Physics 139 Relativity
Relativity Notes 2003

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Notes to be found at:
http://aether.lbl.gov/www/classes/p139/homework/homework.html

1 Differential Geometry

There is a strong connection between the geometry (symmetry) of Minkowski space and dynamics. Further generalization of relativity requires incorporating a new symmetry - the Equivalence Principle. That will be manifested as the more universal Riemannian geometry of simple curved space-time. Riemannian invariance is not limited to orthogonal or even linear transformations but includes all real, single-valued transformations that are continuous with finite first and second derivatives. This will include curved space-times that do not admit a global Cartesian coordinate system.

Through the Equivalence Principle gravity is replaced by space-time curvature. Gravity is replaced by a local effect. Thus we do not need to know the full geometry and topology but only the local differential geometry.

1.1 Invariant Length and the Metric Tensor

We want to generalize the concept of invariant length of Minkowski space. We define the invariant length and the generalized Riemannian metric by the equation

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

where the indices μ and ν are repeated and thus summed and in our case of a 3+1 dimensional geometry cycle from 0 to 3. It is easy to see that $g_{\mu\nu}$ is a tensor. If the transformation for $\vec{x}'$ to $\vec{x}$ is $x^\mu = x^\mu(\vec{x}')$, then

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu = a^\mu_{\nu'} dx'^\nu \tag{2}$$

If ds^2 is the invariant length, then

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\sigma\rho} dx'^\sigma dx'^\rho = g_{\mu\nu} a^\mu_{\sigma'} dx'^\sigma a^\nu_{\rho'} dx'^\rho \tag{3}$$

From this we can conclude

$$g'_{\sigma\rho} = g_{\mu\nu} a^\mu_{\sigma'} a^\nu_{\rho'} \tag{4}$$

These geometry techniques have been more generalized to other dimensions by mathematicians and we will work out some problems and examples in lower dimensions for illustration and ease. This general bilinear form (product of two differentials) of the metric is the class of Riemannian spaces. In Riemannian geometry one must nearly always pay attention to the difference between covariant and contravariant (co and contra variation with respect to the transformations).

For example, consider the metric for a two dimensional space

$$dx^i = (dx, dy) \quad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$dx^i = (dr, d\theta) \quad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{6}$$

$$dx_i = (dr, r^2 d\theta) \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \tag{7}$$

1.2 The Transformation

The general case is any primed coordinate system and any unprimed coordinate systems connected by a transform on $x^i = x^i(x^{i\prime})$. There are no restrictions on the transformation except that it is continuous, real, single-valued, and its derivatives exist over the region of interest. The indices run from 1 to n, the dimension of the space. In differential form the transformation equations are

$$dx^i = a^i_{j\prime} dx^{j\prime} \quad \text{and} \quad dx^{j\prime} = a^{j\prime}_i dx^i \tag{8}$$

where

$$a^i_{j\prime} = [\mathbf{a}] = \frac{\partial x^i}{\partial x^{j\prime}} \quad \text{and} \quad a^{j\prime}_i = \left[\mathbf{a^{-1}}\right] = \frac{\partial x^{j\prime}}{\partial x^i} \tag{9}$$

giving

$$a^{j\prime}_i a^i_{k\prime} = \delta^{j\prime}_{k\prime} \tag{10}$$

The coefficients $a^{j\prime}_i$ and $a^i_{j\prime}$ are called the transformation coefficients and the condition

$$a^{j\prime}_i a^i_{k\prime} = \delta^{j\prime}_{k\prime} \quad \text{or} \quad \left[\mathbf{a^{-1}}\right] \cdot [\mathbf{a}] = [\mathbf{1}] \tag{11}$$

reflects the general requirement that any transformation followed by its inverse is unity.

1.3 Review of Definition of Tensors

The concept of tensors is actually more generalized than simply transforming from one inertial frame to another according to the generalized Lorentz Trnasformation. We can make use of our understanding of the transformation from one coordinate system to another to define a more generalized concept of a tensor.

First let us generate a new vector space with exactly the same properties as the differential forms: xx^i and dx'^i. That is to say that the geometry of those vector spaces has the same properties and thus transformation between coordinate systems as the differential forms.

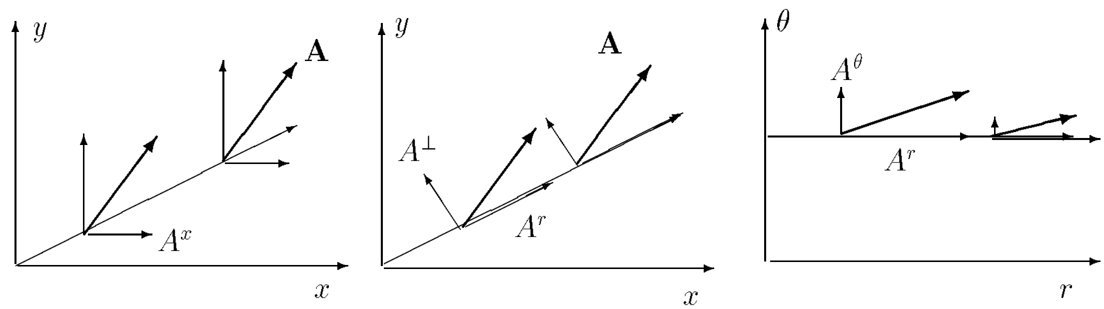
$$A^i = a^i_{j'} dA^{j'} \quad \text{and} \quad A^{j'} = a^{j'}_i A^i \tag{12}$$

The tensor notation is the same as before with the point transformation properties requiring as many $a^i_{j'}$ or $a^{j'}_i$ as there are indices. There is the same issue with upper and lower indicies.

1.4 Parallel Displacement/Transport

When a vector is carried from one point to another without changing its magnitude or direction, it undergoes parallel displacement. In a flat Euclidean space, the idea of of parallel displacement is perfectly clear. In terms of its Cartesian components, it means that the vector's components remain unchanged as it moves from one place to another.

Consider a (contravariant) vector $\vec{A} = (A^x, A^y)$ that is displaced along a line that radiates from the origin. Its components A^x and A^y remain constant under parallel displacement, so they have the same magnitude all along the path. See figure. The vector can also be specified in terms of components that are parallel and perpendicular to the radial line. $\vec{A} = (A^r, A^\perp)$ where the unit component direction vectors are $\hat{r}$ and $\hat{\theta}$ respectively.



(a) (b) (c)

In polar coordinates, the components of $\vec{A} = (A^r, A^\theta))$ in the radial direction also remains the same through out the displacement. That is the value of A^r are constant and the same as in the $(A^r, A^\perp)$ case. However, the A^θ are not the same as $A^\perp$. The relationship between the two is given by $A^\perp = rA^\theta$.

From a variation of this equation and recognizing that $A^\perp$ is constant, δA^θ can be found as a function of r and δr:

$$\delta A^\perp = \delta(rA^\theta) = A^\theta \delta r + r\delta A^\theta = 0 \tag{13}$$

or

$$\delta A^\theta = -\frac{A^\theta}{r}\delta r \tag{14}$$

As the vector moves outward in a $+r$-direction, its θ-component decreases inversely with r.

There is another way of obtaining this result in a more general way. Because $\vec{A}$ is a vector, we can its transformation to polar components from the transformation from polar to Cartesian coordinates

$$x = r cos\theta \qquad y = r sin\theta$$

The corresponding differential interval transformations are

$$dx = cos\theta dr - sin\theta d\theta \qquad dy = sin\theta dr + r cos\theta d\theta$$

or inverted

$$dr = cos\theta dx + sin\theta dy \qquad d\theta = -\frac{1}{r} sin\theta dx + \frac{1}{r} cos\theta dy$$

So the forward and backward transformations are

$$\begin{pmatrix} dr \\ d\theta \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} cos\theta & -r sin\theta \\ sin\theta & r cos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

So vectors can be expected to transform the same way

$$\begin{pmatrix} A^r \\ A^\theta \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} A^x \\ A^y \end{pmatrix} \tag{15}$$

or written explicitly

$$\begin{aligned} A^r &= A^x cos\theta + A^y sin\theta \\ A^\theta &= -\frac{1}{r} A^x sin\theta + \frac{1}{r} A^y cos\theta \end{aligned} \tag{16}$$

Given that A^x, A^y, and θ are constant for a radial translation, a variation of the first equation is $\delta A^r = 0$ as already noted and a variation on the second equation gives

$$\begin{aligned} \delta A^\theta &= \delta\{-\frac{1}{r} A^x sin\theta + \frac{1}{r} A^y cos\theta\} \\ \delta A^\theta &= \frac{1}{r}\{\frac{1}{r} A_x sin\theta - \frac{1}{r} A^y cos\theta\}\delta r = -\frac{A^\theta}{r}\delta r \end{aligned} \tag{17}$$

A vector is parallel displaced, if it moves without changing its magnitude or direction. But it is clear from this example that the directional change is dependent on the coordinate system. The magnitude (i.e. the square root of the invariant length) is unchanged. under coordinate transformation, but when viewed in polar coordinates, its components do vary and hence changes apparent direction. This is not exactly what was defined as parallel transport though it is in some coordinate systems. The definition of parallel transport will have to be broadened.

Define the indexed (not tensor) symbol $,\, ^{k}_{ij}$ called a Christoffel symbol or affine connection. As a vector A^i is parallel displaced over a differential δx^j, its components A^k are changed by an amount determined by the Christoffel symbols in the defining equation

$$\delta A^k = -,\, ^{k}_{ij} A^i \delta x^j \tag{18}$$

The values of these Christoffel symbols in Cartesian coordinates are zero, but they are not zero in polar coordinates.

1.5 Comparison to Curved Surface

We just did the transformation for flat space. Consider embedding a curved two dimensional surface in a three dimensional Euclidean space. Most specifically consider a spherical surface. We know the polar coordinate trnsformations:

$$x = rcos\phi sin\theta \qquad y = rsin\phi sin\theta \qquad z = rcos\theta$$

$$ds^2 = g_{ij}dx^i dx^j = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

Considering a fixed radius to give us a spherical surface, the metric on the surface is

$$ds^2 = r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) = (dx')^2 + (dy')^2$$

where $dx' = rd\theta$ and $dy' = rsin^2\theta d\phi$. In small regions where $|sin\theta|$ is not near zero and changing rapidly, on the surface this looks like a Euclidean formulation. It is only when one looks over larger regions or globally is it apparent that this is a curved surface.

1.6 Geodesic Path

In Riemannian geometry, the term geodesic path means: the path of shortest invariant distance. There is a strong connection between geodesic paths and the idea of parallel displacement. A vector that moves tangent to a geodesic is, at the same time, being parallel displaced as it moves along.

The geodesic path can be found using the variational principle to find the shortest distance between two points, which in Riemannian geometry requires that $\delta \int ds = 0 = \delta \int d\tau$ The calculus of variations gives

$$\frac{d^2x^x}{ds^2} = -\frac{1}{2}g^{ks}\left[g_{is,j} + g_{js,i} - g_{ij,s}\right]\dot{x}^i\dot{x}^j \tag{19}$$

where the dot means derivative with respect to ds (or $cd\tau$) $\dot{x}^k \equiv dx^k/ds$ and $g_{ij,k} = \partial_k g_{ij}$.

Multiplying by

$$d\dot{x}^k = -\frac{1}{2} g^{ks} \left[g_{is,j} + g_{js,i} - g_{ij,s} \right] \dot{x}^i dx^j \tag{20}$$

This equation describes how a contravariant vector $\dot{x}^j$ changes as it moves over a path dx^j. It is the same as the definition of the Christoffel symbol. Clearly,

$$\Gamma^k_{ij} = \frac{1}{2} g^{ks} \left[g_{is,j} + g_{js,i} - g_{ij,s} \right] \tag{21}$$

This equation allows the Christoffel symbols to be calculated from the metric tensor alone. The geodesic equation becomes

$$\frac{d\dot{x}^k}{ds} = -\Gamma^k_{ij} \dot{x}^i \dot{x}^j \tag{22}$$

1.7 Parallel Displacement of Covariant Vectors

Covariant vectors also change direction under parallel displacement. Consider the inner invariant product of two vectors A^i and B_i. When the two are parallel displaced together in a Cartesian system. If δ represents the small variation of the product along the line of displacement, then because it is an invariant,

$$0 = \delta(A^i B_i) = A^k \delta B_k + B_i \delta A^i \tag{23}$$

Substituting $\delta A^i = -\Gamma^i_{jk} A^j \delta x^k$ yielding

$$A^i \delta B_i = -B_k \delta A^k = -B_k(-\Gamma^k_{ij} A^i \delta x^j) A^i \delta B_i = A^i \Gamma^k_{ij} B_k \delta x^j \tag{24}$$

Because this expression is true for any vector A^i, its coefficients on either side of the equation must be equal

$$\delta B_i = \Gamma^k_{ij} B_k \delta x^j \tag{25}$$

The Christoffel symbol gives the changes in a covariant vector as well as the contravariant one:

$$\delta A^k = \Gamma^k_{ij} A^i \delta x^j \qquad \delta A_i = \Gamma^k_{ij} A_k \delta x^j \tag{26}$$

δA^i and δA_i are not vectors. Note that δA^i identically zero in the Cartesian system but non-zero in the polar system. A zero vector cannot be transformed into a non-zero one.

It is also possible to determine how a mixed tensor changes under a parallel displacement. Consider the inner product between tensor T^i_j and two vectors A_i and B^j, while they are parallel displaced together: $\delta(T^i_j A_i B^j) = 0$ since the inner product is invariant.

$$\delta T^i_j = \Gamma^r_{js} T^i_r \delta x^s - \Gamma^i_{rs} T^r_j \delta x^s \tag{27}$$

$$\delta T^i_{jk} = \Gamma^r_{js} T^i_{rk} \delta x^s + \Gamma^r_{ks} T^i_{jr} \delta x^s - \Gamma^i_{rs} T^r_{jk} \delta x^s \tag{28}$$

1.8 Covariant Derivatives

Consider a contravariant vector A'^i in the Cartesian system that is transformed to a Cartesian system

$$A^i = a^i_{i'} A'^i \tag{29}$$

Taking the ordinary derivative of this gives

$$dA^i = d(a^i_{i'})A'^i + a^i_{i'} dA'^i \tag{30}$$

If the Cartesian vector does not change its components in any way as it is parallel displaced, then $dA'^i = 0$ and the change in A^i will be given only by the term $d(a^i_{i'})A^{i'}$. This is precisely what we called δA^i previously in the discussion of parallel transport. Therefore

$$dA^i = \delta A^i + a^i_{i'} dA'^i \quad or \quad a^i_{i'} dA'^i = dA^i - \delta A^i$$

Multiplying through by the inverse transformation yields

$$dA'^i = a^{i'}_i [dA^i - \delta A^i]$$

We can generalize this to additional systems that are not necessarily Cartesian In general the quantity $dA^i - \delta A^i$ will transform as a vector.

$$D \equiv dA^i - \delta A^i$$

The first part represents the change in the vector A component i that is observed in the coordinate system. The second is the change δA^i that is due to the parallel displacement of the vector component A^i. This difference subtracts out the variation that has only to do with the curvilinear characteristics of the coordinate system.

Consider the vector $dx^i/ds = \dot{x}^i$ along a curve defined by the parameter s. Dividing by the invariant interval ds gives

$$\frac{D\dot{x}^i}{ds} = \frac{d\dot{x}^i}{ds} - \frac{\delta \dot{x}^i}{ds} = \frac{d\dot{x}^i}{ds} + \Gamma^i_{jk} \dot{x}^j \dot{x}^k$$

If the line is a straight line in a Cartesian system, then the vector $D\dot{x}^i/ds$ is zero in all coordinates so that

$$\frac{d^2 x^i}{ds^2} = -\Gamma^i_{jk} \dot{x}^j \dot{x}^k$$

1.9 Review: Equations of Motions for Free Particle

If a particle is not being acted on by any forces, thus free of influence, then we can find its equation of motion by several methods: parallel transport, variational principle (geodesic path), and by setting the covariant derivative of the (four) velocity equal to zero becouse there is no force to cause acceleration.

1.9.1 Parallel Transport

We consider that the free particle has a given velocity (or momentum) and if there are no forces acting upon it, then it will simply move forward in parallel transport of its velocity (vector or momentum). This is the simplest way to see that the effect is strictly a local.

The parallel transport of the velocity gives

$$\delta u^k = -\Gamma^k_{ij} u^i \delta x^j$$

Dividing through by $d\tau$ or ds gives

$$\frac{du^k}{d\tau} = -\Gamma^k_{ij} u^i u^j$$

This is a simple equation that says that if the rate of change in the velocity is equal to the transport of that velocity times the effect of the changing coordinate metric.

1.9.2 Geodesic Equation

This comes from the requirement that the path taken is the minimum (extremum) of path between the end points.

$$\frac{du^k}{d\tau} = -\Gamma^k_{ij} u^i u^j$$

where ds is replaced by $d\tau$.

1.9.3 Covariant Derivative of Velocity

If no force is acting on the particle, then the covariant derivative of the velocity should be zero.

$$\frac{Dp^\mu}{d\tau} = F^\mu = 0 \quad \text{implies} \quad \frac{Du^\mu}{d\tau} = 0$$

Thus

$$\frac{Du^k}{d\tau} = \frac{du^k}{d\tau} + \Gamma^k_{ij} u^i u^j = 0$$

Once again we find the geodesic equation through a different conceptual approach.

1.10 Space-Time Differential Geometry

1.11 Spherical Surface as an Example

If we consider a two-dimensional spherical surface, then we can easily see that parallel transport of vector around a closed path does not return to point the same direction, E.g. consider a spherical triangle with one vertex at the north pole and two sides thus being lines of constant longitude (ϕ = constant) and the other connecting parallel to the equator (line of constant latitude). Take vectors parallel (or perpendicular) to the sides and start the parallel transport from the pole. The parallel vector will stay tangent to the constant longitude line (it is a geodesic and so that is clear). Across the constant latitude side the formerly parallel vector will stay perpendicular to that side and at the end will be parallel to the last side. When transported back to the pole it will stay parallel to the last side. Thus back at the same point, the pole, the parallel transported vector will point at a different angle - the angle of the opening of the triangle at the pole. This triangle was constructed so that angle at the pole is the excess angle. That is the angle by which the sum of the angles of the spherical triangle exceeds π radians (180°). Once can use the Christoffel symbols for a spherical surface to show that the result of the parallel transport of a vector acround any spherical triangle on the sphere is displaced by the excess angle. By Gauss's msot beautiful theorem the excess (or deficit angle) is the ratio of the area of the triangle to one-eighth of the area of the sphere

$$\alpha = \frac{2A_{triangle}}{\pi R^2}$$

Intuitively one knows that the curvature tensor for the space defined as on the surface of a sphere should not be equal to zero. Every point on a sphere's surface is intrinsically curved.

1.12 Curvature Measures

If a vector is parallel displaced around a closed loop in a flat space, the it will return to its original magnitude and direction. That should be true independent of the coordinate systems in which the displacement is observed. So if a parallel-displaced vector is changed by a trip around a closed loop, it follows that its region of travel is a "curved" space.

A surface observer can use this procedure to detect the existence of curvature, when visual recognition or embedding in a higher dimensional flat Cartesian space is not part of the mathematical procedure.

$$\delta V^\alpha = \delta x^\mu \delta x^\nu \left[\Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\beta\mu} - \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\beta\nu} \right] V^\beta \tag{31}$$

1.12.1 Christoffel Symbols

$$\Gamma^m_{ij} = \frac{1}{2} g^{mk} \left[g_{ik,j} + g_{jk,i} - g_{ij,k} \right] \tag{32}$$

1.12.2 Curvature Tensor

$$\text{Riemann} \qquad R^k_{ars} = \Gamma^k_{ar,s} - \Gamma^k_{as,r} + \Gamma^b_{ar} \Gamma^k_{sb} - \Gamma^b_{as} \Gamma^k_{rb}$$

The intrinsic curvature at any point can be found in terms of derivatives of the metric at that point.

One can find the curvature by transporting the same vector half way around a differentially small rectangle in both directions and comparing the results. The procedure is to first take the covariant derivative of a vector $\vec{v}$ with respect to x^i, then take the covariant derivative of this result with respect to x^j:

$$D_{x^j}\left(D_{x^i}\vec{v}\right)$$

Then reverse the order and subtract

$$D_{x^j}\left(D_{x^i}\vec{v}\right) - D_{x^i}\left(D_{x^j}\vec{v}\right)$$

The vector $\vec{v}$ can be taken out of the additive terms in this tensor equation, leaving the differential operators in the form of the tensor $[R]$. $[R]$ is called the Riemann-Christoffel curvature tensor.

$$D_{x^j}\left(D_{x^i}\vec{v}\right) - D_{x^i}\left(D_{x^j}\vec{v}\right) = [D_{x^i}\left(D_{x^j}\right) - D_{x^i}\left(D_{x^j}\right)\vec{v} = [R]\cdot\vec{v}$$

or

$$\Delta\vec{v}/\Delta\text{area} = [R]\ \cdot\vec{v}.$$

Remember that when a tensor is zero in one coordinate system, it is zero in all coordinate systems. So if the Riemann-Christoffel tensor is zero at any point in any system of coordinates, then the space at that point is flat. On the other hand, if $[R]$ is nonzero at a point, then the space has an intrinsic curvature at that point.

The Riemann-Christoffel tensor can be explicitly derived by carrying out the operations outlined above. This can be done using contravariant vector A^k. First take the covariant derivative of A^k

$$A^k_{;r} = A^k_{,r} + \Gamma^k_{ar}A^a$$

Now take the next covariant derivative

$$A^k_{;rs} = \frac{\partial}{\partial x^s}(A^k_{;r}) + \Gamma^k_{as}A^a_{;r} - \Gamma^a_{rs}A^k_{;a}$$

where the index s representing the second covariant derivative is placed directly behind the index r representing the first covariant derivative.

$$A^k_{;rs} \equiv \left[A^k_{;r}\right]_{;s}$$

Going back to the definition of the covariant derivative $A^k_{;r}$ above gives

$$\frac{\partial}{\partial x^s}\left[A^k_{,r}\Gamma^k_{ar}A^a\right] + \Gamma^k_{as}\left[A^a_{,r} + \Gamma^a_{br}A^b\right] - \Gamma^a_{rs}\left[A^k_{,a} + \Gamma^k_{ba}A^b\right] \quad (33)$$

where the dummy index b is there to avoid repetitive use of index a in two of the terms. Multiplying out one has

$$A^k_{;rs} = A^k_{,rs} + \Gamma^k_{ar,s}A^a \Gamma^k_{ar}A^a_{,s} + \Gamma^k_{as}A^a_{,r} + \Gamma^k_{as}\Gamma^a_{br}A^b - \Gamma^a_{rs}A^k_{,a} + \Gamma^a_{rs}\Gamma^k_{ba}A^b \qquad (34)$$

This seven term expression is the result of two sequential covariant differentiations of the vector $\vec{A}$ corresponding to the operations $D_{x^j}(D_{x^i}\vec{v})$ acting on vector $\vec{v}$. Now perform the same operations in reverse order and subtract. The reverse order of the covariant differentiation is found by exchanging the indices s and r.

$$A^k_{;sr} = A^k_{,sr} + \Gamma^k_{as,r}A^a + \Gamma^k_{as}A^a_{,r} + \Gamma^k_{ar}A^a_{,s} + \Gamma^k_{ar}\Gamma^a_{bs}A^b - \Gamma^a_{sr}A^k_{,a} + \Gamma^a_{sr}\Gamma^k_{ba}A^b \qquad (35)$$

This is equation is subtracted from the previous and the first, third, fourth, sixth and seventh term in each cancels, leaving the difference between the second and fourth terms:

$$A^k_{;rs} - A^k_{;sr} = \Gamma^k_{ar,s}A^a - \Gamma^k_{as,r}A^a + \Gamma^k_{as}\Gamma^a_{br}A^b - \Gamma^k_{ar}\Gamma^a_{bs}A^b \qquad (36)$$

The dummy indices a and b are now exchanged in the last two terms, and the vector component A^a removed from each term to give

$$A^k_{;rs} - A^k_{;sr} = \left[\Gamma^k_{ar,s} - \Gamma^k_{as,r} + \Gamma^k_{bs}\Gamma^b_{ar} - \Gamma^k_{ar}\Gamma^b_{as}\right] A^a \qquad (37)$$

or one has

$$A^k_{;rs} - A^k_{;sr} = R^k_{ars}A^a \qquad (38)$$

where

$$R^k_{ars} = \Gamma^k_{ar,s} - \Gamma^k_{as,r} + \Gamma^k_{bs}\Gamma^b_{ar} - \Gamma^k_{ar}\Gamma^b_{as} \qquad (39)$$

This fourth-rank mixed tensor - R^k_{ars} - is the Riemann-Christoffel curvature tensor.

1.12.3 Ricci Tensor and Scalar Curvature

$$\text{Ricci} \qquad R_{ij} = R^k_{ijk}$$

$$\text{Scalar} \qquad R = R^i_i$$

We will contract the Riemann-Christoffel tensor in two steps. In the first of these, the contravariant index is contracted with the last of the covariant indices to give the Ricci tensor, which is defined as

$$\begin{aligned} R_{ij} &= R^k_{ijk} \\ R_{ij} &= \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^b_{ij}\Gamma^k_{kb} - \Gamma^b_{ik}\Gamma^k_{jb} \end{aligned} \qquad (40)$$

It is easy to show that this tensor is symmetric

$$R_{ij} = R_{ji} \qquad (41)$$

Other second-rank tensors can be found by contracting other indices; but the Ricci tensor is of special importance because of its symmetry and because of the unique properties of its derivatives. As a result it plays a major role in General Relativity. Since it is symmetric it has the same number of independent components as the metric tensor $g_{\mu\nu}$ and the stress energy tensor $T_{\mu\nu}$ and General Relativity provides a unique way to link them together.