Physics 139 Relativity
Problem Set 4 Due February 13, 2003

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1 Relativistic Kinematics

Cosmic rays have been observed with energies as high as 10^{20} to 10^{21} eV.

Assuming (1) the cosmic ray is a proton (mc^2 = 938 MeV) or (2) an iron nucleus (Z = 26, A = 56),

(a) What are the beta = v/c and gamma for such a cosmic ray?

(b) Assuming a mean, ordered (means directionally aligned in the Galactic plane) magnetic field of 3 microGauss, what is the radius of curvature of such a cosmic ray travelling perpendicular to the plane? How does that compare to the Galactic mean half-height of three thousand light years?

(c) How accurately would such a cosmic ray proton point back to its source?

(d) Show that such an energetic, extragalactic cosmic ray cannot be an iron nucleus, if it comes across intergalactic space. Hint: intergalactic space is filled with cosmic microwave background photons, at T $\simeq$ 3 K and thus about 413 per cc, each with a mean energy of $kT \sim 1/4000$ eV in our frame and the iron nucleus has a mean binding energy of about 8 MeV per nucleon in its frame.

2 Harmonic Motion

In frame S a particle at the origin executes simple harmonic motion in the y direction with frequency f and amplitude a. Determine its position and velocity as a function of time in a reference frame S′ moving with velocity v in the x direction. Calculate the time and distance between successive maxima of y′. Compare the results to the relativistic Doppler Effect.

(Note: I am not asking you to solve the relativistic harmonic oscillator problem where the change in mass must be taken into account.)

3 Rapidity and 3-D Velocity Transformation

The rapidity of a particle, ϕ, is defined as

$$tanh(\phi) = \frac{u}{c}, \qquad \text{or} \qquad \phi = tanh^{-1}\left(\frac{u}{c}\right) \tag{1}$$

where u is the speed (magnitude of 3-D velocity). Note that

$$e^{\phi} = \gamma\left(1 + \frac{u}{c}\right) = \left(\frac{1 + u/c}{1 - u/c}\right)^{1/2}$$

$$cosh\phi = \gamma, \quad sinh\phi = \frac{u}{c}\gamma, \quad tanh\phi = \frac{u}{c}$$

The complicated 3-D velocity transformations we derived in class

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}, \quad u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)}, \quad u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)}$$

and its symmetric under interchange of prime and unprime and v and $-v$

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}, \quad u = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}, \quad u_z = \frac{u'_z}{\gamma(1 + u'_x v/c^2)}$$

can be replaced by a simple formula for the rapidity:

$$\phi(u) = \phi(u') + \phi(v) \tag{2}$$

demonstrating neatly that a velocity boost is simply a rotation.
(a) Show the addition law for rapidity. *Hint:* Show you can write the Lorentz Transformation as

$$\begin{aligned} ct' + x' &= e^{-\phi}(ct + x) \\ ct' - x' &= e^{+\phi}(ct - x) \end{aligned} \tag{3}$$

Then show that the composition (one applied after the other) of two Lorentz transformations of rapidity ϕ_1 and ϕ_2, is a Lorentz transformation of rapidity $\phi_1 + \phi_2$. (It is a Lorentz transformation since $c^2t^2 - x^2 = c^2t'^2 - x'^2$ remains manifestly true.)
(b) Derived the velocity composition rule by transforming between three inertial frames S, S', S'' in standard configuration with each other. Both approaches benefit from this approach. S' has velocity v_1 with respect to S and S'' has velocity v_2 relative to S'.

Show that S'' has velocity $(v_1 + v_2)/(1 + v_1 v_2)$ relative to S. *Hint:* Recall that trigonometric identities can be converted to hyperbolic-function identities by the rule $cosx \rightarrow coshx$, $sinx \rightarrow isinhx$; so in particular,

$$tanh(\phi_1 + \phi_2) = \frac{tanh\phi_1 + tanh\phi_2}{1 + tanh\phi_1 tanh\phi_2}$$

4 Use of Invariants

A proton moving with energy gamma times its rest mass collides with an identical proton moving, also with the same energy gamma times its rest mass, but at right angles to the first proton in the lab. Derive a formula and evaluate for gamma = 10.
(a) What is the center of momentum system total energy of the ensemble?
(b) What is the momentum and energy of one proton in the frame of the other?

5 Netrino Mass Limit from SN 1987A

In 1987 neutrinos were observed in coincidence with the light from a supernova in the Large Magallenic Cloud of distance $d = 30kpc \simeq 100,000$ lightyears. The neutrinos were observed to arrive over a period lasting $\Delta t = 10$ seoonds and those detected had an energy range 10 MeV $< E_\nu <$ 50 MeV. By assuming all the neutrinos started at the same time and that the most energetic (E_1 =50 MeV) arrived first and that the least energetic (E_2 = 10 MeV) arrived last, find a formula for the mass of the neutrino m_ν. By assuming that $m_\nu c^2 << E_\nu$ find a simpler formula and show its dependence on the critical parameters: d, Δt, E_2, E_1.

Evaluate the approximate formula with the numbers given and derive an upper limit on the neutrino rest mass, m_ν, since the neutrinos are expected to be emitted over a fintie time of a few seconds and the most energetic are emitted earlier and less energetic later as the supernova expands and cools.

Optional Problems

6 Neutrino Oscillations

The electron and muon neutrinos are both observed to oscillate as a function of distance from their sources. That is the number of neutrino interactions from a given neutrino species has a rate that oscillates indicating that the neutrino species is oscillating into another and back. One explanation is that two neutrinos have slightly different masses and the quantum mechanical mass eigenstates are mixtures of the species eigenstates. Mixing angle is Θ_M. Show that in the limit that the electron-neutrino rest mass $m_{\nu e}$ and muon-neutrino rest mass $m_{\nu\mu}$ squared difference: $\Delta m^2 = m_{\nu\mu}^2 - m_{\nu e}^2$ is much less than the neutrino energy (E_ν/c^2) or momentum (p_ν/c) that the oscillation of one species to another probability (that the species is present) is given by:

$$P(\nu_e \to \nu_\mu) \simeq sin^2(2\Theta_m) sin^2\left(\frac{\delta m^2}{4}\frac{L}{E_\nu}\right)$$

Hint: Consider the 3-momentum p_ν the same for the two neutrino states (or consider the energy E_ν the same for the two neutrino states).

7 3-D Acceleration Transformation

Derive the equations for transformation of the components of acceleration a_x, a_y, and a_z in system S to a'_x, a'_y, and a'_z in system S' moving with velocity v into the x direction, proceeding directly from the Einstein velocity transformation equations and the Lorentz transformation.