Physics 139 Relativity
Relativity Notes 2003

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http://aether.lbl.gov/www/classes/p139/homework/homework.html

1 General Relativity Principles

First we review Newtonian Gravitation to remind ourselves of what General Relativity must reduce to in the classical limit.

1.1 Newtonian Gravitation

We know very well Newton's three laws of motion and his law of Universal Gravitation. The most famous formula for the first is

$$\vec{F} = m_i\vec{a} \quad F_j = m_i a_j = m_i\frac{d^2x_j}{dt^2} \quad \tilde{F} = m_0\tilde{a} \quad F^\alpha = m_0 a^\alpha = m_0\frac{d^2x^\alpha}{dt^2} \tag{1}$$

where m_i means inertial mass and m_0 means inertial rest mass.

Newton's Universal law of Gravitation is often expressed as

$$F_G = -\frac{GM_G m_G}{r^2} \quad \text{or} \quad \vec{F}_G = -m_G\nabla\Phi \quad F_G^j = \frac{\partial\Phi}{\partial x_j}$$

Where $\Phi = GM_G/r$ for a point mass or $\Phi = \int\int\int \rho_G/r \; dxdydz$ for a distributed density and the subscript G means the source charge of gravity. That is, m_G is the coupling for producing or being acted on (assuming action equals reaction) by gravitational forces.

1.1.1 Weak Equivalence Prinicple

The weak equivalence principle holds that the gravitational mass and inertial mass are exactly the same: $m_i = m_G$. This is well established by experiment to roughtly the 10^{-14} level. We take it as correct here. Then we automatically get an equivalence of acceleration and gravity at a point as far as kinematics are concerned.

Continuing with our discussion of Newtonian gravity assuming the validity of the Weak Equivalence Principle we have for the equations of motion:

$$\frac{d^2x^i}{dt^2} = -\delta^{ij}\frac{\partial\Phi}{\partial x^j} \tag{3}$$

This gives us a simple equation of motion based upon the first partial derivatives of the gravitational potential.

1.2 Tidal Forces Equations

Now it is both intuitive and well known that conventional gravitational fields, e.g. those from the Earth, moon, Sun etc. produce tidal forces. E.g. every thing dropped from rest (or near rest) near the Earth experience an acceleration directed toward the center of the Earth. Over small distances the directions are nearly parallel so everything in free fall (no other forces acting) appears to accelerate down at the same rate and direction. When viewed more globally, it is clear that the accelerations vary with the angle one is separated by around the circumference of the Earth.

We can write and equation for this for the general case by considering two particles one at location x^k and one at location $x^k + \Delta x^k$. The potential will be slightly different but we can expand Φ

$$\Phi(x^k + \Delta x^k) = \Phi(x^k) + \frac{\partial \Phi}{\partial x^k}\Delta x^k + ...$$

around the original point with position vector component x^k.

The equation of motion for the second particle is similar

$$\frac{d^2 x^i + \Delta x^i}{dt^2} = -\delta^{ij}\frac{\partial \Phi(x^k + \Delta x^k)}{\partial x^j} = -\delta^{ij}\left(\frac{\partial \Phi}{\partial x^j} + \frac{\partial^2 \Phi}{\partial x^j \partial x^k}\right)$$

Taking the difference between the original and displace by Δx^k test particles equations of motion, we have

$$\frac{d^2 \Delta x^i}{dt^2} = -\delta^{ij}\frac{\partial^2 \Phi}{\partial x^j \partial x^k} \tag{4}$$

Thus things that are separated by $\Delta \vec{x}$ accelerate together at a rate that is proportional to their separation and to the mixed second partial derivatives of the gravitational potential.

Consider one of Newton's spherical masses for an example.

$$\begin{aligned} \Phi &= -\frac{GM}{r} \\ \frac{d^2 x^i}{dt^2} &= -\frac{\partial \Phi}{\partial x^i} = \frac{GM}{r^2}\hat{n}_i \\ \frac{d^2 \Delta x^i}{dt^2} &= -\delta^{ij}\frac{\partial^2 \Phi}{\partial x^j \partial x^k} = -\left(\delta_{ij} - 3 n_i n_j\right)\frac{GM}{r^3} \end{aligned} \tag{5}$$

1.3 Geodesic Equation

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^{\alpha}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\frac{1}{2}g^{\alpha\sigma}\left[g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{6}$$

Table 1: Gravitational Field Strengths for Astrophysical Objects

At Surface of Object	ϕ/c^2
proton	$\sim 10^{-39}$
Earth	$\sim 10^{-9}$
Sun	$\sim 10^{-6}$
White Dwarf	$\sim 10^{-4}$

Consider the case of very 3-D low velocity, near the instanteous rest frame where $\tilde{U} = \gamma(1, \beta_x, \beta_y, \beta_z)c \simeq (v^0, 0, 0, 0) \approx (1, 0, 0, 0)c$

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu \simeq -\Gamma^\alpha_{00}(v^0)^2 \tag{7}$$

The first part of the equation shows that the geodesic equation is equivalent to the parallel transport of the 4-D velocity vector $\tilde{u}$.

1.4 Geometric Interpretation

Now we have come to understand inertia in curved space-time

Now we continue the Newtonian approximation by considering a static gravitational field in static coordinate system.

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\nu}(g_{\nu 0,0} + g_{0\nu,0} - g_{00,\nu}) = \frac{1}{2} g^{\alpha\nu} g_{00,\nu} \simeq -\frac{1}{2} g_{00,\nu} \tag{8}$$

The standard Newtonian equation of motion for a potential is

$$a^m = \frac{d^2x^m}{dt^2} = \frac{dv^m}{dt} = -\frac{\partial\phi}{\partial x^m} \tag{9}$$

A consistent solution is

$$g_{00,m} = 2\frac{\partial\phi}{\partial x^m} = (2\phi)_{,m} \tag{10}$$

$$g_{00} = 1 + 2\phi/c^2 \tag{11}$$

Note that for a point source or spherically distributed mass, M, within radius $R<r$ $\phi(r) = -GM/r$

$$g_{00} = 1 - \frac{2GM}{c^2 r} \tag{12}$$

1.5 Weak Field Gravity

We are now in a position to understand gravity in the weak field limit through the Equivalence Principle and the idea that gravity is replaced by curved space-time. We can derive the weak field approximation determine its paramaterization from the Newtonian limit. We make use of the observation that Newton's law of gravity holds for situations in which the non-relativistic approximation is justified and thus valid. These are situations in which the space-time region is sufficiently small that it can be approximated by a coordinate system that is nearly flat.

We write the weak field approximation as

$$g_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} + h_{\mu\nu} \tag{13}$$

where $\eta_{\mu\nu}$ is the Minkowski metric (or later in the course this will be expanded to the Robertson-Walker metric which reduces to the Minkowski metric in the small space-time limit) and $h_{\mu\nu} << 1$ is a small perturbation to the metric.

Using the formula for determining the Christoffel symbol from metric tensor one has

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}\left(g_{\lambda\nu,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}\right) \simeq \frac{1}{2}\left(h_{\lambda\nu,\mu} + h_{\lambda\mu,\nu} - h_{\mu\nu,\lambda}\right) \tag{14}$$

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu} + h^{\nu}_{\alpha} h^{\alpha\nu} + \ldots \tag{15}$$

and thus

$$\Gamma^{\alpha}_{\mu\nu} = \eta^{\alpha\lambda}\Gamma_{\lambda\mu\nu} + Order(h^2) \tag{16}$$

The curvature tensor is

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + Order(h^2) \tag{17}$$

The Ricci tensor is

$$R_{\mu\nu} \simeq \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\nu\alpha,\mu} + Order(h^2) = \frac{1}{2}\left(-h_{\mu\nu,00} + \ldots\right) \tag{18}$$

The Ricci scalar curvature is

$$R = h_{\mu\mu,00} + h_{ij,ji} + Order(h^2) \tag{19}$$

A slowly moving particle has

$$\frac{dx^{\mu}}{d\tau} \sim (1,0,0,0) \tag{20}$$

so that the geodesic equation is

$$\frac{d^2 x^i(\tau)}{d\tau^2} = -\Gamma^{i}_{00} \tag{21}$$

In a stationary (no time dependence) system the time derivatives are zero and therefore

$$- \;,\; _{i00} = \frac{1}{2} h_{00,i} \tag{22}$$

so that $-\frac{1}{2}h_{00}$ can be the gravitational potential.

Energy momentum conservation requires that the covariant derivative of the stress-energy tensor be zero:

$$D_\mu T_{\mu\nu} = 0. \tag{23}$$

$$\begin{aligned} -,\, ^{i}_{00} &= -\frac{\partial V(x)}{\partial x^i} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i} \\ -\frac{\partial ,\, ^{i}_{00}}{\partial x^i} &= 4\pi G_N T_{00} \\ \frac{\partial^2 h_{00}}{\partial (ct)^2} &= 8\pi G_N T_{00} \end{aligned} \tag{24}$$

Now this must be expressed in a way that is invariant under general coordinat transformations in order to deterimine the appropriate theory of gravity. Instead of using only one component of the stress-energy tensor $T_{\mu\nu}$ and certain partial derivatives of he connection fields , , we need a relation between covariant tensors. We reason from

$$R_{00} = -\frac{1}{2}\frac{\partial^2 h_{00}}{\partial (ct)^2}; \tag{25}$$

$$R = \frac{\partial^2 h_{ij}}{\partial x^i \partial x^j} + \vec{\partial}^2 (h_{00} - h_{ii}). \tag{26}$$

These equations give us a lead on the relationship between the tensors $T_{\mu\nu}$ and $R_{\mu\nu}$. The most general tensor relation of this thype would be

$$R_{\mu\nu} = A T_{\mu\nu} + B g_{\mu\nu} T^\alpha_\alpha \tag{27}$$

where A and B are constants that have to be determined by comparing to the Newtonian limit. The trace of the energy-momentum (stress-energy) tensor is, in the non-relativisitic approximation

$$T^\alpha_\alpha = -T_{00} + T_{ii}. \tag{28}$$

so that the 00 component can be written as

$$R_{00} = -\frac{1}{2}\vec{\partial}^2 h_{00} = (A+B)T_{00} - B T_{ii} \tag{29}$$

In the newtonian limit the T_{ii} term (the pressure p) vanishes, not only because the pressure of the ordinary (non-relativistic) matter is very small but also because it averages out to zero as a source. In the stationary case

$$0 = \partial_\mu T_{\mu i} = \partial_j T_{ji} \tag{30}$$

$$\frac{d}{dx^1}\int T_{11}dx^2dx^3 = -\int dx^2dx^3(\partial_2 T_{21} + \partial_3 T_{31}) = 0, \tag{31}$$

and therefore, if our source is surrounded by a vacuum

$$\int T_{11}dx^2dx^3 = 0 \rightarrow \int d^3\vec{x}\; T_{11} = 0, \tag{32}$$

and similarly

$$\int d^3\vec{x}\; T_{22} = \int d^3\vec{x}\; T_{33} = 0. \tag{33}$$

This in turn implies

$$A + B = -4\pi G_N. \tag{34}$$

There is an additional piece of information in that the trace of $R_{\mu\nu}$ is $R = (A + 4B)T^{\alpha}_{\alpha}$. Just at the general form of the second rank tensor for the stress energy included a trace component we msut make use of the Einstein curvature tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ to get

$$G_{\mu\nu} = AT_{\mu\nu} - \left(\frac{1}{2}A + B\right) T^{\alpha}_{\alpha} g_{\mu\nu} \tag{35}$$

Then

$$D_\mu G_{\mu\nu} = 0; \quad D_\mu T_{\mu\nu} = 0; \quad \text{therefore} \quad \left(\frac{1}{2}A + B\right) \partial_\nu T^{\alpha}_{\alpha} = 0. \tag{36}$$

For this to hold in general

$$B = -\frac{1}{2}A; \qquad A = -8\pi G_N, \tag{37}$$

The only tensor equation consistent with Newton's equation in a locally flat coordinate frame is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G_N T_{\mu\nu} \tag{38}$$

This is Einstein's General Theory of Relatitivity = gravitation.

Since both the left and right sides of the equation are symmetric under interchange of indices means that there are 10 equations. However, the conservation laws

$$D_\mu G_{\mu\nu} = 0; \quad D_\mu T_{\mu\nu} = 0 \tag{39}$$

give four equations that are automatically satisfied. This means that there are actually 6 non-trivial second-order non-linear partial differential equations. Ultimately one must determine the metric tensor $g_{\mu\nu}$ which has 10 independent components.

$$\begin{aligned} 8\pi G_N T^{\mu}_{\mu} &= R; \\ R_{\mu\nu} &= -8\pi G_N \left(T_{\mu\nu} - \frac{1}{2}T^{\alpha}_{\alpha} g_{\mu\nu}\right). \end{aligned} \tag{40}$$

therefore in regions of space-time where no matter is present one has

$$R_{\mu\nu} = 0 \tag{41}$$

but the complete Riemann tensor $R^{\alpha}_{\beta\gamma\delta}$ will not necessarily vanish.

The Weyl tensor is defined by subtracting from $R_{\alpha\beta\delta\gamma}$ in such a way that contraction on any pair of indices gives zero.

$$C_{\alpha\beta\delta\gamma} = R_{\alpha\beta\delta\gamma} + \frac{1}{2}\left[g_{\alpha\gamma}R_{\delta\beta} - \frac{1}{3}Rg_{\alpha\delta}g_{\beta\gamma} - \text{terms with } \gamma \text{ and } \delta \text{ exchanged}\right] \tag{42}$$

This construction is such that $C_{\alpha\beta\delta\gamma}$ has the same symmetry properties and

$$C^{\mu}_{\beta\mu\gamma} = 0. \tag{43}$$

If one counts the number of independent components at a given point $\tilde{x}$ that $R_{\alpha\beta\delta\gamma}$ has 20 degrees of freedom and $R_{\mu\nu}$ and $C_{\alpha\beta\delta\gamma}$ each have 10.

Though these eqations have been derived in a way that provided a unique match to Newtonian limit, if we allow a small deviation from Newton's laws, then another term is allowed. We have another conservation law (zero covariant derivative) for the metric tensor

$$D_{\mu}g_{\mu\nu} = 0, \tag{44}$$

Thus it is possible to modify General Relativity equation for gravity to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu} \tag{45}$$

where Λ is the cosmological constant. This new term represents a renormalization

$$\delta T_{\mu\nu} \propto g_{\mu\nu} \tag{46}$$

1.6 Isometries

Tensor calculus is largely concerned with how quantities change under coordinate transformations. It is of particular interest when a quantity does not change, i.e. remains invariant, under coordinate transformations. For example, coordinate which leave a metric invariant are of importance since they contain information about the **symmetries** of the underlying Riemannian manifold. Just as in an ordinary Euclidean space, there are two sorts of transformations: **discrete** ones, like reflections, and **continuous** ones, like translations and rotations. In most applications, these latter types are the more important ones and they can in principle be obtained systematically by obtaining the so-called Killing vectors of the metric.

A metric g_{ab} is **from-invariant** or simply **invariant** under the transformation $x^a \to (x')^a$, if

$$g'_{ab}(\vec{y}) = g_{ab}(\vec{y}) \quad \text{for all coordinates} y^c, \tag{47}$$

that is, the transformed metric $g'_{asb}(\vec{x}')$ is the **same** function of its argument $\vec{x}'$ as the original metric $g_{ab}(\vec{x})$ is of its argument $\vec{x}$. Then a transformation leaving g_{ab} form-invariant is called an **isometry**. Since g_{ab} is a covariant tensor, it transforms according to the equation above, or equivalently (interchanging primes and unprimes)

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g'_{cd}(x'). \tag{48}$$

Then, using the equation from above, $x^a \to x'^a$ will be an isometry, if

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g_{cd}(x'). \tag{49}$$

Consider all quantities appearing in this equation to be functions of x using $x'^a = x'^a(x)$. In general, this condition is very complicated, but it may be greatly simplified, if we consider the special case of an **infinitesimal** coordinate transformation

$$x^a \to x'^a = x^a + \epsilon X^a(x) \tag{50}$$

where ϵ is small and arbitrary and X^a is a vector field. Differentiating gives

$$\frac{\partial x'^a}{\partial x^b} = \delta^a_b + \epsilon \partial_b X^a \tag{51}$$

Now substituting into the transformation equation and applying Taylor's theorem

$$\begin{aligned} g_{ab}(x) &= (\delta^c_a + \epsilon\partial_a X^c)(\delta^d_b + \epsilon\partial_b X^d) g_{cd}(x^e + \epsilon X^e) \\ &= (\delta^c_a + \epsilon\partial_a X^c)(\delta^d_b + \epsilon\partial_b X^d)[g_{cd}(x) + \epsilon X^e \partial_e g_{cd}(x) + \ldots] \\ &= g_{ab}(x) + \epsilon[g_{ad}\partial_d X^d + g_{bd}\partial_a X^d + X^e\partial_e g_{ab}] + O(\epsilon^2). \end{aligned} \tag{52}$$

Working to first order in ϵ and subtracting $g_{ab}(x)$ from each side, it follows that the quantity in the square brackets must vanish. This quantity is simply the Lie derivative of g_{ab} with respect to X, namely,

$$L_X g_{ab} = X^c \partial_e g_{ab} + g_{ad}\partial_b X^d + g_{bd}\partial_a X^d \tag{53}$$

Now we can replace ordinary derivative and so, the condition for an infinitesimal isometry becomes

$$L_X g_{ab} = X^c \nabla_e X_a + \nabla_a X_b = 0. \tag{54}$$

These equations are called **Killing's equations** and any solution of them is called a **Killing vector field** X^a. The metric is dragged into itself' by the vector field X^a.

Theorem: An infinitesimal isometry is generated by a Killing vector $X^a(x)$ **satisfying** $L_X g_{ab} = 0$.

It is sufficient to restrict attention to infinitesimal transformations because it is possible to build up any finite transformation with non-zero jacobian (i.e. a continuous transformation) by an integration process involving an infinite sequence of infinitesimal transformations.

2 The Schwarzschild Solution from Symmetry

2.1 Stationary Solutions

A metric will be stationary, if there exists a special coordinate system in which the metric is visibly time-independent, i.e.

$$\frac{g_{ab}}{\partial x^0} \equiv = 0, \tag{55}$$

where x^0 is a timelike coordinate. In an arbitrary coordinate system the metric will probably depend explicitly on all the coordinates; so we need to make the statement coordinate independent. Define a vector field

$$X^a \equiv \delta^a_0 \tag{56}$$

in the special coordinate system, then,

$$L_X g_{ab} = X^c g_{ab,c} + g_{ac} X^c_{,b} + g_{bc} X^c_{,a} \equiv \delta^c_0 g_{ab,c} = g_{ab,0} = 0 \tag{57}$$

$L_X g_{ab}$ is a tensor, or if it vanishes in one coordinate system, it vanishes in all coordinate systems. Hence, X^a is a **Killing vector field**. Conversely, a given **timelike** Killing vector field X^a, then there always exists a coordinate system which is **adapted** to the Killing vector field,, that in which the last equation holds, and then

$$0 = L_X g_{ab} \equiv g_{ab,0}, \tag{58}$$

and so the metric is stationary. This is a coordinate-independent definition.

A space-time is said to be stationary, if and only if, it admits a time like Killing vector field.

2.2 Hypersurface-orthogonal vector fields

To discuss static solutions in a coordinate-independent way, we need to introduce the concept of a hypersurface-orthogonal vector field. The equation of a **family** of hypersurfaces is given by

$$f(x^a) = \mu \tag{59}$$

where different members of the family correspond to different values of μ. Consider two neighboring points with coordinates x^a and $(x^a + dx^a)$, respectively, lying in one of the hypersurfaces, S.

$$\mu = f(x^a + dx^a) = f(x^a) + \frac{\partial f}{\partial x^a} dx^a \tag{60}$$

to first order. Thus

$$0 = \frac{\partial f}{\partial x^a} dx^a \tag{61}$$

evaluated at x^a. Define the **covariant vector field** n_a to the family of hypersurfaces by

$$n_a \equiv \frac{\partial f}{\partial x^a} dx^a \tag{62}$$

then becomes

$$n_a dx^a = g_{ab} n^a dx^b = 0$$

which tells us that n^a is orthogonal to the infinitesimal contravariant vector field dx^a. Since dx^a lies in S by construction, it follows that n^a is orthogonal to S and is therefore known as the **normal vector field** to S at x^a. Any other vector field X^a is said to be **hypersurface-orthogonal**, if it is everywhere orthogonal to the family of hypersurfaces, in which case it must be proportional to n^a everywhere, i.e.

$$X^a = \lambda(x) n^a$$

for some proportionality factor λ, which in general will vary from point to point.

3 Gravitational Lensing

The gravitational attraction of mass (stress-energy) deflects light. Because of this there can be multiple pathways for light to come from its source to an observer. It is possible for an intervening mass to produce multiple images of a distant source. When a mass concentration produces multiple images that is called strong *gravitational lensing*. When a mass (energy) concentration produces an image of a source that is distorted (magnified and sheared) that is generally called weak *gravitational lensing*. Gravitational lensing image can give information about the source, about the object acting as lens, and about the intervening large-scale geometry of the universe when source, lens, and observer are at cosmological distances from one another.

Realistic gravitational lenses may be clusters of distant galaxies withot any special symmetries. Light may propagate through them as well as around them. However, we first consider simple lensing by an concentrated spherical mass to illustrate the main features of lensing.

3.1 Lens Geometry and Image Position

The deflection angle α for a light ray passing by a mass M at an impact parameter $b >> R_s = 2GM/c^2$ is

$$\alpha = \frac{4GM}{c^2 b} = \frac{2R_s}{b}. \tag{63}$$

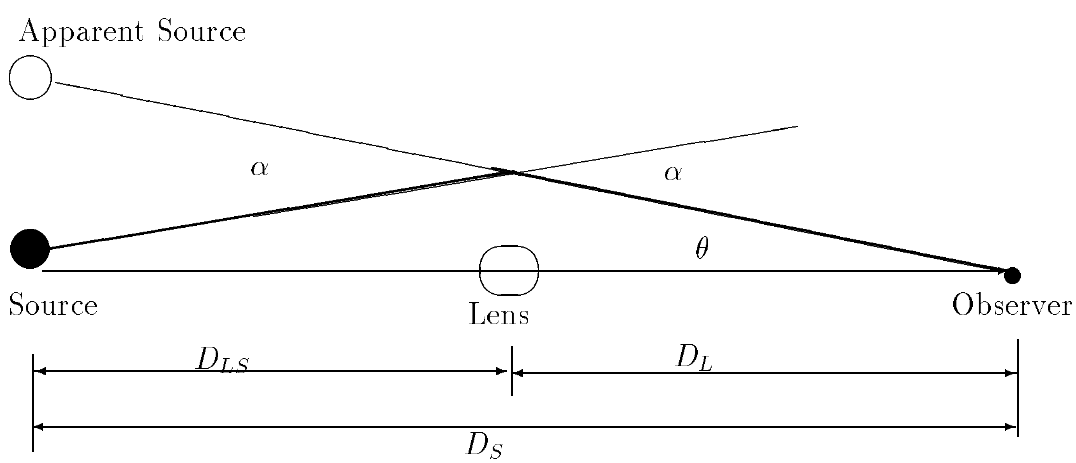
Proof of this is shown in lecture and is a homework problem.

For example and illustration, consider lensing by a stellar object, a galaxy, or a galactic cluster.

Object:	Planet	Star	Galaxy	Cluster
Mass	$0.01M_\odot$	$M_\odot$	$10^{11}M_\odot$	$10^{13}M_\odot$
R_s	0.03 km	2.95 km	0.03 lyr (10^{11} km)	3.12 lyr
Distance		10 kpc	10^9 lyr	7.2×10^{13}
		3×10^4 lyr $\sim 10^{17}$ km	3×10^{22} km	
θ_E		$\sim 10^{-3''}$	2.6×10^{-6} $0.5''$	$5''$

Table 2: Sample Parameters for Astrophysical Lenses

The following diagram explains the various quantities used in the the lensing formulae.



This figure shows the basic geometry of an astrophysical gravitational lens in the thin lens approximation. The Source is located a distance D_S from the observer and distance D_{LS} from the lens. Thus the distance from the lens to the source is

D_L. The heavy line from source to observer shows the approximate path of light bundle. The light bundle is bent through an angle α shown schematically (thin lens approximation) as a sharp bend at the distance of the lens. The apparent location extrapolating the incoming light bundle backwards appears at angle θ relative to line of sight to the lens. Not shown (I could not seem to draw it) is a straight line between the source and observer which is at an angle β to the line of sight to the lens.

The actual characteristic distance over which the bulk of the light bending occurs is of order of the Schwarzschild radius R_S which is typically much less than the other distances. Thus the then lens approximation will be fairly accurate. It is an excellent approximation that the light rays propagate as straight lines in flat space when far from the lens, and all the deflection occurs at the lens.

In realistic simulations, all the angles involved are very small, so that one can find transverse distances as angles times longitudional distance:

$$b \simeq D_L \Theta \simeq D_L \beta + D_{LS}(\alpha + \beta - \theta) \qquad \theta D_S = \beta D_S + \alpha D_{LS} \tag{64}$$

This can be rewritten as

$$\theta = \beta + \frac{\theta_E^2}{\theta} \qquad \theta_E^2 \equiv 2R_S \left(\frac{D_{LS}}{D_S D_L} \right) \tag{65}$$

θ_E is called the Einstein angle.

The solutions to these equations gives the angular locations of two images

$$\theta_\pm = \frac{1}{2} \left(\beta \pm \left(\beta^2 + 4\Theta_E^2 \right)^{1/2} \right). \tag{66}$$

The arrangement of these two images as produced by a spherical mass is shown in the accompanying figure. There are two images on opposite sides of the position of the lens. One is at a position greater than the Einstein angle and one closer than the Einstein angle. If the source and lens are directly aligned with the observer ($\beta = 0$), then the images make a complete ring at the Einstein angle. In the limit of a small but finite-sized spherical lens, there is a third image behind the lens in addition to the two solutions.

The image locations are independent of the frequency of the light by the Equivalence Principle.

By measuring the angles $\theta_\pm$ between the position of the lens and the position of the images, the Einstein angle θ_E can be determined. If the distance to the lens is estimated, then the mas of the lens is determined.

Many sources have finite size and a defined shape. Because gravitational lensing is not true lensing but the power of the bending declines with impact parameter, the images not only are magnified but also sheared. The symmetry about the observer-lens axis implies that the light ray's value of azimuth, ϕ about that axis is unchanged by the deflection of the lens. the azimuthal angular width of the image $\Delta\phi$ is thus preserved. The polar width $\Delta\theta$ is changed by an amount that can be determined by differentiating the formula for $\theta_\pm$ to find

$$\Delta\theta_\pm = \frac{1}{2}\left(1 \pm \frac{\beta}{(\beta^2 + 4\theta_E^2)^{1/2}}\right)\Delta\beta \tag{67}$$

Thus the images of something like a galaxy will be elongated and distorted. The change in total brightness is called the magnification and is directly related to how much the image area as surface brightness is conserved.

$$\begin{aligned}
\frac{I_\pm}{I_*} &= \frac{\Delta\Omega_\pm}{\Delta\Omega_*} = \left|\frac{\theta_\pm\Delta\theta_\pm\Delta\phi}{\beta\Delta\beta\Delta\phi}\right| \\
&= \left|\left(\frac{\theta_\pm}{\beta}\right)\left(\frac{d\theta_\pm}{d\beta}\right)\right| = \frac{1}{4}\left(\frac{\beta}{(\beta^2 + 4\theta_E^2)^{1/2}} + \frac{(\beta^2 + 4\theta_E^2)^{1/2}}{\beta} \pm 2\right) \\
\frac{I_{total}}{I_*} &= \frac{I_+ + I_-}{I_*} = \frac{1}{2}\left(\frac{\beta}{(\beta^2 + 4\theta_E^2)^{1/2}} + \frac{(\beta^2 + 4\theta_E^2)^{1/2}}{\beta}\right)
\end{aligned} \tag{68}$$

The total intensity is always greater than unity. The gravitational lens always enhances total brightness for small angles to the lens. If the source is close to the observer-lens axis so that β is fairly small, this enhancement can be large. There are microlensing events that show a factor of ten increasse of star light. There are cases of distant galaxies being enhanced by significant amounts.