density is given by

$$\rho_{\mathrm{c}} = 5.99\langle\rho\rangle = 1.43\,\frac{M}{R^3}.$$

For a self-gravitating sphere with $P \propto \rho^{4/3}$, hydrostatic equilibrium requires that

$$P_{\mathrm{c}} = 11.0\,\frac{GM^2}{R^4}, \quad \text{and} \quad \rho_{\mathrm{c}} = 54.2\langle\rho\rangle = 12.9\,\frac{M}{R^3}$$

A careful calculation of the degenerate electron gas pressure gives

$$\begin{aligned} P_{\mathrm{e}} &= \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2 n_{\mathrm{e}}^{5/3}}{m_{\mathrm{e}}} = 0.0485\,\frac{h^2 n_{\mathrm{e}}^{5/3}}{m_{\mathrm{e}}} \quad (\text{non}-\text{relativistic}) \\ &= \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hc\; n_{\mathrm{e}}^{4/3} = 0.123\; hc\; n_{\mathrm{e}}^{4/3} \quad (\text{relativistic}). \end{aligned}$$

To a very good approximation, a white dwarf is charge neutral so that the number density of the electrons is equal to the sum over ion type i of the charge Z_i of the ion species, times the number density n_i of the ith ion species:

$$n_{\rm e} = \sum_i Z n_i = \sum_i \frac{Z_i}{A_i} n_b = \left\langle \frac{Z}{A} \right\rangle n_b$$

Because electrons are so light

$$n_{\rm e} = \left\langle \frac{Z}{A} \right\rangle \frac{\rho}{m_b}.$$

Show that

$$P_{\rm e} = 0.0485 \; \frac{h^2}{m_{\rm e}} \; \left(\frac{Z}{A}\right)^{5/3} \; \frac{\rho^{5/3}}{m_b^{5/3}}$$

and that

$$R = 0.114 \; \frac{h^2}{G m_{\rm e} m_b^{5/3}} \; \left(\frac{Z}{A}\right)^{5/3} M^{-1/3}$$

where Z/A is now the average over all ion species. Notice that the volume is inversely proportional to the mass.

Compute R and ρ_c for $M = 0.5 M_\odot$ and $M = 1.0 M_\odot$. It is helpful to have the non-relativistic ($\gamma = 5/3$) relations:

$$\frac{R}{10^4 \text{ km}} = 1.122 \times \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{-1/6} \left(\frac{A}{2Z}\right)^{-5/6}$$

and

$$\frac{M}{M_\odot} = 0.4964 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{1/2} \left(\frac{A}{2Z}\right)^{-5/2} = 0.7001 \left(\frac{R}{10^4 \text{ km}}\right)^{-3} \left(\frac{A}{2Z}\right)^{-5}.$$

For the relativistic electron gas the degenerate pressure is

$$P_{\rm e} = 0.123 \; hc \; n_{\rm e}^{4/3}.$$

It is easy to show that

$$P_{\rm e} = 0.123 \; \frac{hc}{m_b^{4/3}} \; \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}.$$

Set the central pressure needed to that available from relativistic electron degeneracy. Show that R cancels out of each equation and solve for M to get Chandrasekhar's mass limit:

$$M_{\rm Ch} = 0.2 \left(\frac{Z}{A}\right)^2 \left(\frac{hc}{G m_b^2}\right)^{3/2} m_b.$$

Again, the maximum mass of a white dwarf is set by fundamental constants. Evaluate this as you did in the Landau case for a star made of helium or hydrogen to find its maximum mass and radius. Once again, if you do not want to put in the proton/neutron mass and G, h, and c, it will help to have the relations:

$$\frac{M}{M_\odot} = 1.4587 \left(\frac{2Z}{A}\right)^2$$

and

$$\frac{R}{10^4 \text{ km}} = 3.347 \times \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{-1/3} \left(\frac{A}{2Z}\right)^{2/3}$$

Note again that the mass M is independent of ρ_c and R in the extreme relativistic limit.

Concluding Remarks: After Chandrasekhar's original work a number of corrections were noticed. There is an electrostatic interaction between the electrons and ions which give smaller radii and higher central densities; though, in general these corrections are relatively small.

General Relativity induces a dynamical instability for white dwarfs when their radii become smaller than 1.1×10^3 km.

In thermonuclear reactions, the thermal energy of the reacting nuclei overcomes the Coulomb repulsion between them so that nuclear reactions can proceed. A white dwarf does not have sufficient mass and pressure to get past burning hydrogen to helium. This is why it can eventually cool and become a degenerate electron gas and begin its long slow collapse. If it reaches sufficiently high density, even at zero temperature, the high density and zero point energy of the nuclei in a lattice can lead to an appreciable rate of *pycnonuclear* nuclear reactions ("pyknos" is "dense" in Greek). Hamada and Salpeter estimated that in 10^5 years hydrogen would be converted to ^{4}He via pycnonuclear reactions above a density of 5×10^4 g cm^{-3}. Higher densities can burn the helium to Carbon and at even higher densities the Carbon may burn to Magnesium. These calculations of rate are highly uncertain, but it seems that the maximum white dwarf radius is $3.90 \times 10^{-2} R_\odot$ at $M = 2.2 \times 10^{-3} M_\odot$ for cold carbon stars.

Following the discovery of the neutron and its beta decay to a proton and electron

$$n \rightarrow p \; + \; e^- \; + \; \bar{\nu}_e$$

it was realized that at very high densities electrons would react with protons to form neutrons via inverse beta decay:

$$p \; + \; e^- \rightarrow n \; + \; \nu_e.$$

If all of a white dwarf's electrons and protons were converted to neutrons, what would be the equation for the radius of the neutron star? Hint: Set $Z/A = 1$ in the original equations and replace the electron mass with a neutron mass. What radius do you get for a $M = 1.4 M_\odot$ neutron star?

6 Neutron Stars

The first calculation of neutron star models was performed by UCB locals Oppenheimer and Volkoff (1939) who derived the relativistically correct hydrostatic equation in a Schwarzschild metric. Here one has to use the metric for the interior $M(r)$ that was used in Problem 3. Then Einstein's equations give

$$\frac{dM(r)}{dr} = 4\pi\rho$$

(nothing new there),

$$\frac{dP}{dr} = -\frac{\rho GM(r)}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi Pr^3}{GM(r)}\right)\left(1-\frac{GM(r)}{r}\right)$$

(note that when $GM/r \ll 1$ and $P/\rho \ll 1$, this reduces to the non-relativistic, Newtonian case of the previous problem), and

$$\frac{d\Phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr}\left(1+\frac{P}{\rho}\right)^{-1}$$

These are the Oppenheimer-Volkoff equations of hydrostatic equilibrium.

Low density neutron stars with the ideal neutron gas equation of state ($P = K\rho^{5/3}$) give

$$\frac{R}{1\ \text{km}} = 14.64\left(\frac{\rho_c}{10^{15}\ \text{g cm}^{-3}}\right)^{-1/6}$$

and

$$\frac{M}{M_\odot} = 1.102\left(\frac{\rho_c}{10^{15}\ \text{g cm}^{-3}}\right)^{1/2} = \left(\frac{15.12\ \text{km}}{R}\right)^3.$$

Thus, in the Oppenheimer-Volkoff calculation, there is no minimum neutron star mass; in reality, neutrons become unstable to beta decay at sufficiently low density. They also found a maximum mass of $0.7M_\odot$. Calculate the radius and central density for this neutron star. How does this $\gamma = 5/3$ result compare with the relativistic limit of $R = 9.6$ km and $\rho_c = 5 \times 10^{15}$ g cm^{-3}?

Comment: The real equation of state for neutrons as they approach nuclear densities is stiffer than the relativistic case, because repulsive nuclear forces become strong (as they must if nuclei are to be stable). It is more like the non-relativistic equation of state or stiffer. Thus, it is likely that neutron stars of $1.4M_\odot$ are possible, and are perhaps favored by nature.

7 Black Holes

To explore the Schwarschild geometry further, consider the motion of a freely moving test particle. Such a particle must move on a geodesic in space-time. This means that

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}$$

is an extremum. In the case of particles with mass it is a maximum. The Euler-Lagrange equations that result from the calculus of variations give

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}g_{\alpha\beta,\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0.$$

Using the simple relation from the metric,

$$\frac{dg_{\mu\nu}}{d\tau} = g_{\mu\nu,\alpha}\frac{dx^\alpha}{d\tau},$$

this becomes

$$\frac{d^2x^\sigma}{d\tau^2} + \frac{1}{2}g^{\sigma\mu}\left(2g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Since this looks so terrible, consider the connection to classical mechanics. The Lagrangian $\mathcal{L}$ is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\, m\; g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\
&= \frac{1}{2}m\Bigg[\left(1 - \frac{2GM(r)}{c^2r}\right)\left(\frac{dt}{d\tau}\right)^2 \\
&\qquad - \left(1 - \frac{2GM(r)}{c^2r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\Bigg].
\end{aligned}$$

A particle will follow a path that makes the action $A = \int \mathcal{L} d\tau$, an extremum. The Euler-Lagrange equations are:

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} - \frac{\partial\mathcal{L}}{\partial x^\alpha} = 0.$$

For θ, ϕ, and t these equations are, respectively:

$$\begin{aligned}
\frac{d}{d\tau}(r^2\dot{\theta}) &= r^2\sin\theta\cos\theta\dot{\phi} \\
\frac{d}{d\tau}(r^2\sin^2\theta\dot{\phi}) &= 0 \\
\frac{d}{d\tau}\left[\left(1 - \frac{2GM}{c^2r}\right)\dot{t}\right] &= 0.
\end{aligned}$$

Instead of using the r equation it is easier to use the fact that

$$g_{\alpha\beta}u^\alpha u^\beta = 1 \quad \rightarrow \quad g_{\alpha\beta}p^\alpha p^\beta = m^2.$$

Now, orient the coordinate system so that initially the particle is moving in the equatorial plane ($\theta = \pi/2,\ \dot{\theta} = 0$). Then the particle will stay in the equatorial plane and

$$p_\phi \equiv r^2\dot{\phi} = \text{constant} = l,$$
$$p_t \equiv (1 - \frac{2GM}{c^2 r})\dot{t} = \text{constant} \equiv E_\infty,$$

where E_∞ is a constant of integration, and is identified as the energy at infinity. It is related to the local energy $E^2_{\text{local}} = p^t p_t = g^{00} p_t p_t$ by the redshift factor $(1 - 2GM/c^2 r)^{-1/2}$. That is,

$$E_\infty = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} E_{\text{local}} \approx \left(1 - \frac{GM}{r}\right) E_{\text{local}}.$$

The final expression corresponds to the weak field limit.

Gravitational Redshift For an alternate derivation of the redshift formula, use the fact that E_∞ is constant along the photon path to show that

$$\frac{\nu_{\text{emitted}}}{\nu_{\text{received}}} = \frac{(1 - 2GM/c^2 r_{\text{received}})^{1/2}}{(1 - 2GM/c^2 r_{\text{emitted}})^{1/2}}.$$

What is it approximately for an observer at a very large distance receiving the photons? Explain why the Schwarzschild black hole radius (horizon) is sometimes called the "surface of infinite redshift"? How does this compare to the slowing of clocks, $dt/d\tau$?

The physical interpretation of l is that it is the angular momentum:

$$l = E_{\text{local}} r v^\phi,$$

and it is conserved because the potential is radial. Hereafter, let E denote E_∞. It is convenient to consider the cases of massive and massless particles separately. For a massive particle we can renormalize the energy E and the angular momentum l in to quantities per unit mass.

$$E' = E/m, \qquad l' = l/m$$

so that we can find a simple equation of motion with the form of an effective potential.

Then using the relation $m^2 = g_{\alpha\beta} p^\alpha p^\beta$ show that

$$\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 &= E'^2 - \left(1 - \frac{2GM}{c^2 r}\right)\left(1 + \frac{l'^2}{r^2}\right), \\
\frac{d\phi}{d\tau} &= \frac{l}{r^2}, \quad \text{and} \\
\frac{dt}{d\tau} &= \frac{E'}{1 - (2GM/c^2 r)}.
\end{aligned}$$

Show that a local observer at r finds that the velocity of a radially freely-falling particle released from rest at infinity is given by

$$v^r = \left(\frac{2GM}{r}\right)^{1/2},$$

which has precisely the same form as the Newtonian velocity. For a Schwarzschild mass equal to the Sun's, how long will it take to get to $r = 0$ from r_s?

The equation for the velocity of r is simply the equation expressing conservation of energy. Thus we can express the radial equation as

$$\left(\frac{dr}{d\tau}\right)^2 = E'^2 - V(r)$$

where the effective potential is

$$V(r) = \left(1 - \frac{2GM}{c^2 r}\right)\left(1 + \frac{l'^2}{r^2}\right).$$

It is easy to show that this reduces to the Newtonian expression when $2GM/c^2 r << 1$.

The potential will be at a maximum or minimum when $\partial V/\partial r = 0$. Show that $GMr^2 - l'^2 r + 3GMl'^2 = 0$ for $\partial V/\partial r = 0$, and hence there are no maxima or minima of V for $l' < 2\sqrt{3}GM$.

Show that $V_{\max} = 1$ for $l' = 4GM$. Circular orbits occur when $\partial V/\partial r = 0$ and $dr/d\tau = 0$. Using the equations above gives

$$\begin{aligned} l'^2 &= \frac{GMr^2}{r - 3GM} \\ E'^2 &= \frac{(r - 2GM)^2}{r(r - 3GM)}. \end{aligned}$$

Thus, circular orbits exist down to $r = 3GM/c^2 = 1.5r_s$. The limiting case corresponds to a photon (massless particle) orbit ($E' = E/m \to \infty$). Circular orbits will be stable if V is concave up; that is, $\partial^2 V/\partial r^2 > 0$ and unstable if $\partial^2 V/\partial r^2 < 0$. Why? Show that Schwarzschild orbits are stable if $r > 3r_s = 6GM/c^2$.

The fractional binding energy per unit mass of a particle in the last stable orbit at $r = 3r_s = 6GM/c^2$ is easily found from the last equation for the energy per unit mass:

$$\begin{aligned} E'^2 &= \frac{(r - r_s)^2}{r(r - 1.5r_s)} = 8/9 \\ E'_{\text{binding}} &= 1 - E' = 1 - (8/9)^{1/2} = 5.72\%. \end{aligned}$$

This is the fraction of the rest-mass energy released when a particle, originally at rest at infinity, slowly spirals towards a black hole to the innermost stable orbit, and then plunges into the black hole. Thus, the conversion of the rest mass to other forms of energy is potentially more efficient for accretion onto a black hole than for nuclear burning, which releases a maximum of only 0.9% of the rest mass for burning H to Fe. This is the basis for invoking black holes as the energy source in many models seeking to explain astronomical observations of the huge energy output from compact regions (e.g. Cygnus X-1, AGNs, quasars, double-lobe radio galaxies).

To estimate accretion rates one needs to know the capture cross section for particles falling in from infinity. This is simply

$$\sigma_{\rm capture} = \pi b_{\rm max}^2$$

where $b_{\rm max}$ is the maximum impact parameter of a particle that is captured. We can define the impact parameter b as

$$b = \lim_{r\to\infty} \ r \ \sin\phi.$$

The first two Euler-Lagrange (or geodesic) equations of motion combine together in the limit $r\to\infty$ to give

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 \cong \frac{E'^2-1}{l'^2}.$$

Substituting $r \cong b/\phi$ gives,

$$\frac{1}{b^2} = \frac{E'^2-1}{l'^2},$$

or in terms of the velocity at infinity, $E' = (1-v_\infty^2)^{-1/2}$,

$$l' = bv_\infty(1-v_\infty)^{-1/2} \to bv_\infty \ \ \text{for} \ \ v_\infty << 1.$$

A non-relativistic particle moving towards the black hole ($E' \cong 1, \ v_\infty << 1$) is captured if $l' < 4GM$ (as shown earlier in the problem). Thus the maximum capture impact parameter is

$$b_{\rm max} = \frac{4GM}{v_\infty},$$

which gives a capture cross section

$$\sigma_{\rm capture} = \frac{4\pi(2GM/c^2)^2}{v_\infty^2} = \frac{4\pi r_s^2}{v_\infty^2}.$$

Compare this value with the geometrical capture cross section of a particle by a gravitating sphere of radius R in Newtonian theory:

$$\sigma_{\rm Newtonian} = \pi R^2\left(1+\frac{2GM}{v_\infty^2 R}\right).$$

Thus, a black hole captures nonrelativistic particles like a Newtonian sphere with radius $R = 8GM/c^2 = 4r_s$.

Now consider massless particles like photons. Since the mass is zero we cannot use $d\tau$ because it is zero also and zero over zero gives us an undefined answer. To get the action one uses an arbitrary variable, e.g. s, to parameterize the trajectory in 4-space. The equations of motion become

$$\begin{aligned} \frac{dt}{ds} &= \frac{E}{1-2GM/r} \\ \frac{d\phi}{ds} &= \frac{l}{r^2} \\ \left(\frac{dr}{ds}\right)^2 &= E^2 - \frac{l^2}{r^2}\left(1-\frac{2GM}{r}\right). \end{aligned}$$

By the Equivalence Principle, the particle's world line should be independent of its energy. If we introduce a new parameter $s' = l\ s$ and $b \equiv l/E$ (this is equivalent to dividing the mass out to get an effective potential), the equations of motion become

$$\begin{aligned} \frac{dt}{ds'} &= \frac{1}{b(1-2GM/r)} \\ \frac{d\phi}{ds'} &= \frac{1}{r^2} \\ \left(\frac{dr}{ds'}\right)^2 &= \frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2GM}{r}\right). \end{aligned}$$

The worldlines depend only on the parameter b, which is the particle's impact parameter, and not on E or l separately. Note that in the limit $m \to 0$ the two impact parameters are the same and thus the quantity defined previously is consistent.

The photons can be understood in terms of an effective potential in the same way the massive particles were earlier.

$$V_{\text{massless}} = -\frac{1}{r^2}\left(1-\frac{2GM}{r}\right)$$

and

$$\left(\frac{dr}{ds'}\right)^2 = \frac{1}{b^2} - V_{\text{massless}}(r)$$

Show that $V(r)$ has a maximum of $[1/27(GM)^2]$ at $r = 3GM$. Thus any massless particle heading towards a black hole with $b^2 < 27(GM/c^2)^2$ will be captured. The capture cross section for massless particles, including photons, from infinity is thus

$$\sigma_{\text{massless}} = \pi b_c^2 = 27\pi(GM/c^2)^2.$$

The Galaxy may have as many as 10^9 black holes with masses on the scale of $M_\odot$ confined in a volume of about 10^{15} cubic light years. Light coming from the center of the Galaxy must travel about 3×10^5 light years. What is its chance of getting absorbed by a black hole on the way to us?

8 The fate of a Man who falls in a Black Hole

Consider the plight of an experimental astrophysicist (or cosmic tourist) who stands on the surface of a freely falling star as it collapses to $R = 0$. As the collapse proceeds toward $R = 0$, the various parts of the astrophysicist's body experience different gravitational forces. His feet, which are on the surface of the star, are attracted toward the star's center by an infinitely mounting gravitational force. While his head, which is farther away from he center, is accelerated downward by a somewhat smaller, though ever rising force. The difference between the two accelerations (tidal force) mounts higher and higher as the collapse proceeds, finally becoming infinite as R reaches zero. The astrophysicist's body, which cannot withstand such extreme forces, suffers unlimited stretching between head and food as R drops to zero.

But this is not the only indignity. Simultaneous with the head-to-foot stretching, the astrophysicist is pulled by the gravitational field into regions of spacetime with ever-decreasing circumferential area $4\pi r^2$. Tidal gravitational forces compress the astrophysicist on all sides simultaneously as they stretch him from head to foot. The circumferential compression is actually more extreme than the longitudinal stretching; so the astrophysicist, in the limit $R \to 0$, is crushed to zero volume and indefinitely extended length.

This discussion can be put on a mathematical footing as follows.

There are three stages in the deconstruction of the astrophysicist: (1) the early stage, when his body successfully resists the tidal forces; (2) the intermediate stage of gradual succumbing; and (3) the final stage where the particles in the astrophysicist's body act independently and crush together.

During the early stage, the tidal forces are given by the geodesic equation evaluated in the astrophysicist's orthonormal frame. In this frame the nonvanishing components of the Riemann curvature tensor are:

$$\begin{array}{cc} R_{\tau\rho\tau\rho} = -2GM/r^3, & R_{\tau\theta\tau\theta} = R_{\tau\phi\tau\phi} = GM/r^3, \\ R_{\theta\phi\theta\phi} = 2GM/r^3, & R_{\rho\theta\rho\theta} = R_{\rho\phi\rho\phi} = -GM/r^3 \end{array} \tag{2}$$

The geodesic equation says that two *freely moving* particles, momentarily at rest in the astrophysicist's local inertial frame and separated by the 3-vector

$$\vec{\xi} = \xi^j \vec{\mathbf{e}}_{\mathbf{j}} \tag{3}$$

must accelerate apart with relative acceleration given by

$$D^2\xi^j/d\tau^2 = -R^j_{\tau k\tau}\xi^k = -R_{j\tau k\tau}\xi^k = -R_{\tau j\tau k}\xi^k \tag{4}$$

From the components of the curvature tensor, one has

$$\begin{array}{rcl} D^2\xi^\rho/d\tau^2 & = & +(2GM/R^3)\xi^\rho \\ D^2\xi^\theta/d\tau^2 & = & -(GM/R^3)\xi^\theta \\ D^2\xi^\phi/d\tau^2 & = & -(GM/R^3)\xi^\phi \end{array} \tag{5}$$

To apply these equations to the astrophysicist's body, idealize it as a homogeneous cylinder of mass $\mu = 75$ kg, length $\ell = 1.8$ m in the $\vec{\mathbf{e}}_\rho$ direction, and width and depth of 0.2 m (area A). Then calculate the stresses that must be set up in this idealized body to prevent its particles from moving along diverging and converging geodesics.

From the equations is tis evident that the principal directions of stress are radial to the star (stretching) and radial to the cylindrical axis (compression). A volume element of his body with mass $d\mu$ located at a height h above the center of mass would accelerate with $a = (2GM/r^3)h$ away from the center of mass, it it were allowed to move freely. To prevent this acceleration, the astrophysicist's muscles and connecting tissue must exert a force

$$dF = a d\mu = (2GM/r^3) h d\mu \tag{6}$$

This force contributes to the stress across the horizontal plane ($\theta - \phi$) through the center of mass. The total force across that plane is the sum of forces on all the mass elements above the center.

$$F = \int_{\text{above plane}} a d\mu = \int_0^{H/2} \frac{2GMh}{r^3} \frac{\mu}{\ell A} A dh = \frac{1}{4} \frac{GM\mu\ell}{r^3} \approx 4 \times 10^{19} \frac{M/M_\odot}{(r/1 \text{ km})^3} \text{ dynes.} \tag{7}$$

The stress is this force divided by the cross-sectional area A, with a minus sign since it is a tension rather than a pressure.

$$T_{\rho\rho} = -\frac{1}{4} \frac{GM\mu\ell}{Ar^3} \approx 10^{15} \frac{M/M_\odot}{(r/1 \text{ km})^3} \text{ dynes/cm}^2 \tag{8}$$

The components of stress in the θ and ϕ are

$$T_{\theta\theta} = T_{\phi\phi} = +\frac{1}{8} \frac{GM\mu\ell}{\ell r^3} \approx +0.7 \times 10^{13} \frac{M/M_\odot}{(r/1 \text{ km})^3} \text{ dynes/cm}^2 \tag{9}$$

Note that one atmosphere of pressure is 1.01×10^6 dynes/cm^2. The human body cannot withstand a tension or pressure of more than 100 atmospheres without breaking. Consequently, and astrophysicist on a freely collapsing star of one solar mass will be killed by tidal forces, when the star's radius is $R \sim 200$ km which is much larger than the Schwarzschild radius of 3 km.

By the time the stars is smaller than its gravitational radius, the baryons of the astrophysicist's body are moving along geodesics. His muscles and bones have given away completely. In this final stage of collapse, the timelike geodesics are curves along which the Schwarzschild "time"-coordinate, t, is almost constant. The astrophysicist's feet touch the star's surface at one value of t, i.e. $t = t_f$ – while his head moves along the curve $t = t_h > t_f$. Consequently, the length of the astrophysicist's body increases according to the formula

$$\ell_{astrophysicist} = [g_{11}(R)]^{1/2}(t_h - t_f) = (2GM/c^2R)^{1/2}(t_h - t_f) \propto R^{-1/2} \propto (\tau_{collapse} - \tau)^{-1/3} \tag{10}$$

Here $\tau = -\int^R |g_{rr}|^{1/2} dr$ + constant is the proper time as it would be measured by the astrophysicist, if he were still alive, and $\tau_{collapse}$ is the time at which he arrives at $r = 0$. The gravitational field also constrains the baryons of the astrophysicist's body to fall along world lines of constant θ and ϕ during the final stages of collapse. His cross-sectional area decreases according to the law

$$A_{astrophysicist} = [g_{\theta\theta}(R) g_{\phi\phi}(R)]^{1/2} \Delta\theta \Delta\phi \propto R^2 \propto (\tau_{collapse} - \tau)^{-4/3} \tag{11}$$

By combining these relations, one sees that the volume of the astrophysicist's body decreases, during the last few moments of collapse, according to

$$V_{astrophysicist} = \ell_{astrophysicist} A_{astrophysicist} \propto R^{3/2} \propto (\tau_{collapse} - \tau) \tag{12}$$

The crushing of matter to infinite density by an infinitely large gravitational forces occurs not only on the surface of the collapsing star, but at any other point along the $r = 0$ singularity outside the surface of the star. Hence any foolish tourist who ventures below the Schwarzschild radius $r_s = 2GM/c^2 r$ is doomed to destruction.

9 Kerr-Newman Black Hole

9.1 "A Black Hole has no hair"

There is a nearly proved theorem that the external field of a black hole is determined uniquely by the mass, charge, and angular momentum that went into the horizon. The heart of this argument is a theorem by R. H. Price that states that if a physical property of a slightly disturbed body is described by a field with spin s, then only moments to one order lower, that is up to order $s - 1$, survive.

For example, electromagnetic information is carried by the photon which has spin 1. Price's theorem therefore allows only electrical information of zero spin to survive. The gravitational information is thought to be carried by the spin 2 graviton. Hence the only surviving gravitational information is order zero and one. These bits are carried by the mass (order zero) and the angular momentum (one).

Because classical physics depends only on gravity and electromagnetism, we can argue that, so far as the known interactions of classical physics are concerned, the most general black hole is characterized by its mass, charge and angular momentum. This is a conjecture prompted by Price's theorem. so far a general proof that any body, however, large its initial irregularities, will reach the same state has not been given.

If this is the case, then when a mass M gets compacted into a region whose circumference in every direction is $\lesssim 4\pi GM/c^2$, the external gravitational field of a horizon (black hole), after all the "dust" and gravitational waves have cleared away, is almost certainly the Kerr-Newman generalization of the Schwarzschild geometry. The Kerr-Newman geometry is

$$ds^2 = \frac{\Delta}{\rho^2}\left[cdt - asin^2\theta d\phi\right]^2 - \frac{sin^2\theta}{\rho^2}\left[(r^2 + a^2)d\phi - acdt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 \tag{13}$$

where

$$
\begin{aligned}
\Delta &\equiv r^2 - 2GMr/c^2 + a^2 + Q^2 \\
\rho^2 &\equiv r^2 + a^2 cos^2\theta \\
a &\equiv S/M \equiv \text{ angular momentum per unit mass}
\end{aligned} \tag{14}
$$

The horizon of the Kerr-Newman black hole occurs at a radial coordinate given by

$$R_+ = \frac{GM}{c^2} + \frac{1}{c^2}\sqrt{G^2M^2 - GQ^2 - a^2} \tag{15}$$

where $a = cS/M$. Notice that for R_+ to be a real number, the quantity under the square root must be positive; that is

$$G^2M^2 - GQ^2 - a^2 \geq 0. \tag{16}$$

If this quantity is positive, there appears from the mathematics to be another horizon at

$$R_- = \frac{GM}{c^2} - \frac{1}{c^2}\sqrt{G^2M^2 - GQ^2 - a^2} \tag{17}$$

However, since $R_- < R_+$, the external observer is concerned only with R_+. IN the particular case when the quantity under the square root is zero, $R_- = R_+$.

There is no horizon if the quantity under the square root is negative. Unlike the Schwarzschild black hole, there is he intriguing possibility of the external observer being in a position to witness the final state of gravitational collapse – the singularity. The singularity is then called naked. Do black holes of this type exist? Or is there a cosmic censorship that permits only those black holes to exist that have horizons concealing the singular fate of the collapsing body from the external observer. This is another open question.

10 Kerr Solution Black Hole

Since most actual black holes are likely to have a very low net electric charge a useful astrophysical solution is the Kerr solution for a rotating black hole. The exact solution of the nonlinear Einstein equations is:

$$ds^2 = c^2dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2)sin^2\theta d\phi^2 - \frac{2GMr}{\rho^2}\left[cdt - asin^2\theta d\phi\right]^2 \tag{18}$$

where

$$
\begin{aligned}
\Delta &\equiv r^2 - 2GMr/c^2 + a^2 + Q^2 \qquad Q = 0 \\
\rho^2 &\equiv r^2 + a^2 cos^2\theta \\
a &\equiv S/M \equiv \text{ angular momentum per unit mass}
\end{aligned} \tag{19}
$$

The limiting case of large r reduces to:

$$ds^2 \cong \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 + \frac{2GM}{c^2 r}\right) dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{2gMa^2}{r} sin^4 d\phi^2 + \frac{4GMa}{r} sin^2\theta d\phi dt \tag{20}$$

or substituting in $r^2 sin^2 d\phi = xdy - ydx$ one has

$$ds^2 \cong \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 + \frac{2GM}{c^2 r}\right) dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{4GMa}{r^2}(xdy - ydx)dt \tag{21}$$

The first four terms are exactly the same as in the Schwarzschild case and show that M is the total mass of the system. The last term corresponds to the off-diagonal components in the metric that we saw for the frame dragging effect of a rotating body so that $S = Ma$ or $a = S/M$ is the angular momentum per unit mass.

The Kerr geometry is much more complicated than the Schwarzschild geometry, and depends drastically on whether $GM > |a|$ or $GM < |a|$. One complication of the Kerr geometry is the presence of an infinite redshift surface when $g_{00} = 0$. This corresponds to

$$g_{00} = 1 - 2GMr/\rho^2 = 0 \quad or \quad r^2 + a^2cos^2\theta - 2GMr = 0. \tag{22}$$

The solution to this is

$$r_{\inf\pm} = GM \pm \sqrt{(GM)^2 - a^2cos^2\theta} \tag{23}$$

There are two distinct infinite redshift surfaces. They do not correspond to any physical singularity. The vanishing of g_{00} tells us that a particle cannot be at rest at these surfaces; only light emitted in the radial direction can be at rest.

In the region between these two surfaces, the value of g_{00} is negative. Hence t is not a time-like coordinate and the t independence of the metric does not necessarily imply that the geometry is truly time independent. At radius $r < GM - \sqrt{(GM)^2 - a^2cos^2\theta}$, the coordinate t again reverts to the character of a time coordinate.

In the Kerr geometry, the infinite redshift surfaces do not coincide with the event horizons.Where are the event horizons? Their locations are given by the requirement that $dr/dt = 0$ for light (where $ds^2 = 0$) independent of the value of θ and ϕ. This corresponds to

$$r_{Horizon\pm} = GM \pm \sqrt{g^2M^2 - a^2} \tag{24}$$

At $r_{Horizon\pm}$ a light signal necessarily has zero velocity in the radial direction, the light cones lie along the surfaces $r = constant$, and light cannot escape from these surfaces.

Neither at the infinite redshift surfaces nor at the horizons does the Kerr geometry develop a local singularity; in freely falling geodesic coordinates, the curvature remains finite. The only true singularity occurs at $r = 0$. Note that in spheriodal coordinates, $r = 0$ is a disc centered on the origin. In rectangular coordinates $r = 0$ corresponds to $x^2 + y^2 = a^2 sin^2\theta$, $z = 0$.. Since θ varies between 0 and $\pi/2$, this corresponds to $x^2 + y^2 \leq a^2$. This is necessary for there to be a net moment of inertia.

10.1 Can an External Body Gain Energy from a Rotating Black Hole

Is there any way in which an external observer can determine the presence of a rotating black hole? Suppose the observer comes closer and closer to the black hole, while keeping an eye on the distant stars of the Universe. The distant stars provide a background against which the rotation of the black hole can, in principle, be measured via the frame dragging effect. Will the observer be able to arrange it so that the distant stars do not appear to rotate? As the black hole is approached, the observer will find an increasing tendency to get carried away in the same sense in which the black hole is rotating. To keep stationary, he will need to apply a force against this tendency, a force that increases as the black hole is approached. A stage known as the static limit will come when he will be swept away by the black hole no matter how hard he tries to counteract this rotational sweeping force. When this happens, he has entered a zone called the *ergosphere*.

The ergosphere is not really spherical but has a shape that changes with latitude l.

$$R_l = \frac{GM}{c^2} + \frac{1}{c^2}\sqrt{G^2M^2 - GQ^2 - a^2 sin^2 l} \tag{25}$$

Why the name *ergosphere*? The nomenclature comes from the possibility that a black hole may afford energy extraction in this region. (The Greek work *ergo* means work.) Roger Penrose (1969) suggested that a projectile fired from outside into the the ergosphere begins to rotate with the black hole so that the projectile acquires more rotational energy than it originally possessed. The projectile can now break up into two pieces. Of these, one may fall into the black hole singularity, whereas the other may come out of the ergosphere. The piece coming out may then have more energy than the original projectile.

In the Penrose mechanism, the black hole contributes part of its rotational energy so that the black hole itself is slowed down, a process that can continue until the black hole has given away all its rotational energy. The ergosphere will then no longer exist. Starting from a Kerr black hole, one ends with a Schwarzschild black hole. The schwarzschild black hole represents the finally irreducible state in which external processes can only increase the energy of the black hole instead of decreasing it.

10.2 Area of Black Hole

For a Schwarzschild black hole the area is simply

$$A_s = 4\pi R_s^2 = \frac{16\pi G^2 M^2}{c^4} \tag{26}$$

For the Kerr-Newman black hole, the area is

$$A = 4\pi\left(R_+^2 + \frac{a^2}{c^4}\right) = 4\pi\left[\left(\frac{GM}{c^2} + \frac{1}{c^2}\sqrt{G^2M^2 - GQ^2 - a^2}\right)^2 + \frac{a^2}{c^4}\right] \tag{27}$$

Note that the more involved expression for area is an indication of the fact that the non-Euclidean geometry of the Kerr-Newman black hole is much more complex than that of the Schwarzschild black hole.

11 Free-Fall Times Question

Imagine that the thermal heat is removed from the Sun instantly and its constituents behaved as non-interacting dust and would begin pressure-free collapse from the present state. Calculate the time of a particle on the Sun's surface ($R = 7 \times 10^8$ m) to reach $r = 0$. Plot the radius as a function of time. Note how quickly the later part of the collapse take. How would the time and curve look for a one solar mass White Dwarf? Use $R_{WD} = 0.5 \times 10^4$ km $= 5 \times 10^6$ m. How about for a one solar mass neutron star? With $R_{NS} = 10$ km. Do your calculation using the particle rest time τ so that simple Newtonian mechanics works.

12 Eddington-Finkelstein Coordinates

As a star collapses its surface will eventually reach and fall through the critical value $r = 2GM/c^2$ on the way to forming a black hole. As noted the Schwarzschild coordinate singularity at $r = 2GM/c^2$ is an impediment to understanding this simple gravitational collapse. A new coordinate system that covers the $r = 2GM/c^2$ surface regularly (without) singularity is desirable. One such set is the Eddington-Finkelstein coordinates, which are particularly suited to exploring gravitational collapse. Skipping over the mathematics of getting new coordinates note that the new metric is

$$ds^2 = (1 - 2GM/c^2r)dv^2 - 2dvdr - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{28}$$

where $v \equiv ct + r + (2GM/c^2)ln\,(r/(2GM/c^2) - 1)$ is a coordinate such that the past light cones centered on the star are the surfaces $v = constant$. To see this, consider a radial displacement $v = constant$, $\theta = constant$, $\phi = constant$. Because $dv = 0$, $d\theta = 0$, and $d\phi = 0$, $(dx^\alpha) = (0, dr, 0, 0)$; because there is no term in dr^2 in the metric, the displacement is zero, i.e. it is a light ray. Note that on a plot of t versus r the line $v = constant$ is a 45° line toward $r = 0$

Consider radial light rays in the Eddington-Finkelstein metric. Deduce that the displacements dv and dr along the light rays are related by

$$\left[2dr - \left(1 - (2GM/c^2 r)\right) dv\right] dv = 0 \tag{29}$$

Hence show that the ingoing light rays are given by $dv = 0$ and the outgoing light rays by $dr = (1 - (2GM/c^2 r))\, dv/2$; and so confirm that the local light cones increasingly tilt toward the v-axis (asymptotically close in on the 45° slant) as one moves in from a large radius toward a small radius. In particular show that the future light cone point 100% inward for $r \leq r_s = 2GM/c^2$. Make a sketch of a collapsing star surface and resulting singularity in Eddington-Finkelstein coordinates (v vs. r). Include a sequence in r of light cones for a time v when the dust has settled and the singularity has come to equilibrium.

Table 1: Laws of Black Hole Physics & Thermodynamics

Law	Black Holes	Thermodynamics
1st	Energy & Momentum are conserved in every physical process.	Energy is conserved (Heat is form of energy.)
2nd	In all physical processes involving BHs the total surface area of all the participating BHs can never decrease	In any physical process the entropy of all participating systems taken together can never decrease
3rd	By no finite series of operations can one make the surface gravity of a black hole zero	Cannot get to absolute zero
0th	The surface gravity κ is constant over the event horizon of a stationary axially symmetric BH.	Thermal Equilibrium $A \equiv B$ & $B \equiv C$, implies $A \equiv C$ Tend to thermal equilibrium

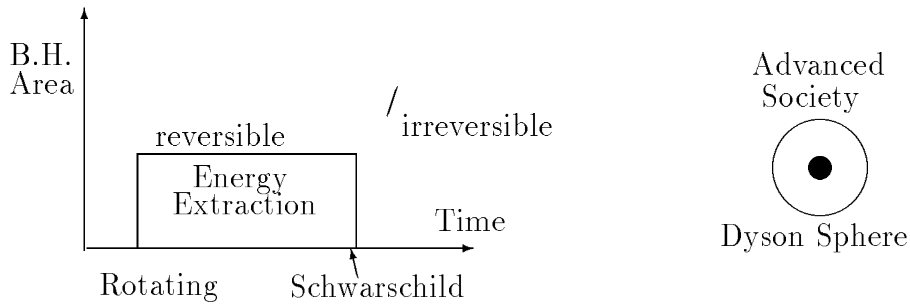
13 Laws of Black Hole Physics

13.1 Black Hole Threat/Link to Thermodynamics

The black hole absorbs everything but does not let anything leave! A black hole seems like an ideal heat sink for running a Carnot engine. We can dump in heat energy and get work! Imagine an advanced society that has built a complete Dyson sphere around a black hole rather than a star.

Dyson speculated that a sufficiently technologically advanced society might build a complete sphere around a star and use the star's radiated power to run the machines of that advanced society. These would be difficult to detect since they would absorb the full star light and only reject waste heat.

Here we have gone one step further and assumed the advanced society has built its sphere around a rotating black hole. They first extract rotational energy from the black hole in the optimally efficient manner which keeps the area of the black hole constant by adding a little waste garbage and extracting angular momentum energy. The effective Schwarschild radius increases by the added energy just enough to compensate for the loss in surface area that would occur for the change in angular momentum. This is a reversible phase since the surface area of the black hole is held constant.



Once the society has extracted all the angular momentum energy and transformed the rotating black hole to a Schwarschild black hole with no more angular momentum energy to extract. Then the energy extraction process must be irreversible.

Suppose our advanced engineers fill a box with the waste heat in the form of thermal (blackbody) radiation and lower it down to the event horizon, slide open the bottom and let out the radiation. Since the horizon is the surface of infinite redshift, the radiation has zero frequency and thus all the heat (thermal radiation) has been turned into useful work. This argument works for all the garbage or any mass that the advanced civilization wants to lower into the black hole but radiation is a good example since this clearly violates the second law of thermodynamics as I am turning all the heat into useful work, unless $T_{BH} = 0$ K. In the last case, any civilization that has access to a black hole can violate the second law at will and run a perpetual motion machine. How do we get out of this conundrum?

One must take into account the size of the box used to transport the radiation. To hold radiation of wavelength λ, the box must be of size comparable to the wavelength:

$$d \approx \lambda \approx \frac{\hbar c}{kT} \tag{30}$$

Now when the bottom of the box is at the horizon, the mean radiation is $d/2$ above the horizon and the energy left is $Egd/2c^2$ where $g = \kappa = GM/R_s^2 = c^4/4GM$. Thus the available work is

$$W = E\left(1 - \frac{c^2 d}{8GM}\right) \cong E\left(1 - \frac{\hbar c^3}{8GMkT}\right) \cong E\left(\frac{T - T_{BH}}{T}\right) \tag{31}$$

Thus one has

$$T_{BH} \cong \frac{\hbar c^3}{8GMk} \cong 10^{-7}\mathrm{K}\left(\frac{M_\odot}{M}\right) \cong 1.2 \times 10^{26}\mathrm{K}\left(\frac{1\ \mathrm{gm}}{M}\right) \tag{32}$$

This result from T_{BH} was found by Bekenstein in 1973 & 4, who proposed not only to associate this temperature to a black hole but also and entropy so that the laws of thermodynamics could be applied to the whole process.

The heat absorbed by a black hole equals the increase in mass of the black hole, i.e. $\delta Q = \delta M c^2$ so that

$$\delta S = \frac{\delta Q}{T_{BH}} = \frac{\delta M c^2}{T_{BH}} = \frac{8kGM\delta M}{\hbar c} \tag{33}$$

Alternately, this can be described in terms of a change in the surface area of the black hole.

$$\begin{aligned} A &= 4\pi r_s^2 \\ \delta S &= 8\pi r_s \delta r_s = 32\pi G^2 M \delta M / c^4 \\ S &= \frac{kc^3}{4\pi\hbar G} A \ + \ \text{constant} \end{aligned} \tag{34}$$

$(\delta M c^2 = T_{BH}\delta S + \Omega \delta J)$

The formation of a black of a black hole entails a very large increase in entropy. We have this directly from the formula and we can understand this in an information science point of view. When we put anything in a black hole the only information that seems to be preserved is the total mass, angular momentum, and electric charge. Thus vast amounts of information are destroyed increasing entropy greatly. The question of what really happens to this information is a currently hot topic in black hole physics.

If a black hole has a temperature does it radiate? The surprising answer to this that contrary to classical black hole physics, quantum mechanics says that this is possible. This remarkable discovery was made by Steven Hawking in 1973. The reasoning is as follows. Imagine the formation of a particle-antiparticle pair in a strong field. I.e. like the electric field given at the beginning of this document. The Uncertainty principle gives us the possibility of creating an electron-positron pair and if they can be far enough apart, then the gain in energy in the electric field is comparable or larger than the rest mass energy they can be made real. That will happen for a sufficiently high electric field and it will also happen for a sufficiently strong gravitational field.

But with gravity there is an extra fillip. At the horizon surface of a black hole, by definition, a particle there has zero net energy. Its gravitational binding energy just cancels its rest mass and kinetic energy. Therefore creating virtual particles as the horizon is simple; however, they are trapped at the horizon and any perturbation will send them back in. Thus one needs to have a positive net energy to make the real. However, if one creates a pair of massless particle, such as photons, gravitons, or neutrinos, then if one of the pair appears inside the horizon it will have negative total energy and the one outside can have positive total energy. Since the particle is massless it can then escape the black hole provided its momentum is in the outward direction.

One can calculate this process carefully by simple quantum mechanics to find that the barrier penetration factor for the pair is $e^{-8\pi GME/\hbar c^3}$ for $E \gg \hbar c^3/GM$. The flux of particles incident on the horizon is proportional to the number of quantum states: $4\pi p^2 dp/h^3$ which when multiplied by the barrier penetration factor for the pair (one in one out) gives

$$\text{outgoing flux of particles} \propto \frac{4\pi p^2 dp}{h^3} e^{-8\pi GME/\hbar c^3}$$

This is exactly the form of a thermal spectrum in the limited case of large energy ($E \gg kT$). The penetrating factor $e^{-8\pi GME/\hbar c^3}$ is the Boltzmann factor $e^{E/kT}$, with temperature:

$$T = T_{BH} = \frac{\hbar c^3}{8\pi GMk}$$

If the energy is not large compared to kT_{BH} the penetration factors is somewhat more complicated, and depends on the wave nature and whether the particle is a fermion or a boson. The result is still thermal distributions. But at low energy (frequency) the Black Hole is actually a greybody. Not all the particles liberated at the horizon escape to infinity. Some of the particles are back scattered by gravitational potential surrounding the black hole and are absorbed by the black hole. Thus the black hole and its surrounding potential form a thermal radiator with less than perfect emissivity; a gray body rather than blackbody.

The radiated power is the product of the black hole area times the Stephan-Boltzmann constant and the black hole temperature to the fourth power. (We can neglect the gray-body emissivity that occurs for low energies.)

$$\text{Radiated Power} = \sigma_{SB} T_{BH}^4 \, 4\pi r_s^2 \cong -10^{47} (\frac{1\text{gm}}{M})^2 \text{erg s}^{-1} \cong -10^{26} (\frac{M}{1\text{gm}})^{-2} \text{gm s}^{-1} c^2$$

The lifetime of a black hole will then be

$$\tau \cong \frac{M}{dM/dt} \simeq 10^{-26} s \times (M \text{ 1 gm})^3$$

Approximately how long does it take for a 1 gm black hole to evaporate?

What Black hole mass evaporates in one second?

How long does it take for a solar mass (10^{33} gm) to evaporate?

What is the temperature of a Planck Mass, $M_{Pl} = \sqrt{\hbar c/G} = 2.16 \times 10^{-5}$ gm, black hole?

Supposed that black holes were produced at the beginning of the universe. What mass black hole would now, approximately 10^{10} years later be in the final stage of evaporating?

What was its typical temperature? I.e. its temperature for most of its life.

Therefore what typical energy photons will it have been radiating? ($k = 8.617 \times 10^{-5}$ eV/K)