Physics 139 Relativity

Thomas Precession   February 1998

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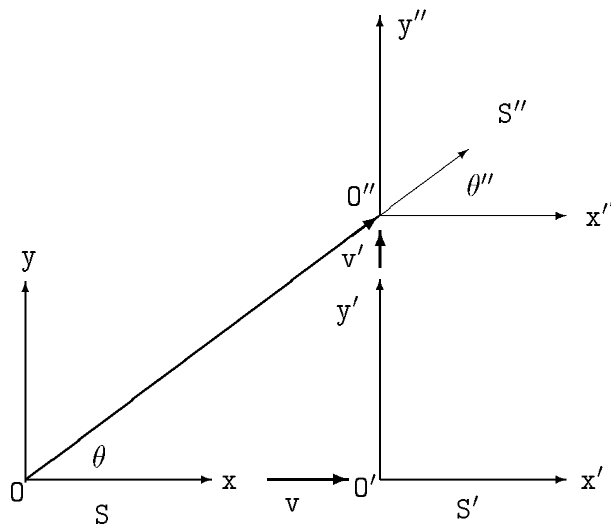
1 Thomas Precession

Thomas Precession is a kinematic effect discovered by L. T. Thomas in 1926 (L. T. Thomas *Phil. Mag.* **3**, 1 (1927)). It is fairly subtle and mathematically sophisticated but it has great importance in atomic physics in connection with spin-orbit interaction. Without including Thomas Precession, the rate of spin precession of an atomic electron is off by a factor of 2. Later we will see that there is a similar effect for gravitational fields.

The effect is connected with the fact that two successive Lorentz transformations in different directions are equivalent to a Lorentz transformation plus a three dimensional rotation. This rotation of the local frame of rest is the kinematic effect that causes the Thomas precession.

For the lecture we will not do the full mathematical treatment, since it is rather involved. Instead we will show by a simple example how the rotation and thus precession comes about.

Make two successive Lorentz transformations in orthogonal directions: from S to S′ with velocity v along the x axis, followed by a transformation from S′ to S″ with velocity v' along the y' axis, as shown by the following diagram.



The line from the origin O of S to the origin O'' of S″ making and angle θ in S and an angle θ'' in S″. We can calculate the angles in the two frames by applying the Lorentz transformations and evaluating them in each frame.

$$
\begin{array}{cc}
x' = \gamma(x - vt) & x = \gamma(x' + vt') \\
t' = \gamma(t - vx/c^2) & t = \gamma(t' + vx'/c^2) \\
y' = y & y = y' \\
 & \\
y'' = \gamma'(y' - v't') & y' = \gamma'(y'' + vt'') \\
x'' = x' & x' = x'' \\
\text{where} & \\
\gamma = 1/\sqrt{1 - v^2/c^2} & \gamma' = 1/\sqrt{1 - (v'/c)^2}
\end{array}
\tag{1}
$$

Combing these equations one finds:

$$
\begin{aligned}
y'' &= \gamma'[y - v'\gamma(x - vt)] \\
x'' &= \gamma(x - vt)
\end{aligned}
\tag{2}
$$

Now we can calculate the angle θ made by the line between origins. For a Galilean transform one would have

$$
tan\theta = \frac{y}{x} = \frac{v't}{vt} = \frac{v'}{v}
\tag{3}
$$

but Special Relativity shows us that 3-D velocities do not transform like 3-D vectors. So we must calculate carefully.

$$
tan\theta = \frac{y}{x} = \frac{y'}{vt} = \frac{\gamma'(y'' + v't'')}{vt}|_{y''=0} = \frac{\gamma' v' t''}{vt}
\tag{4}
$$

$$
t = \gamma(t' + vx'/c^2)|_{x''=x'=0} = \gamma\gamma'(t'' + v'y''/c^2)|_{y''=0} = \gamma\gamma' t''
\tag{5}
$$

so that

$$
tan\theta = \frac{\gamma' v' t''}{v\gamma\gamma' t''} = \frac{v'}{\gamma v}
\tag{6}
$$

Note that this answer is very near the Galilean result but with the factor of $1/\gamma$ which reminds us of aberration.

Now we calculate θ'':

$$
tan\theta'' = \frac{y''}{x''} = \frac{\gamma'[y' - v't']}{x'}
\tag{7}
$$

where x'' and y'' are the coordinates of the origin O of system S in the S$''$ system. Thus

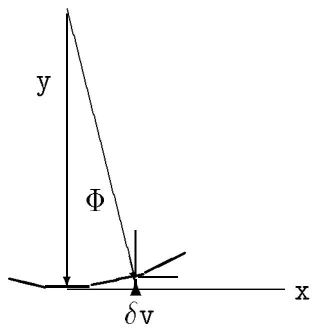
$$
tan\theta'' = \frac{\gamma'[y - v']}{x'}|_{y=0} = -\frac{\gamma' v' t'}{x'} = -\frac{\gamma' v' t'}{\gamma(x - vt)}|_{x=0} = \frac{-\gamma' v' t'}{-\gamma vt}
\tag{8}
$$

$$
t' = \gamma(t - vx/c^2)|_{x=0} = \gamma t;
\tag{9}
$$

$$
tan\theta'' = \frac{\gamma' v'}{v}
\tag{10}
$$

This looks again similar to the Galilean angle except for the extra factor of γ'.

Now consider a particle on a curved path



At a certain time it is at the origin O of our system S. Put the x axis parallel to the path, and y axis toward the center of curvature. At $t = 0$, the rest frame S′ is moving in the x direction with velocity v. At a slightly later time its rest frame S″ is moving perpendicular to x' in the y direction with velocity $v' = \delta v$.

Define

$$\delta\theta = \theta'' - \theta = tan^{-1}\left(\frac{v'\gamma'}{v}\right) - tan^{-1}\left(\frac{v'}{\gamma v}\right) \tag{11}$$

For a very short time interval the motion is circular. That is fit the local curve with a tangent circle with appropriate radius of curvature.

$$\begin{array}{cc} v_x = \omega R cos\phi & v_y = \omega R sin\phi \\ v_x = v & v_y = \delta v = v' \end{array} \tag{12}$$

so

$$tan\phi = \frac{v'}{v}$$

$$\delta\theta = \theta'' - \theta = tan^{-1}\left(\gamma' tan\phi\right) - tan^{-1}\left(\frac{tan\phi}{\gamma}\right) \tag{13}$$

Choose ϕ to be very small;

$$\phi = \frac{\delta S}{R} = \frac{v\delta t}{R}$$

Then

$$\delta\theta \approx \frac{v\delta t}{R}\left(\gamma' - \frac{1}{\gamma}\right) \tag{14}$$

$$\omega_T = \frac{\delta\theta}{\delta t} \approx \frac{v}{R}\left(\gamma' - \frac{1}{\gamma}\right)$$

In a circle the acceleration is

$$a = \frac{v^2}{R} \qquad \text{so that} \qquad \frac{v}{R} = \frac{a}{v}$$

giving

$$\omega_T = \frac{a}{v}\left(\gamma' - \frac{1}{\gamma}\right)$$

Suppose we are in a non-relativistic region $v << c$, like an electron in an atom:

$$\gamma' - \frac{1}{\gamma} = \frac{1}{\sqrt{1-(v'/c)^2}} - \sqrt{1-(v/c)^2} \approx 1 + \frac{1}{2}(\frac{v'}{c})^2 - 1 + \frac{1}{2}(\frac{v}{c})^2 \sim \frac{1}{2}(\frac{v}{c})^2$$

since $tan\phi = v'/v << 1$. Putting this back into the expression for ω_T

$$\omega_T \approx \frac{a}{v}\frac{v^2}{2c^2} = \frac{va}{2c^2}$$

Thus $\theta'' > \theta$, thus a counter-clockwise rotation, implying

$$\vec{\omega}_T = \frac{\vec{v}\times\vec{a}}{2c^2} \tag{15}$$

The rigorous result is

$$\vec{\omega}_T = \frac{\gamma^2}{\gamma+1}\frac{\vec{v}\times\vec{a}}{2c^2} \tag{16}$$

2 Spin-Orbit Interaction of Electron with Nucleus in an Atom

Now we are set to apply this kinematic effect to spin precession in an atom. In its own rest frame the electron "sees" the nucleus flying by.

The electron's magnetic moment, $\vec{\mu}$, and spin angular momentum, $\vec{S}$, are related by

$$\vec{m}u = \frac{e}{m_c c}\vec{S} \tag{17}$$

The torque on the magnetic moment is

$$\vec{\tau} = \frac{d\vec{S}}{dt} = \vec{\mu}\times\vec{B}' \tag{18}$$

where $\vec{B}'$ is the magnetic field in the e^- frame.

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v}_e}{c}\times\vec{E}\right) \tag{19}$$

Where $\vec{B}$ is the magnetic field and $\vec{E}$ is the electric field in the nucleus rest frame. $v/c << 1$ so that $\gamma \approx 1$,

$$\frac{d\vec{S}}{dt} = \vec{\mu}\times\left(\vec{B} - \frac{\vec{v}_e}{c}\times\vec{E}\right) \tag{20}$$

arises from the interaction energy

$$U' = -\vec{\mu} \cdot \left(\vec{B} - \frac{\vec{v}_e}{c} \times \vec{E} \right) \tag{21}$$

If $\vec{E}$ is due to a spherically symmetrical charge distribution – as for a one-electron atom or one outside a closed shell – then

$$e\vec{E} = -\vec{\nabla} V(r) - \frac{\vec{r}}{r}\frac{dV}{dr}. \tag{22}$$

Then

$$U' = -\frac{e}{m_e c}\vec{S} \cdot \vec{B} + \frac{e}{m_e c^2}\vec{S} \cdot \vec{v} \times \left(-\frac{\vec{r}}{r}\frac{dV}{dr} \right) \tag{23}$$

$$\vec{S} \cdot \vec{v} \times \vec{(} - r) = +\vec{S} \cdot \vec{f} \times \vec{v}$$

$$\begin{aligned} U' &= -\frac{e}{m_e c}\vec{S} \cdot \vec{B} + \frac{e}{m_e^2 c^2}\vec{S} \cdot (\vec{r} \times \vec{v}) \frac{1}{r}\frac{dV}{dr} \\ &= -\frac{e}{m_e c}\vec{S} \cdot \vec{B} + \frac{e}{m_e c^2}\vec{S} \cdot \vec{L}\frac{1}{r}\frac{dV}{dr} \end{aligned} \tag{24}$$

since $m\vec{r} \times \vec{v} = \vec{L} \equiv$ angular momentum. This second term is the spin-orbit interaction.

Now, if the electron rest frame is **rotating** – Thomas angular velocity $\vec{\omega}$, $d\vec{S}/dt \neq \vec{\mu} \times \vec{B}'$. The general kinematic result from classical physics is:

$$\frac{\partial}{\partial t}|_{\text{rotation coordinates}} = \frac{\partial}{\partial t}|_{\text{inertial coordinates}} - \vec{\omega} \times \tag{25}$$

as an operator on <u>any</u> vector. So

$$\frac{\partial \vec{S}}{\partial t}|_{\text{rotation coordinates}} = \frac{\partial \vec{S}}{\partial t}|_{\text{inertial coordinates}} - \vec{\omega} \times \vec{S} \tag{26}$$

With this expression the interation energy is changed to:

$$U = U' - \vec{S} \cdot \vec{\omega}_T \tag{27}$$

where $\vec{\omega}_T$ is proportional to the centripetal acceleration due to E_r.

$$\begin{aligned} \vec{\omega}_T &\approx \frac{1}{2c^2}\vec{v} \times \vec{a} = \frac{1}{2c^2}\vec{v} \times \left(\frac{e\vec{E}}{m_e} \right) \\ &= \frac{1}{2mc^2}\vec{v} \times \left(-\frac{\vec{r}}{r}\frac{dV}{dr} \right) \end{aligned}$$

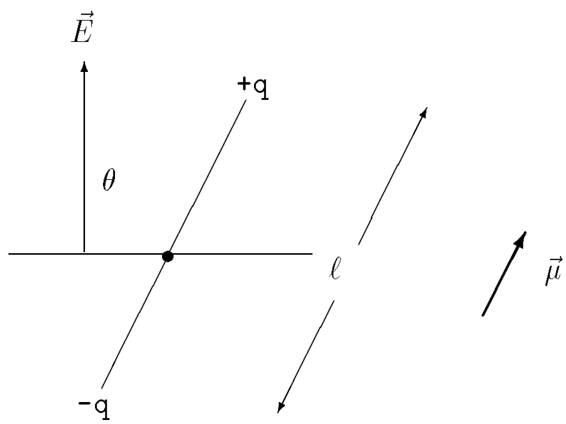
$$= \frac{1}{2m_e c^2}(\vec{r}\times\vec{v})\frac{1}{r}\frac{dV}{dr} = \frac{\vec{L}}{2m_e^2c^2}\frac{1}{r}\frac{dV}{dr} \tag{28}$$

Thus

$$\begin{aligned} U &= U' - \frac{1}{2m_e^2c^2}\vec{S}\cdot\vec{L}\frac{1}{r}\frac{dV}{dr} \\ &= -\frac{e}{m_e c}\vec{S}\cdot\vec{B} + (1-\frac{1}{2})\frac{1}{m_e^2c^2}\vec{S}\cdot\vec{L}\frac{1}{r}\frac{dV}{dr} \end{aligned} \tag{29}$$

The -1/2 is the famous one half. Including it, the observed fine-structure spacings in atomic spectra, due to electron spin, are correctly predicted.

This schematic gives a heuristic indication of how the torque arises.



The force on each charge (positive and negative) is $F = qE$. The magnetic moment is $\mu = g\ell$. The net torque is

$$\tau = qE\ell sin\theta = \mu E sin\theta$$

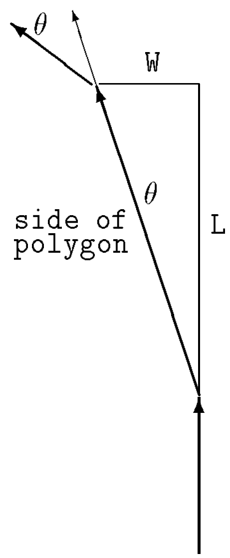
The energy relative to $\theta = \pi$ is

$$\Delta E = -2qE\frac{\ell}{2}cos\theta = -\vec{\mu}\cdot\vec{E}$$

3 A Simple Derivation of the Thomas Precession

The follwoing derivation is based upon a suggestion by E.M. Purcell.

Imagine an aricraft flying in a large circular orbit. Approximate the orbit by a polygon of N sides, with N a very large number. As the aircraft traverses each of the N sides, it changes its angle of flight by the angle $\theta = 2\pi/N$ as shown in the figure.



After the aircraft has flown N segments, it is back at its starting point. IN the laboratory frame, the aircraft has rotated through an angle of 2π radians. However in the aircraft's instantaneous rest frame, the triangles shown have a Lorentz-contraction along the direction it is flying but not transversely. Thus at the end of each segment, in the aircraft frame, the aircraft turns by a larger angle than the laboratory $\theta = 2\pi/N$, but by an angle $\theta' = \gamma\theta = W/(L/\gamma) = 2\pi\gamma/N$. After all N segements in the aircraft instanteous rest frame the total angle of rotation is $2\pi\gamma$.

The difference in the reference frame is

$$\Delta\theta = 2\pi(\gamma - 1)$$

Since N has dropped out of the formula for the angle and angle difference, one can let it go to infinity and the motion is circular and the formula is for the rate of precession.

$$\frac{\omega_P}{\omega} = \frac{\Delta\theta/T}{2\pi/T} = \gamma - 1$$

This equation, dispite the simplicity of the derivation, is the exact expression for the Thomas precession . The equation does not include the oscillationg term because the derivation neglected the fact that the front and rear of the inertial bars are not accelerated simultaneously.