Table 1: Planck Dimensional Quantities

Planck	$Q_P = f(\hbar, G, c)$	Value
Length	$L_P = \left(\frac{\hbar G}{c^3}\right)^{1/2}$	1.6×10^{-35} cm
Time	$t_P = \frac{L_P}{c} = \left(\frac{\hbar G}{c^5}\right)^{1/2}$	5.4×10^{-44} sec
Mass	$M_P = \left(\frac{\hbar c}{G}\right)^{1/2}$	2.2×10^{-5} g
Energy	$E_P = \left(\frac{\hbar c^5}{G}\right)^{1/2}$	1.22×10^{19} GeV
Temperature	$T_P = \frac{E_P}{k} = \left(\frac{\hbar c^5}{k^2 G}\right)^{1/2}$	1.42×10^{32} K
Density	$\rho_P = \frac{c^5}{\hbar G^2}$	5.16×10^{93} g/cm^3

P139 - Relativity
Problem Set 15: Quantum Gravity & Cosmology 2002 May

G. F. SMOOT
Department of Physics,
University of California, Berkeley, USA 94720

1 Quantum Gravity

This course ends with a summary of the issues of quantum gravity. These include quantum effects in classical gravity such as Hawking radiation, quantum fluctuations in Inflation, and radiation seen by an accelerating observer. There are problems of extreme fluctuations in the metric and the formulation of a consistent quantum theory of gravity. This shows both fundamental flaws in the classical gravity and quantum mechanics which are the two major edifices of 20th century physics. We take a brief excursion into zero point radiation showing that underneath there is a deep connection between the structure of space-time (the vacuum), gravity, and quantum mechanics. Then we head to the *wave equation for the Universe* and the *wave function of the Universe* as the grand finale.

1.1 Curvature/Horizon Radiation

In our discussion of the laws of black holes and the parallel to thermodynamics we found an expression for the effective temperature and entropy of a black black hole. Then we saw that Steven Hawking (1975 "Particle Creation by Black Holes" Commun. Math. Physics 43, 199-220) showed that quantum effects produced a thermal radiation from the surface gravity and horizon. For your homework you did

Table 2: Quantum Mechanics in a classical curved spacetime

A) Hawking Radiation	gravitational field	$kT_H = \hbar\kappa/2\pi c$
B) Unruh Radiation	acceleration (Eq. prin.)	$kT_U = \hbar a/2\pi c$
zero point radiation		
C) de Sitter Radiation	horizon	$kT_S = \hbar H/2\pi$
D) Inflation perturbations	small horizon	$\delta\phi = H/2\pi$
	$\langle h^2_{GW}\rangle^{1/2} = \sqrt{16\pi}\frac{\delta phi}{m_p} = \sqrt{\frac{2}{\pi}}\frac{H}{m_P}$	$\left\langle h^2_\phi\right\rangle^{1/2} = \frac{V\delta phi}{V' m_P^2}$

a heuristic calculation of the spontaneous creation of particles in a field (electric, magnetic, or gravitational) to see how this happens. The effective temperature of the black hole from both approaches is given by the formula:

$$kT = \frac{\hbar\kappa}{2\pi c} = \frac{\hbar c^3}{8\pi GM} \qquad T \simeq 6\times 10^{-8}\left(\frac{M_\odot}{M}\right)\ \mathrm{K} \tag{1}$$

$$S' = S + \frac{1}{4}k\frac{c^3 A}{G\hbar} \tag{2}$$

Both this calculation and the estimate of quantum fluctuations in Inflation were made in a classical, though curved, geometry. That is the background metric was well-defined and smooth in the region of interest. The metric itself did not undergo fluctuations in the straight forward approach. It is also possible to derive the fluctuations in the Inflation case as fluctuations in the scale factor or in the scalar field driving inflation.

1.2 Quantum Fluctuations from Inflation

The existence of a horizon - a result of the de Sitter accelerating universe of inflation - sets a scale for quantum mechanical uncertainty to act. If the expansion rate is ($v = Hr$, $H \equiv \dot{a}/a$), then there is a distance d_H between two points at which they are moving apart at the speed of light $d_H = c/H$. Points that are further apart than that are out of causal contact. The uncertainty principle would imply that there will be quantum fluctuations in energy and momentum on the scale of $\Delta E \simeq \Delta pc \sim \hbar H$. The first Friedmann equation gives us a relationship between the energy density ρ and H:

$$H^2 \rightarrow 8\pi G\rho/3 = \frac{8\pi}{3m_P^2}V(\phi)$$

If H is sufficiently large, then one can expect that fluctuations in the inflaton field ϕ will result in fluctuations in the metric, curvature fluctuations of order $\delta g_{\mu\nu} = h_{\mu\nu} \sim= 8\pi G\Delta E = \hbar H = \left(\frac{E_{Inflation}}{E_P}\right)^2$.

Although a rigorous calculation entails many subtleties, especially the hypersurface- or gauge-dependence of the fluctuation amplitude when it is inflated outside the horizon, there is a useful mnemonic for the final, correct answer. First, the scalar curvature fluctuations on a given scale as that scale is stretched outside the horizon during the de Sitter (inflationary) epoch can be understood as a local variation in the universe's scale factor a.

$$\delta \ln a = \frac{d\ln a}{d\phi}\delta\phi = \frac{d\ln a}{dt}/\frac{d\phi}{dt}\ \delta\phi = H/\frac{d\phi}{dt}\ \delta\phi; \qquad H \equiv \frac{d\ln a}{dt}. \tag{3}$$

In a de Sitter background, the rms fluctuation in the inflaton is $\langle\delta\phi^2\rangle^{1/2} = H/2\pi$. Tensor fluctuations in the metric are produced by the same basic process and in rough equipartition but with a coupling of of $\sqrt{16\pi G}$:

$$h_{GW} = \sqrt{16\pi G}\delta\phi_{GW} = \sqrt{16\pi}\delta\phi_{GW}/m_{Pl}, \tag{4}$$

where each component of the metric undergoes fluctuations with rms amplitude $\langle\delta\phi_{GW}^2\rangle^{1/2} = H/2\pi$. (One must include the fact that there are two tensor degrees of freedom but only one scalar.)

How do these curvature fluctuations evolve to the surface of last scattering? For both scalar and tensor fluctuations, the amplitude for a given wavelength is frozen from when the wavelength is stretched outside the horizon during the de Sitter epoch to when it re-enters the horizon during the Friedmann-Robertson-Walker epoch that follows. Hence, the amplitude upon re-entering the horizon simply equals the amplitude when exiting during the de Sitter epoch. Hence, for gravity waves, the amplitude on the surface of last scattering is determined from Eq. (4), $\langle h_{GW}^2\rangle^{1/2} \sim (2/\pi)^{1/2}H/m_{Pl}$. For the scalar field fluctuations, the curvature perturbation δ ln a during the de Sitter epoch can be related to the fluctuations in the density through the relativistic continuity equation

$$\frac{d\rho}{dt} + 3H(\rho + p/c^2) = 0; \qquad H \equiv \frac{dln(a)}{dt}, \tag{5}$$

where p is the pressure. Multiplying through by δt we have

$$\delta\rho + 3(\rho + p/c^2)\delta ln(a) = 0 \ \text{ or } \ \delta\mathrm{ln(a)} = -\frac{1}{3}\frac{\delta\rho}{(\rho + \mathrm{p/c^2})}. \tag{6}$$

(The last expression has been shown to be gauge-invariant (Bardeen et al.1983).) The curvature perturbation when a given scale re-enters the horizon in the Friedmann-Robertson-Walker epoch equals $\delta ln(a) = -\frac{1}{3}\frac{\delta\rho}{(\rho+p/c^2)}$ when the scale was stretched beyond the horizon during the de Sitter epoch. From the slow roll condition, Eq. (**??**), and using $H^2 = 8\pi V(\phi)/3m_{Pl}^2$, one finds

$$\delta \ln a = -\frac{3H^2}{V'(\phi)}\ \delta\phi = -\frac{8\pi V}{V'(\phi)m_{Pl}^2}\ \delta\phi. \tag{7}$$

Thus the scalar (density) metric fluctuations amplitude depends not only on the energy in the scalar field (on the expansion rate) but also on the gradient of the field's potential.

In sum, we estimate that the ratio of rms gravity wave curvature fluctuations to those from the scalar field is

$$\frac{< h_{GW}^2 >^{1/2}}{< h_\phi^2 >^{1/2}} = \sqrt{\frac{2}{\pi}} \frac{V'(\phi) m_{Pl}}{V}, \tag{8}$$

where this expression is to be evaluated when the scale of interest was stretched beyond the horizon during the de Sitter epoch. For cosmologically observable scales, this corresponds to roughly 60 e-foldings before the end of inflation.

1.3 The Holographic Principle

Consider the following argument on the maximum entropy allowed in a finite volume of space. This leads to the conclusion that all the information in 4-D spacetime actually resides on a 3-D surface. According to the Schwarzschild solution, the mass of a black hole is given by its radius: $r_s = (G/c^2)M$. Hence, the mass M contained within a sphere of radius R obeys

$$M \lesssim R \quad \text{natural units} \quad M \lesssim (c^2/G)R$$

The result for the entropy (and radiation) had an ultraviolet cutoff that there not be more than one Planck mass per Planck volume. For larger regions this cut off would permit $M \sim R^3$, in violation of our previous Schwarzschild limit $M \lesssim R$. Hence our cut-off was too lenient to prevent black hole formation on larger scales.

For example, consider a spher of radius $R = 1$ cm, or 10^{33} Planck lengths. Suppose that the field energy in the enclosed region saturated the naive cut-off in each of the 10^{99} Planck cells. Then the mass within the sphere would be 10^{99}. But the most massive object that can be localized to the sphere is a black hole of radius and mass 10^{33}.

Usign the spherical entropy baound, $A/4$ degrees of freedom are sufficient to describe fully any stable region in asymptotically flat space enclosed by a sphere of area A.

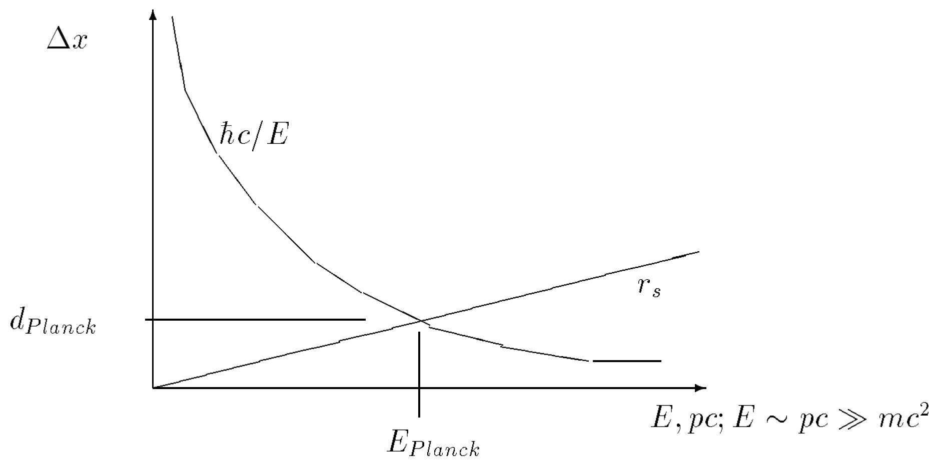
Holographic Principle: A region with boundary of area A is fully described by no more than A/4 degrees of freedom, or about 1 bit of information per Planck area. A fundamental theory, unlike local field theory, should incorporate this result.

1.4 The Generalized Uncertainty Principle

The Heisenberg Uncertainty Principle tells us that the fundamental uncertainty in position and thus spatial resolution is related to and limited by the uncertainty of the momentum related with that direction.

$$\Delta x \Delta p_x \geq \hbar/2 \qquad \text{or} \qquad \Delta x \geq \hbar/2\Delta p_x \tag{9}$$

That means if we want to probe and resolve to a distance Δx we must have a minimum available momentum $\Delta p_x \geq \hbar/\Delta x$. Thus:



We also know that if we concentrate energy E within its Schwarschild radius, we can get no information from inside the Schwarschild radius.

$$r_s = 2GE/c^2 \tag{10}$$

These two limits cross each other at the Planck energy E_{Planck} and distance d_{Planck} which can be calculated by setting the two distances equal and solving for the energy and then feeding back to get the distance.

$$E_{Planck} = \left(\frac{\hbar c^5}{G}\right)^{1/2} = 1.22 \times 10^{19}\text{GeV} \tag{11}$$

$$l_P \equiv d_{Planck} = \left(\frac{G\hbar}{c^3}\right)^{1/2} = 1.6 \times 10^{-35}\text{m} \tag{12}$$

This would then be the logical end point of black hole evaporation. I.e. either the final and smallest black which falls apart or a stable quantum relic. Understanding this issue is one of the motivations for quantum gravity.

We can write the Generalized Uncertainty Principle in a suggestive form (there is a factor of two floating here $\Delta x \Delta p \geq \hbar/2$):

$$\Delta x \geq \frac{\hbar/2}{\Delta p_x} + l_P^2 \frac{\Delta p_x}{\hbar/2} \tag{13}$$

or in units of Planck lengths

$$\begin{aligned} \frac{\Delta x}{l_P} &= \frac{l_P \Delta p}{\hbar/2} + \frac{\hbar/2}{l_P \Delta p} \\ \Delta x &= \Delta p + \frac{1}{\Delta p} \qquad \text{natural units} \end{aligned} \tag{14}$$

Note the symmetry or inverse correspondence of Δp_x in units of $\hbar/2$ and the prominence of the Planck length l_P. This is suggestive of the idea of duality that $1/x$ is equivalent to x.

Note a similar argument can be used to go to more generalized coordinates. A general rule for commutation relations in non-commuting geometry is

$$[x_i, p_j] = i\hbar\delta_{ij}\left(1 + \frac{l_P^2 E^2}{4}\right)^{1/2} = i\hbar\delta_{ij}\left(1 + \frac{l_P^2(p^2 + m^2)}{4}\right)^{1/2} \tag{15}$$

which leads to the uncertainty relation

$$\Delta x^\mu \Delta p^\mu \gtrsim \frac{1}{2}\left\langle\sqrt{1 + l_p^2 p^2/4}\right\rangle \tag{16}$$

This may not be the full story. One could argue that in four dimensions In particular, consider time, or proper time uncertainty. Using the commutation relation we would have

$$\Delta t \Delta E \geq \hbar\left(1 + \frac{l_P^2 \Delta E^2}{4}\right)^{1/2}$$

$$\Delta t \geq \frac{\left(1 + \frac{l_P^2 \Delta E^2}{4}\right)^{1/2}}{\Delta E}$$

We know that to probe short times requires high energies but if the information gets hidden in a black hole we have to wait until the black hole evaporates to get access to the information. The usual time-energy uncertainty principle is

$$\Delta\tau \geq \frac{\hbar}{2\Delta E}$$

a more extended version for the Planck length is set by the light travel time $\tau_{Planck} = l_P/c$ plus the decay time for a Planck-sized black hole.

$$\tau_{total} = \tau_{Planck} + \tau_{BH} = \left(\frac{G\hbar}{c^5}\right)^{1/2} + \sim 10^3 \tau_{Planck}$$

Geometry (topology) appears to persist longer than light propagation time across the external dimensional scale.

1.5 Zero Point Radiation

Consider a classical vacuum with all matter and thermal radiation removed. Is there anything else in the vacuum. The classical answer is yes. There is what we shall call the zero point radiation which is an isotropic, homogeneous radiation field with spectral intensity proportional to the frequency cubed.

1.5.1 Casmir Force

The first indication that we have that this field must exist is the measured Casmir force. If two uncharged metal plates are placed in a very cold vacuum, there is a force that attracts one plate toward the other in an amount proportional to the area of the plates and the inverse fourth power of the separation.

$$F_{\text{Casmir}} = \frac{\pi^2}{240} \frac{\hbar c}{d^4} A = 0.2\text{mg} \ \frac{A}{d^4} \ \frac{0.5 \times 10^{-4}\text{cm}}{1 \ \text{cm}^2} \tag{17}$$

Exercise Show that this is the force law if there is a radiation field with $I \propto \nu^3$.

1.5.2 Consistent with Special Relativity

Show that an isotropic, homogeneous radiation field with $I \propto \nu^3$ is the only radiation field that is identical for all Lorentz-frame observers. That is that one cannot determine one's absolute velocity by measuring the intensity, angular distribution, or spectrum of this radiation. (Hint: Show that I/ν^3 is proportional to the number of photons in a quantum state at frequency ν and thus is a conserved number.)

1.5.3 Ideal Harmonic Oscillator

Suspend an electron from an ideal spring fixed on the inside wall of an ultracold, ultrahigh vacuum chamber. (i.e. perfect vacuum and no thermal radiation)

If the electron is displaced from its equilibrium position, then it will begin to oscillate and the acceleration will cause it to radiate. The back reaction of the radiation on the electron will damp down the oscillations to match the radiated energy and the electron oscillations will asymptotically approach zero amplitude.

Now if you include the effect of the radiation field with $I \propto \nu^3$, show that the electron continues to oscillate randomly with an amplitude that corresponds to the Uncertainty Principle and with a rms energy equal to the zero point energy of the harmonic oscillator. Thus the name *zero point radiation* even though no quantum mechanics is thus far involved in this classical vacuum radiation field.

This is made manifest in the Lamb-Reherford shift in hydrogen atom energy. Because of zero point fluctuations, the nominal path of an electron around proton (nucleus) has a jitter to it around the nominal mean radius. Because the potential $V(r) = -e^2/r$, the effective potential energy for the electron is given by averaging the Taylor series expansion:

$$\langle V \rangle = -e^2 \left(\frac{1}{\langle r \rangle} + \nabla V_{\langle r \rangle} \langle r - < r > \rangle + \frac{1}{2} \nabla^2 V_{\langle r \rangle} \left\langle r^2 - < r >^2 \right\rangle + h.o.t. \right)$$

Clearly the linear term averages to zero effect but the second order (mean square working on curvature) results in a shift from nominal equilibrium.

1.6 Vacuum Energy

The zero point energy ($h\nu$/2E) for each of the modes will add up to a tremendous amount of energy

$$\rho_{vac} = \frac{\hbar}{2\pi^2 c^5} \int \omega^3 d\omega = 4\pi h \int \nu^3 d\nu$$

If we use a cut off at the Planck length (for wavelength) we find a density of zero point fluctuation energy in vacuum of order 10^{94} gm/cm^3. To be compared with typical nuclear densities of about 10^{14} gm/cm^3.

1.6.1 Uniformly Accelerating Observer Radiation

Show that an uniformly accelerating observer will see two radiation fields the zero point radiation with $I \propto \nu^3$ and a thermal spectrum of radiation with

$$T_{acceleration} = \frac{\hbar a}{2\pi c} \qquad T \simeq 4 \times 10^{-23} a \ \mathrm{K}/(\mathrm{cms}^{-1}) \tag{18}$$

Showing this relation then shows consistency with the Equivalence Principle since a surface gravity $\kappa = g_s$ gives exactly the same temperature. This also shows that black body radiation (Planckian distribution) arises classically from relativity without recourse to quantum mechanics. Therefore we can conclude that some how gravity/space-time and quantum mechanics are related at some deep fundamental level.

1.7 Quantum Field Theory & Issues

Now that we know gravity and quantum mechanics are deeply related, then we are ready to create a quantum field theory for gravity. First what is a field theory? Two examples of field theory are:

Newtonian Gravity:

$$\vec{F} = \vec{F}_g m = \frac{GM}{r^2} m \hat{r} \tag{19}$$

where $\vec{F}_g$ is the gravitational force field.

Electromagnetism:

$$F_\mu = F_{\mu\nu} j^\nu \tag{20}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor.

We showed as part of the homework that the fields of two objects create a force through the mechanism of distorting the field lines to the minimum energy configuration so that there is a net force because of the field distortion.

Then quantum mechanics started with the first quantization of the relevant measurable quantities of the particle or system under consideration.

The second quantization is the quantization of the fields allowing them to be treated as force carrying particles that are interchanged. This is the concept behind the Feynman diagrams of particle interactions and this approach is called *Quantum Field Theory*. A quantum field theory for gravity calls for a force carrier particle given the name the graviton which is expected to be massless to obtain the $1/r^2$ force law and to have spin 2 in order to always be attractive.

This all sounds great so what are the issues holding us back from a full field theory of quantum gravity?

We will need a third quantization: an operator that creates and annihilates universes. This is daunting in that one does not think of seeing universes created and destroyed regularly. But wait there is more:

(1) Quantum field theories are based on the assumption that the wave function commute:

$$\left[\hat{\psi}(x), \hat{\psi}(x')\right] \equiv \hat{\psi}(x)\hat{\psi}(x') - \hat{\psi}(x')\hat{\psi}(x) = 0 \tag{21}$$

That is to say that if x and x' are causally disconnected, then a measure at x' of $\hat{\psi}$ cannot influence the value of $\hat{\psi}$ at x. However what is the wave function for gravity? It is going to be the probability amplitude of the metric. Does

$$[\hat{g}_{ab}(x), \hat{g}_{ab}(x')] = 0? \tag{22}$$

Only when x and x' are in a space-like relation. We only know this is true when we know what g_{ab} is and we are trying to find its wave function and thus uncertain value.

(2) Superposition is taken for granted in quantum field theory. The wave function is routinely written as the sum over the orthonormal basis set of the wave equation $\psi = \sum$ states which assumes linearity However, gravity is non-linear and even more of a problem the curvature of spacetime and a graviton are not readily separable especially when the field strength variations are large.

insert picture showing curved space, add a recognizable graviton, and then a chaotic structure of spacetime and defy the reader to find the graviton.

If something does not intervene by the Planck scale, the metric is so deformed that one is looking at a picture of spacetime foam rather than a nice simple Riemannian space. Topology changes are possibilities. Were this the case, then because of the non-linearity of gravity, one could find a dispersion relation where more energetic particles interact with and back react on the existing curvature:

$$p^2 c^2 = E^2 \left[1 + \xi \frac{E}{E_p} + \epsilon \left(\frac{E}{E_p} \right)^2 + ... \right]$$

$$v = \frac{\partial E}{\partial p} \simeq c \left[1 - \xi \frac{E}{E_p} - \epsilon \left(\frac{E}{E_p} \right)^2 + ... \right]$$

Thus a sharp pulse of light or particle of high energies coming over great distances would be dispersed according to the energy.
(3) Time: The entire causal structure of spacetime is destroyed when one attempts to quantize $g_{\mu\nu}$. Microcausality ... need background metric ...

...... superspace as a desired solution

$$ds^2 = c^2 dt^2 - g_{0a} dt dx^a - g_{ab} dx^a dx^b$$

$$i \frac{\partial \psi}{\partial t} = H \psi$$

$$\psi(\vec{x}, t) = N \int_c \delta x(t) \; e^{iS(x(t))}$$

where N is the normalization and C is the class of paths which are weighted in a away that reflects the projection of the system.

1.8 Wave Equation for the Universe

The usual approach to find a wave equation is to define an action

$$I_{Hilbert} = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \, [R + 2\Lambda]$$

Then by calculus of variation one finds the Lagrangian or Hamiltonian and then the momenta replaced by operators (derivatives) yielding.

$$\left[\hbar^2 \frac{8\pi}{G} G_{ijkl} \frac{\delta^2}{\delta g_{ij} \delta g_{lk}} \frac{g^{1/2}}{8\pi/G} \left({}^{(3)}R(x) - 2\Lambda - T \right) \right] \psi(g_{ij}) = 0 \qquad (23)$$

where

$$G_{ijkl} = \frac{1}{2} g^{-1/2} \left(g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl} \right) \quad \text{and} \quad T = T_0^0(\phi, -i\partial/\partial\phi) \qquad (24)$$

This is not such a bad equation since it is for three instead of four dimensions and g_{ab} is symmetric tensor giving us only six unknown functions. This is still a bit much for this class but fortunately we can appeal to some boundary conditions and symmetry.

1.9 Wave Function for the Universe

For a homogeneous, isotropy universe with a constant vacuum energy density ρ_v this reduces to a one dimensional problem. We get a wave equation which has as its classical analog the Friedman equation, $(\dot{a}/a)^2 + 1/a^2 = \Lambda/3$, for a vacuum energy dominated universe (aka as DeSitter Space).

$$\left[\frac{d^2}{da^2} - a^2\left(1 - H^2a^2\right)\right]\psi(a) = 0 \qquad H^2 = 8\pi G\rho_v/3 \tag{25}$$

which has solution of the form $a(t) = a_0 cosh(ct/a_0)$, where $a_0 = \sqrt{\Lambda/3}$. This is the form of the Schrödinger wave equation for a particle of zero energy with coordinate $a(t)$ (the scale factor for the universe) in potential $U(a) = a^2(1 - H^2a^2)$. The classically allowed region is $a \geq H^{-1}$. The solution to this equation is a linear combination of Airy functions $Ai[z(a)]$ and $Bi[z(a)]$, where $z(a) = (3\pi^2a_0^2/4G)^{2/3}(1 - a^2/a_0^2)$.

Plot sample wave functions on this graph. The wave function one obtains is set by the boundary conditions that one sets for the universe which in turn set the coefficients for the Airy functions $Ai[z(a)]$ and $Bi[z(a)]$. The Hartle-Hawking, Vilenkin, and Linde wavefunctions are

$$\begin{aligned}
\Psi^{HH} &\propto Ai[z(a)] \\
\Psi^{V} &\propto Ai[z(a)]Ai[z(0)] + iBi[z(a)]Bi[z(0)] \\
\Psi^{L} &\propto \frac{1}{2}\left(Ai[z(a)] + Bi[z(a)]\right) \\
\Psi^{yours} &\propto c_1Ai[z(a)] + c_2Bi[z(a)]
\end{aligned} \tag{26}$$

Chose your boundary condition and set the complex coefficients c_1 and c_2 to match your boundary conditions and show your results on the plot of the potential along with the wave functions of Hartle-Hawking, Vilenkin, and Linde. *Hint: It is good to have your solution contain the expanding universe in the classical region.*

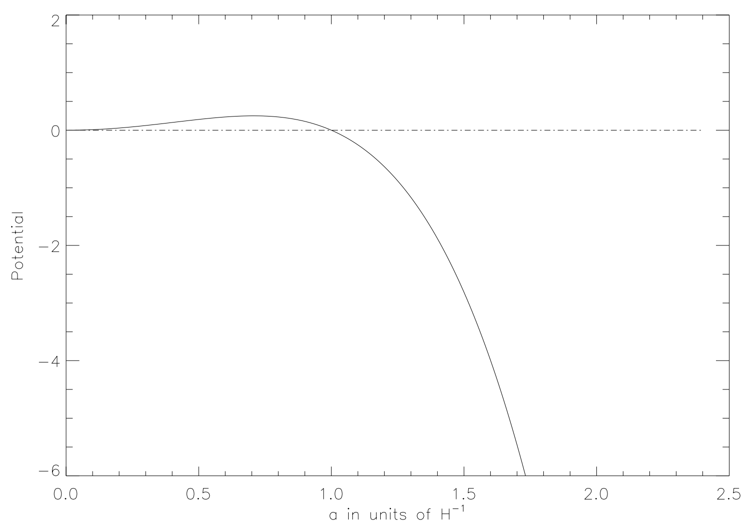


Figure 1: Potential for DeSitter Space Universe