Physics 139 Relativity: Review

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1 The Principles of Relativity

(1) Galileo → Poincare: No experiment, without reference to the outside world, can determine its absolute velocity.
(2) Maxwell Electromagnetism or the constancy of the speed of light.
or the geometrical approach
(1) Space-time is 3 + 1 dimensions
(2) Metric is $ds^2 = c^2dt^2 - dl^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$
Results in:
(a) Time dilations by factor $\gamma = 1/\sqrt{1-\beta^2}$
(b) Length contraction by factor $\gamma = 1/\sqrt{1-\beta^2}$
(c) Clock synchronization effect: $-v\Delta x/c^2$, **The clock that is farther behind in space is further ahead in time.**

1.1 The Lorentz Transformation

$$\begin{aligned} t' &= \gamma\left[t - vx/c^2\right] & t &= \gamma(t' + vx'/c^2) & tanh(\xi) &= v/c \\ x' &= \gamma\left[x - vt\right] & x &= \gamma(x' + vt') & cosh(\xi) &= \gamma \\ y' &= y & y &= y' & sinh(\xi) &= \gamma\beta \\ z' &= z & z &= z' & ct' \pm x' &= e^{\mp\xi}(ct \pm x) \end{aligned} \quad (1)$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cosh\xi & -sinh\xi & 0 & 0 \\ -sinh\xi & cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (2)$$

1.2 Doppler Effect

$$\begin{aligned} \frac{\nu'}{\nu} &= \gamma(1 - \vec{\beta}\cdot\hat{r}) = \gamma(1 - \beta cos\theta) & \frac{\lambda}{\lambda'} &= \gamma(1 - \beta cos\theta) \\ \frac{\nu}{\nu'} &= \gamma(1 - \vec{\beta}\cdot\hat{r}') = \gamma(1 + \beta cos\theta') & \frac{\lambda'}{\lambda} &= \gamma(1 + \beta cos\theta') \\ 1 &= \gamma(1 - \beta cos\theta)\gamma(1 + \beta cos\theta') \\ \nu_{\rm obs} &= \frac{\nu_{\rm source}}{\gamma(1 - \beta cos\theta_{\rm obs})} = \nu_{\rm source}\gamma\left(1 + \beta cos\theta_{\rm source}\right) \end{aligned} \quad (3)$$

1.3 Aberration of Light

$$tan\theta = \frac{sin\theta'}{\gamma\left(cos\theta' + v/c\right)} \tag{4}$$

where S′ is the frame of the emitter and S is the frame of the receiver.

1.4 Velocity Addition; Acceleration Transform

$$\begin{aligned}
u_x &= \frac{u'_x + v}{1 + u'_x v/c^2} & a_x &= \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x \\
u_y &= \frac{u'_y}{\gamma(1 + u'_x v/c^2)} & a_y &= \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^2} a'_y - \frac{\frac{u'_y v}{c^2}\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x \\
u_z &= \frac{u'_z}{\gamma(1 + u'_x v/c^2)} & a_z &= \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^2} a'_z - \frac{\frac{u'_z v}{c^2}\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x
\end{aligned} \tag{5}$$

1.5 Four Vectors

$$\begin{aligned}
\tilde{x} &= (ct, \vec{x}), && \text{position} \\
\tilde{u} &= d\tilde{x}/d\tau \quad u^\alpha = dx^\alpha/d\tau, && \text{velocity} \\
\tilde{a} &= d\tilde{u}/d\tau = d^2\tilde{x}/d\tau^2 \quad a^\alpha = d^2x^\alpha/d\tau^2, && \text{acceleration} \\
\tilde{p} &= m_0\tilde{u} = (E/c, \vec{p}), && \text{momentum} \\
\tilde{k} &= \tilde{p}/h = (\nu, \vec{k}), && \text{wave number vector} \\
\tilde{j} &= \rho_0\tilde{u} = (\rho c, \vec{j}) && \text{electric current} \\
\tilde{A} &= (\phi, A_x, A_y, A_z) && \text{Electromagnetic Potential}
\end{aligned} \tag{6}$$

1.6 Relativistic Kinematics & Invariants

$$|\tilde{p}|^2 c^2 = \tilde{p} \cdot \tilde{p} c^2 = p^\alpha p_\alpha c^2 = \left(m_0 c^2\right)^2$$

Conservation of four momentum.

$$\sum \tilde{p}_{in} = \sum \tilde{p}_{out}$$

For two particles

$$\tilde{p}_1 + \tilde{p}_2 = \text{constant}$$

$\tilde{p}_1 \cdot \tilde{p}_2$ is an invariant (same in all reference frames) yielding the following relation for the energy, E_{21}, of particle 2 in the rest frame of particle 1:

$$E_{21} = \frac{\tilde{p}_1 \cdot \tilde{p}_2}{m_{o1}}$$

Energy-Momentum Tensor: $T^{\mu\nu} = \rho u^\mu u^\nu$ Fluid: $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$

2 Electromagnetism

Electromagnetic Field Transformation

$$\begin{array}{cc} \vec{\mathbf{E}}'_{||} = \vec{\mathbf{E}}_{||} & \vec{\mathbf{B}}'_{||} = \vec{\mathbf{B}}_{||} \\ \vec{\mathbf{E}}'_{\perp} = \gamma\left(\vec{\mathbf{E}}_{\perp} + \vec{\beta}\times\vec{\mathbf{B}}\right) & \vec{\mathbf{B}}'_{\perp} = \gamma\left(\vec{\mathbf{B}}_{\perp} - \vec{\beta}\times\vec{\mathbf{E}}\right) \\ \vec{\mathbf{E}}' = \mathbf{E}_{||} + \gamma\left(\vec{\mathbf{E}}_{\perp} + \vec{\beta}\times\vec{\mathbf{B}}\right) & \vec{\mathbf{B}}' = \mathbf{B}_{||} + \gamma\left(\vec{\mathbf{B}}_{\perp} - \vec{\beta}\times\vec{\mathbf{E}}\right) \end{array} \tag{7}$$

For the special case of a Lorentz boost in the x direction

$$\begin{array}{cc} E'_x = E_x & B'_x = B_x \\ E'_y = \gamma\left(E_y - \beta B_z\right) & B'_y = \gamma\left(B_y + \beta E_z\right) \\ E'_z = \gamma\left(E_z + \beta B_y\right) & B'_z = \gamma\left(B_y - \beta E_y\right) \end{array} \tag{8}$$

$$\tilde{A} = (\phi, A_x, A_y, A_z) \qquad A^{\mu} = \frac{1}{c}\int\int\int \frac{j^{\mu}}{r} d^3V \tag{9}$$

$$F^{\mu\nu} = \partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu} \quad \text{or negative} \quad F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \tag{10}$$

$$\begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{11}$$

are F^{μ}_{ν}, $F_{\mu\nu}$, and $F^{\mu\nu}$, the electromagnetic field tensor in mixed, covariant, and contravariant form respectively.

2.1 Maxwell's Equations

$$F^{\nu}_{\mu,\nu} = F^{;\nu}_{\mu\nu} = \frac{4\pi}{c}j_{\mu} \text{ equivalent to } \vec{\nabla}\cdot\vec{E} = 4\pi\rho \text{ and } \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \tag{12}$$

$$F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0 \text{ equivalent to } \vec{\nabla}\cdot\vec{B} = 0 \text{ and } \vec{\nabla}\times\vec{E} - \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \tag{13}$$

For electromagnetism the four-force density and Lorentz force equations are

$$f_{\mu} = F^{\nu}_{\mu}j_{\nu} = F^{\nu}_{\mu}F^{\sigma}_{\nu,\sigma} \quad \text{or} \quad \frac{dp^{\alpha}}{d\tau} = F^{\alpha}_{\beta}j^{\beta} = qF^{\alpha}_{\beta}u^{\beta} \qquad \vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{14}$$

where $j_{\nu} = (\rho c, j_x, j_y, j_z)$ and $f_{\mu} = (W/c, f_x, f_y, f_z)$. $W = \vec{E}\cdot\vec{j}$ is the power density.

Note that a particle with charge q moving with momentum p in a uniform magnetic field will move in a circle with radius R given by $R = p/qB$ where, if p is in GeV and B is in Tesla, $r = (10/3)(p/GeV)/[(q/e)(B/T)]m$

2.2 Stress-Energy Tensor - $T^{\mu\nu}$

$$T^{\mu\nu} = F^{\mu}_{\alpha} F^{\alpha\nu} - \frac{1}{4}\delta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \tag{15}$$

$$T_{\mu\nu} = \frac{1}{4\pi}\begin{bmatrix} \frac{E^2+B^2}{2} & E_yB_z - E_zB_y & E_zB_x - E_xB_z & E_xB_y - E_yB_x \\ E_yB_z - E_zB_y & \frac{(E_x^2-E_y^2-E_z^2)+(B_x^2-B_y^2-B_z^2)}{2} & E_xE_y + B_xB_y & E_xE_z + B_xB_z \\ E_zB_x - E_xB_z & E_xE_y + B_xB_y & \frac{(E_y^2-E_x^2-E_z^2)+(B_y^2-B_x^2-B_z^2)}{2} & E_yE_z + B_yB_z \\ E_xB_y - E_yB_z & E_xE_z + B_xB_z & E_yE_z + B_yB_z & \frac{(E_z^2-E_x^2-E_y^2)+(B_z^2-B_x^2-B_y^2)}{2} \end{bmatrix} \tag{16}$$

2.3 Accelerated Charge/Synchrotron Radiation

$$P = \frac{2q^2}{3c^3}\vec{a}\cdot\vec{a} = \frac{2q^2}{3c^3}\vec{a}'\cdot\vec{a}' = \frac{2q^2}{3c^3}\left(a'^2_\perp + a'^2_\parallel\right) = \frac{2q^2}{3c^3}\gamma^4\left(a^2_\perp + \gamma^2 a^2_\parallel\right) \tag{17}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|a|^2 sin^2\Theta = \frac{q^2}{4\pi c^3}\frac{a^2_\perp + \gamma^2 a^2_\parallel}{(1-\beta cos\theta)^4} sin^2\Theta' \tag{18}$$

Evaluation for perpendicular and parallel cases yields:

$$\begin{aligned} \frac{dP_\perp}{d\Omega} &= \frac{q^2 a^2_\perp}{4\pi c^3}\frac{1}{(1-\beta cos\theta)^4}\left[1 - \frac{sin^2\theta cos^2\phi}{\gamma^2(1-\beta cos\theta)^2}\right] \\ \rightarrow_{\gamma\gg 1} &\approx \frac{4q^2 a^2_\perp}{\pi c^3}\gamma^{\mathbf{8}}\frac{1 - 2\gamma^2\theta^2 cos2\phi + \gamma^4\theta^4}{(1+\gamma^2\theta^2)^6} \end{aligned} \tag{19}$$

$$\frac{dP_\parallel}{d\Omega} = \frac{q^2 a^2_\perp}{4\pi c^3}\frac{sin^2\theta}{(1-\beta cos\theta)^6} \rightarrow_{\gamma\gg 1} \approx \frac{4q^2 a^2_\parallel}{\pi c^3}\gamma^{\mathbf{10}}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^6} \tag{20}$$

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3} = \frac{2}{3}\frac{\beta^2\gamma^2 q^4 B^2}{m_o^2 c^3} \tag{21}$$

3 Uniformly Accelerating Frame

$\eta = 1 + \kappa x$, $\gamma^* = 1/\sqrt{\eta^2 - \beta^2}$, $\kappa = g/c^2$, or $\eta = 1 + gx/c^2$.

$$(cd\tau)^2 = \left(1 + \frac{gx}{c^2}\right)^2 (cdt)^2 - \left(d\vec{\ell}\right)^2 \tag{22}$$

Transformation equations

$$\begin{aligned} x' &= -\frac{1}{\kappa} + (x + 1/\kappa)cosh(\kappa\tau) = -\frac{c^2}{g} + \left(x + \frac{c^2}{g}\right)cosh(g\tau/c) \\ t' &= (x + 1/\kappa)sinh(\kappa\tau) = (x + \frac{c^2}{g})sinh(g\tau/c) \end{aligned} \tag{23}$$

$$\text{Local Coordinates}: \quad \beta^i_L = \frac{dx^i}{\eta d\tau} \quad \beta^i_L = \frac{1}{\eta}\beta^i \tag{24}$$

4 Differential Geometry

(1) **Riemannian Geometry: Invariant Interval** $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$
(2) Parallel Displacement/ Christoffel Symbol

$$\delta A^\nu = -\Gamma^\nu_{\alpha\beta} A^\alpha \delta x^\beta \qquad \delta A_\alpha = \Gamma^\nu_{\alpha\beta} A_\nu \delta x^\beta \qquad \Gamma^m_{ij} = \frac{1}{2} g^{mk} \left[g_{ik,j} + g_{jk,i} - g_{ij,k} \right] \quad (25)$$

(3) Geodesic Path

$$\frac{d^2 x^\sigma}{ds^2} + \Gamma^\sigma_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \qquad \frac{du^\sigma}{d\tau} = -\Gamma^\sigma_{\alpha\beta} u^\alpha u^\beta \quad \text{or} \quad \delta u^\sigma = -\Gamma^\sigma_{\alpha\beta} u^\alpha \delta x^\beta \quad (26)$$

Geodesic Deviation Equation: General and in free-fall

$$\frac{D^2 \chi^\alpha}{d\tau^2} = -R^\alpha_{\beta\gamma\delta} u^\beta \chi^\gamma u^\delta \qquad \frac{d^2 \chi^\alpha}{d\tau^2} = -R^\alpha_{\tau\beta\tau} \chi^\beta \quad (27)$$

(4) Covariant Derivative

$$\text{covariant derivative of } A^\mu \equiv \frac{DA^\mu}{Dx^\alpha} \equiv A^\mu_{;\alpha} = \frac{\partial A^\mu}{\partial x^\alpha} + \Gamma^\mu_{\beta\alpha} A^\beta = A^\mu_{,\alpha} + \Gamma^\mu_{\beta\alpha} A^\beta \quad (28)$$

(5) Curvature: Riemann, Ricci, Scalar

$$R^k_{ars} = \Gamma^k_{ar,s} - \Gamma^k_{as,r} + \Gamma^b_{ar} \Gamma^k_{sb} - \Gamma^b_{as} \Gamma^k_{rb} \qquad \text{Ricci}: \ R_{ij} = R^k_{ijk} \qquad \text{Scalar}: \ R = R^i_i \quad (29)$$

5 General Relativity

5.1 Equivalence Principle

No experiment can distinguish between a uniformly accelerating reference frame and a uniform gravitational field. Including a complete equivalence between gravitational and inertial mass.

5.2 Weak Field Gravitation: $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$

$$(cd\tau)^2 = \left(1 + \frac{2\phi}{c^2}\right)(cdt)^2 - \left(1 - \frac{2\phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right) \quad (30)$$

5.3 Einstein Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv G_{\mu\nu} = -8\pi G T_{\mu\nu}/c^4 \qquad R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\sigma_\sigma\right) \quad (31)$$

5.4 Schwarzschild Solution

$$ds^2 = (1 - r_s/r)\, c^2 dt^2 - (1 - r_s/r)^{-1}\, dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{32}$$

where $r_s = 2GM/c^2$.

5.5 Gravitational Lenses

Impact parameter approximation along constant z' path gives bending angles and magnification

$$\begin{aligned} \vec{\alpha} &= -\frac{2}{c^2}\int \vec{\nabla}_\perp \phi dz' \qquad \alpha_x = -\frac{2}{c^2}\int \frac{\partial\phi}{\partial x'}dz' \qquad \alpha_y = -\frac{2}{c^2}\int \frac{\partial\phi}{\partial y'}dz' \\ I(\vec{\theta}) &= I(\vec{\beta}) = I(\vec{\theta} - \nabla\psi) \qquad \mu = \left| det\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right|^{-1} = \left[(1-\kappa)^2 - \gamma^2\right] \end{aligned} \tag{33}$$

Schwarzschild metric mass, bending angle, Einstein Ring radius angle,

$$\alpha = \frac{4GM}{c^2 b} \qquad \theta_E = \left(\frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}\right)^{1/2} \qquad \theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right) \tag{34}$$

$$\mu_\pm = \left|\frac{1}{1 - (\theta_E/\theta_\pm)^4}\right| \qquad \mu = m_+ + m_- = \frac{u^2+2}{u\sqrt{u^2+4}}, \quad u = \beta/\theta_E \tag{35}$$

Isothermal or constant rotation velocity curve

$$\alpha = 4\pi\left(\frac{v}{c}\right)^2 \quad \theta_E = 4\pi\left(\frac{v}{c}\right)^2\frac{D_{ds}}{D_s} \quad \theta_\pm = \theta_E \pm \beta \quad \mu_\pm = \left|1 \mp \frac{\theta_E}{\theta_\pm}\right|^{-1} = \left|1 \pm 4\pi\left(\frac{v}{c}\right)^2\frac{D_{ds}}{\beta D_s}\right| \tag{36}$$

5.6 Weak Field Gravity Waves

total power radiated in quadrupole mode

$$-\frac{dE}{dt} = \frac{G}{5c^5}\dddot{Q}^{kl}\dddot{Q}^{kl} \qquad Q^{kl} = \int (x^k x^l - \frac{1}{3}r^2\delta^{kl})\rho(t - r/c, \mathbf{x}')d^3x \tag{37}$$

For a body in orbit around a Schwarzschild metric mass

$$-\frac{dE}{dt} = \frac{G}{5c^5}\dddot{Q}^{kl}\dddot{Q}^{kl} \sim \frac{G}{c^5}(\frac{M}{R})^2 v^6 \sim L_{cGW}(\frac{R_{Schwarzschild}}{R})^2(\frac{v}{c})^6 \tag{38}$$

$$L_{GW} = \frac{L_{internal}^2}{L_{cGW}} \qquad L_{cGW} \equiv \frac{c^5}{G} = 3.63 \times 10^{59}\ \mathrm{erg\ s^{-1}} = 2.03 \times 10^5 \mathrm{M}_\odot c^2 \mathrm{s}^{-1} \tag{39}$$

$$h \sim \epsilon^{2/7} \frac{R_{schwarzschild}}{r} \sim 3 \times 10^{-18} (\frac{\epsilon}{0.1})^{2/7} \frac{(M/\mathrm{M}_\odot)}{(r/10kpc)}. \tag{40}$$

Vibrating Quadrupole

$$\gamma_{GW} \equiv -\frac{1}{E}\frac{dE}{dt} = \frac{32G}{15c^5} m b^2 \omega^4 \tag{41}$$

Two orbiting masses

$$-\frac{dE}{dt} = \frac{32G}{5c^5}\left[\frac{m_1 m_2}{m_1+m_2}\right]^2 R^4\omega^6 = \frac{32G}{5c^5}\mu^2 R^4 \omega^6 = \frac{32G^4}{5c^5R^5}(m_1m_2)^2(m_1+m_2) \tag{42}$$

$$\frac{dR}{dt} = -\frac{64G^3}{5c^5R^3} m_1 m_2 (m_1+m_2) \tag{43}$$

$$\frac{d\omega}{dt} = -\frac{3\omega}{2R}\frac{dR}{dt} = \frac{96}{5}[\frac{G(m_1+m_2)}{c^2R^3}]^{3/2}\frac{Gm_1m_2}{c^2R} = \frac{96}{5}\frac{G}{c^5}\omega^3\frac{Gm_1m_2}{R} \tag{44}$$

The fall of a test particle into Schwarzschild black hole of mass M gives

$$\Delta E = 0.0104 \frac{\mu^2 c^2}{M}. \tag{45}$$

$$T^{00} = \frac{1}{16\pi}\frac{c^2}{G} < h_+^2 + h_\times^2 > \tag{46}$$

5.6.1 Frame Dragging

$$\vec{\Omega}_{frame} = G\frac{3\left\langle \hat{n}\hat{n}\cdot\vec{S}\right\rangle - \vec{S}}{r^3} = \frac{G}{r^3}\left[3\vec{r}\vec{r}\cdot\vec{S}/r^2 - \vec{S}\right]$$

5.7 Black Holes

$$ds^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{47}$$

The Reissner-Nördstrom metric

$$ds^2 = \left(1 - \frac{2GM}{c^2r} + \frac{GQ^2}{r^2}\right)dt^2 - \left(1 - \frac{2GM}{c^2r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega^2 \tag{48}$$

The Kerr-Newman geometry is

$$ds^2 = \frac{\Delta}{\rho^2}\left[cdt - asin^2\theta d\phi\right]^2 - \frac{sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - acdt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 \tag{49}$$

where $\Delta \equiv r^2 - 2GMr/c^2 + a^2 + Q^2$, $\rho^2 \equiv r^2 + a^2cos^2\theta$, and $a \equiv S/M \equiv$ angular momentum per unit mass

Black Hole Thermodynamics and Hawking Radiation

$$kT = \frac{\hbar\kappa}{2\pi c} = \frac{\hbar c^3}{8\pi GM} \qquad T \simeq 6\times 10^{-8}\left(\frac{M_\odot}{M}\right)\ \mathrm{K} \tag{50}$$

$$\delta S = \frac{\delta Q}{T_{BH}} = \frac{\delta M c^2}{T_{BH}} = \frac{8kGM\delta M}{\hbar c} \qquad S = \frac{kc^3}{4\pi\hbar G}A\ +\ \mathrm{constant} \tag{51}$$

$$\text{Radiated Power} = \sigma_{SB}T_{BH}^4\ 4\pi r_s^2 \cong -10^{47}(\frac{1\mathrm{gm}}{M})^2 \mathrm{erg\ s^{-1}} \cong -10^{26}(\frac{M}{1\mathrm{gm}})^{-2}\mathrm{gm\ s^{-1}}c^2$$

The lifetime of a black hole will then be

$$\tau \cong \frac{M}{dM/dt} \simeq 10^{-26}s\times(M/1\ \mathrm{gm})^3$$

5.8 Cosmology

The spacetime interval for a homogeneous, isotropic universe is The Robertson-Walker metric in (r,θ,ϕ) form is

$$ds^2 = c^2dt^2 - a(t)^2\left[\frac{dr^2}{1-k\,(r/R)^2} - r^2\left(d\theta^2 + sin^2\theta^2 d\phi^2\right)\right] \quad \text{where} \quad k = \left\{\begin{matrix} +1 & \text{closed} \\ 0 & \text{flat} \\ -1 & \text{open}\end{matrix}\right. \tag{52}$$

$$\mathbf{(\frac{\dot{a}}{a})^2 - \frac{8\pi}{3}G\rho = -\frac{k}{a^2} + \frac{\Lambda}{3}} \qquad \mathbf{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P/c^2\right) + \frac{\Lambda}{3}} \tag{53}$$

$Stuff$	$P = w\rho$	$\rightarrow$	$\rho \propto a^{-3(1+w)}$	$a \propto t^{2/[3(1+w)]}$	$t_0 = \frac{2}{3(1+w)}H_0^{-1}$
$Radiation$	$P = 1/3\ \rho$	$\rightarrow$	$\rho \propto a^{-4}$	$a \propto t^{1/2}$	$t_0 = \frac{1}{2}H_0^{-1}$
$Matter$	$P = 0$	$\rightarrow$	$\rho \propto a^{-3}$	$a \propto t^{2/3}$	$t_0 = \frac{2}{3}H_0^{-1}$
$Curvature$	$-1/3$	$\rightarrow$	$\rho \propto a^{-2}$	$a \propto t$	$t_0 = H_0^{-1}$
$Vacuum\ Energy$	$P = -\rho$	$\rightarrow$	$\rho = constant$	$a(t) \propto e^{Ht}$	$t_0 = \infty$

$$R = \frac{cH^{-1}}{(\Omega - 1 + \Lambda/3H^2)^{1/2}} \qquad V(a) = -\frac{1}{2}\left(\sum_x \frac{\Omega_x a^2}{a^{3(1+w_x)}}\right) \qquad w = constant \tag{54}$$

If w depends upon a (or time), then $\Omega_x(a_2) = \Omega_x(a_1)e^{-\int_{a_1}^{a_2} 3(1+w)dln(a)}$

$$\Omega_x = \frac{\rho_x}{\rho_c} \qquad \rho_c = \frac{3H^2}{8\pi G} \qquad \rho_{0c} = 1.00\times 10^{-29}\left(\frac{H_0}{71km/s/Mpc}\right)^2 gm/cm^3$$

5.8.1 cosmic string

$$\begin{aligned} ds^2 &= dt^2 - dr^2 - (1-4G\mu)^2\,r^2\,d\theta^2 - dz^2 \\ &\approx dt^2 - dr^2 - (1-8G\mu)\,r^2\,d\theta^2 - dz^2, \end{aligned}$$