Physics 139 Relativity
Relativity Notes 1999

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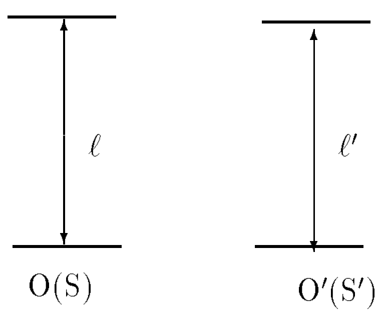
Notes to be found at
http://aether.lbl.gov/www/classes/homework/homework.html

3 Properties of Spatial and Temporal Measurements – From the Postulates

3.1 Comparison of Meter Sticks and Clocks in Relative Motion

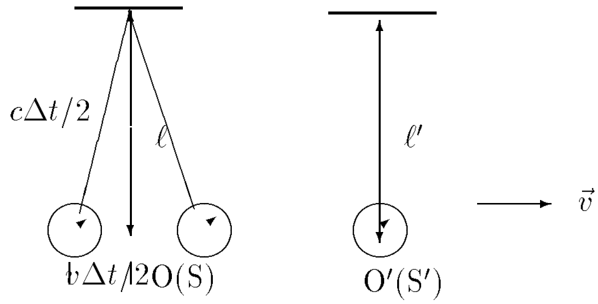
3.1.1 Meter Sticks ⊥ Motion



By the first postulate the lengths $\ell = \ell'$, since, if one were shorter , it might be absolutely at rest and the other moving with respect to it, or at least they would be distinguishable.

3.1.2 Clock Rates

Consider a clock made by counting reflections between two parallel mirrors moving perpendicularly.



For frame O:

$$\begin{aligned}
\left(\frac{c\Delta t}{2}\right)^2 &= \ell^2 + \left(\frac{v\Delta t}{2}\right)^2 \\
(\Delta t)^2\left(c^2 - v^2\right) &= 4\ell^2 \\
\Delta t &= \frac{2\ell}{\sqrt{c^2 - v^2}} \\
&= \frac{2\ell}{c}\frac{1}{\sqrt{1-(v/c)^2}}
\end{aligned} \tag{1}$$

For frame O′:

$$\Delta t' = \frac{2\ell'}{c} = \frac{2\ell}{c} \tag{2}$$

So one has

$$\Delta t = \frac{\Delta t'}{\sqrt{1-(v/c)^2}} \tag{3}$$

This is called **Time Dilation**. The time for frame O is greater than the time for O′, so that an observer in frame O claims that frame O′'s clocks are running more slowly.

This is termed Time dilation of a moving clock.

Comments:

(1) In Ether theory

$$\Delta t = \Delta t' = \frac{2\ell}{c\sqrt{1-(v/c)^2}}$$

Since both observers would agree on the actual path length through the Ether.

Same for Ether Theory with Lorentz contraction since there is no contraction perpendicular to $\vec{v}$.

(2) Emission Theory gives

$$\Delta t = \Delta t' = \frac{2\ell}{c}$$

(3) What does O′ say about the clocks in frame S?

O′ says clocks in S run slow. This is necessary but the first postulate; Do the experiment the other way and remember that the systems cannot be distinguishable.

(4) Is all this consistent?

Observer O uses two clocks, O′ uses one! O′ blames O's "wrong result" on O's clocks not being properly synchronized. (The second clock is set later.) Thus the systems are not symmetrical and identical. O′ agrees with O as to all the clock readings but explains this differently.

(5) Can the rate of a moving clock be tested experimentally?

Yes. The earliest good work was by Ives and Stilwell using "canal rays". (1928)

Doppler effect makes the result unless the observation is made perpendicular to $\vec{v}$ or one uses the average of parallel and anti-parallel light. The latter approach is better. They used Dempster's velocity selector.

$$\lambda = \frac{\lambda_0 \left(1 \pm \frac{v}{c}\right)}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{4}$$

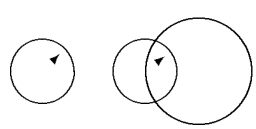
The upper term represents the Doppler effect and the lower the time dilation.

Other early measurements include: Nereson and Rossi published in Physical Review 64, 199 (1943) and the direct test with mesons by Neher and Stever published in Physical Review 58, 756 (1940).

The height was chosen so that matter traversed was the same difference in rate so decay of cosmic ray mesons in 12,000 feet.

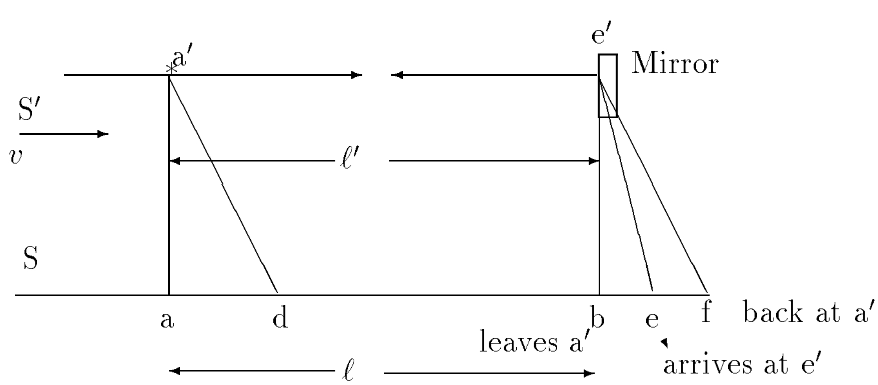
(6) The nature of clocks: Clocks may be mechanical, electrical, chemical, radioactive, biological, atomic, nuclear, etc.: **All clocks obey the same law of time dilation.**

(7) How can one compare clocks in two different systems?



Example, put one clock on a rotating wheel with velocity v and compare after each revolution. The moving clock runs slow by the $\gamma = 1/\sqrt{1 - v^2/c^2}$ factor.

3.1.3 Meter Sticks || Motion



S$'$ sends light signal to mirror and back, with time $\Delta t'$ in distance $2\ell'$. $\Delta t' = 2\ell'/c$ by second postulate.

By time dilation which is just established:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} = \frac{2\ell'}{c\sqrt{1 - v^2/c^2}} \tag{5}$$

But from the second postulate directly

$$\Delta t = \frac{\overline{ae} + \overline{ed}}{c}$$

$$\overline{ae} = \ell + \overline{be} = \ell + \overline{ae}\frac{v}{c}$$

$$\overline{ae}(1 - \frac{v}{c}) = \ell$$

$$\overline{ae} = \frac{\ell}{1 - \frac{v}{c}}$$

$$\overline{de} = \ell + \overline{be} - \overline{de} = \ell + \overline{ae}\frac{v}{c} - [\overline{ae} + \overline{de}]\frac{v}{c} = \ell - \overline{de}\frac{v}{c};$$

$$\overline{de}(1 + \frac{v}{c}) = \ell$$

$$\overline{de} = \frac{\ell}{1 + \frac{v}{c}}.$$

$$\Delta t = \frac{\ell}{c}\left[\frac{1}{1 - \frac{v}{c}} + \frac{1}{1 + \frac{v}{c}}\right]$$

$$\Delta t = \frac{2\ell}{c\,(1 - v^2/c^2)}$$

But

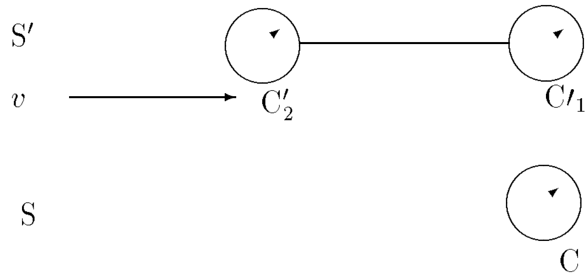
$$\Delta t = \frac{2\ell'}{c\sqrt{1 - v^2/c^2}},$$

so the equation

$$\ell = \ell'\sqrt{1 - v^2/c^2} \tag{6}$$

give the Lorentz-Fitzgerald contraction.

3.1.4 Setting of Clocks



S notes the clock readion on C and on C'_1 when C'_1 passes C **and** the readings on C and C'_2 when C'_2 passes C. By the first postulate, O and O′ agree on $|v|$.

$$t'_2 - t'_1 = \frac{\ell'}{v}, \qquad t_2 - t_1 = \frac{\ell}{v}$$

with the same v.

But $\ell = \ell'\sqrt{1 - v^2/c^2}$ (just established) and

$$t_2 - t_1 = \frac{t'_2 + \Delta t'_2 - t'_1}{\sqrt{1 - v^2/c^2}}$$

where the time $\Delta t'_2$ was just established previously in which $\Delta t'_2$ is the correction made by O to the setting by O′ of clock C'_2 in order to get clock synchronization in frame S′. ($\Delta t'_2$ turns out to be negative; O finds t'_2 to be ahead, and must subtract $|\Delta t'_2|$ from its reading.)

$$\frac{t'_2 + \Delta t'_2 - t'_1}{\sqrt{1 - v^2/c^2}} = \frac{\ell}{v} = \frac{\ell'}{v}\sqrt{1 - v^2/c^2}$$

$$\frac{t'_2 - t'_1}{\sqrt{1 - v^2/c^2}} + \frac{\Delta t'_2}{\sqrt{1 - v^2/c^2}} = \frac{\ell'}{v}\sqrt{1 - v^2/c^2}$$

$$= \frac{\ell'/v}{\sqrt{1 - v^2/c^2}} + \frac{\Delta t'_2}{\sqrt{1 - v^2/c^2}} = \frac{\ell'}{v}\sqrt{1 - v^2/c^2}$$

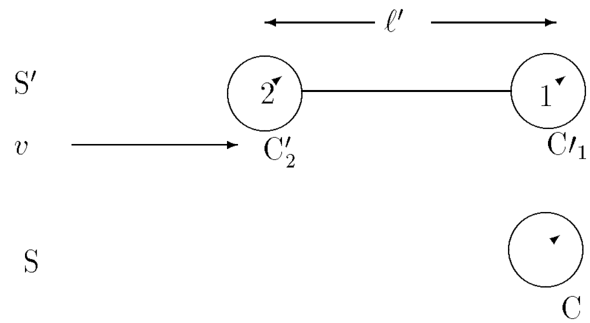
Solving for $\Delta t'_2$ yields

$$\Delta t'_2 = \frac{\ell'}{v}\left[\left(\sqrt{1 - v^2/c^2}\right)^2 - 1\right] = -\frac{\ell' v}{c^2}.$$

O says clock C'_2 is set **ahead** by $\ell' v/c^2$ in the time units used by O′.

The clock behind in space is ahead in time.

Now by the Second Postulate A new experiment:



S′ sends a beam of light from 1 to 2 and times how long it takes.

$$t_2' - t_1' = \ell'/c$$

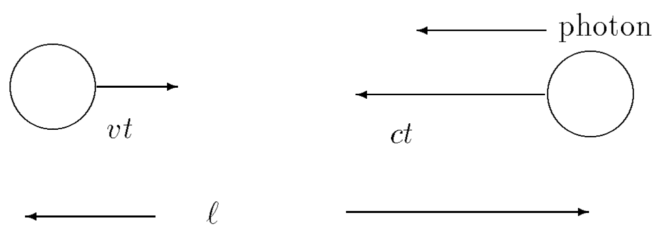
Waht does O calculate?

$$t_2 - t_1 = \frac{t_2' + \delta t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

remembering that time dilation is previously established. There is a new $\delta t_2'$ for this experiment, the correction calculated by O for C_2'.

$$t_2 - t_1 = \frac{\ell}{c+v} = \frac{\ell' \sqrt{1 - v^2/c^2}}{c+v}$$

making use of the established Lorentz contraction.



$$ct + vt = \ell$$
$$t = \frac{\ell}{c+v}$$

$$\frac{t_2' + \Delta t_2' - t_1'}{\sqrt{1 - v^2/c^2}} == \frac{\ell' \sqrt{1 - v^2/c^2}}{c+v}$$

$$\delta t_2' = \frac{\ell' \left(\sqrt{1 - v^2/c^2}\right)^2}{c+v} - (t_2' - t_1')$$

$$= \frac{\ell' \left(\sqrt{1 - v^2/c^2}\right)^2}{c+v} - \frac{\ell'}{c}$$

$$= \ell' \left[\frac{1 - v^2/c^2}{c + v} - \frac{1}{c}\right] = \ell' \frac{c - v^2/c^2 - c - v}{c(c + v)}$$

$$= -\frac{\ell' v(1 + v/c)}{c^2(1 + v/c)} = -\frac{\ell' v}{c^2}$$

Which is exactly the same as deduced from the First Postulate. O says "Clock behind in space is ahead in time."

3.1.5 Operational Explanation of Perceived Synchronization Defect

1) O′ synchronizes two identiacal clocks at the same place.

2) He carries one slowly to the rear; or both slowly away from each other and they stay synchronized as he sees it.

3) O says thaat the rear clock moves ahead in time because it runs fater while being moved back to its final position.

Rate of front clock $\equiv r_1'$ and rate of rear clock $\equiv r_2'$ as seen by O.

$$r_1' = r_o \sqrt{1 - v^2/c^2}$$

$$r_2' = r_o \sqrt{1 - (v - \Delta v)^2/c^2}$$

$$\Delta r' \simeq r_o \{\left[1 - (v - \delta v)^2/(2c)\right] - \left[1 - v^2/(2c^2)\right]\}$$

$$= r_o \left[1 - v^2/c^2 + v\Delta v/c^2 - 1 + v^2/c^2\right]$$

$$\Delta r' = r_o \ \frac{v \Delta v}{c^2}$$

Distance moved is $\ell' = \Delta V \Delta t'$ where $\Delta t'$ is the time to move the clock.

$$\Delta r' = r_o \ \frac{v \ell'}{c^2 \Delta t'}$$

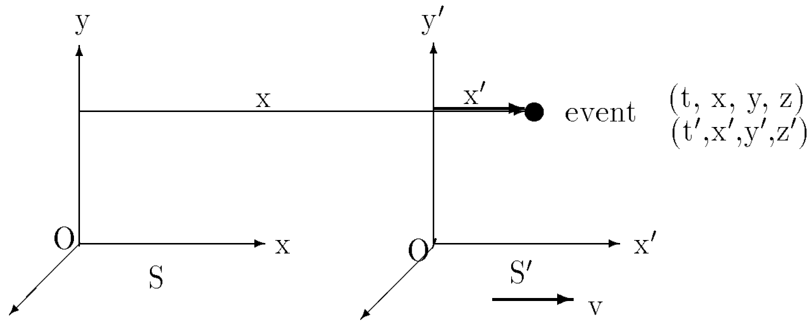
$\Delta r' \Delta t'$ is he synchronization "error" which is $\delta t' = \ell' f / c^2$. The rear clock is ahead in time.

CONCLUSION One must abandon the notion of simultaneity of time for observers in relative motion.

4 The Lorentz Transformation

The Lorentz transformations are demanded and supported by experimental observations. The Lorentz transformation equations can readily be derived from length contraction and time dilation after taking a short detour to dicuss clock synchronization.

Consider two frames of reference: S, the laboratory frame and S′ a frame of reference moving vith velocity $\vec{v}$ in the $\hat{x}$ direction as shown in the following figure.



Arrange things so that at $t = 0$ and $t' = 0$ that the two origins O and O′ coincide.

Consider each reference system to a an actual lattice of meter sticks and clocks, e.g. each reference system is filled with these space and time measuring devices at every point.

Get clocks in S to agree. Identical clocks set by sending out a light pulse from origin O and also from the midpoint between any two clocks. Check times, reflect back to midpoint, if pulses arive together, then clocks agree.

System S′ dos the same with his clocks.

We have for the space-time event in the figure above

$$x = vt + x' \times \sqrt{1 - v^2/c^2} \tag{7}$$

where the second term takes into account length contraction of a moving frame. We can use our arguments about the transverse directions to show that they are unchanged and then have the spatial Lorentz transformations:

$$\begin{aligned} x' &= \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt) \\ y' &= y \\ z' &= z \\ t' &= t\sqrt{1 - v^2/c^2} + \text{sychronization effect} \end{aligned} \tag{8}$$

4.1 Synchronizing Clocks in Moving Frame

Our approach to get our system of reference made of a grid of meter sticks and synchronized clocks requires that we synchronized the clocks. An approach to synchronizing the clocks is: bring the clock together, match their readings, then move into place. Move them slowly and gently so as not to disturb their operation.

Consider the simple case of two clocks brought together at the origin of the moving system S′. When they are together, from the laboratory frame S both clocks read same time and are going slow by a factor $\sqrt{1 - (v/c)^2}$ as a result of

time dilation. Now very slowly and genterly move one clock back (in negative x'-direction; toward the laboratory system origin) a distance ℓ in elapsed time $\ell = \delta v\ \tau$. The clock at the origin has its rae slo by $\sqrt{1 - v^2/c^2}$ relative to the laboratory frame. Clock moving back in negative x'-direction has its rate slowed by the factor $\sqrt{1 - (v - \delta v)^2/c^2}$

$$f_A = \sqrt{1 - v^2/c^2} f_o \qquad f_B = \sqrt{1 - (v - \delta v)^2/c^2} f_o \tag{9}$$

The difference in the clocks' rates is

$$\begin{aligned} f_A - f_B &= f_o \left[\sqrt{1 - v^2/c^2} - \sqrt{1 - (v - \delta v)^2/c^2} \right] \\ &= \frac{f_o}{\sqrt{1 - v^2/c^2}} \left[1 - \frac{v^2}{c^2} - \left((1 - \frac{v^2}{c^2})(1 - \frac{(v - \delta v)^2}{c^2} \right)^{1/2} \right] \\ &= \frac{f_o}{\sqrt{1 - v^2/c^2}} \left[1 - \frac{v^2}{c^2} - \left((1 - \frac{v^2}{c^2})(1 - \frac{v^2}{c^2} + \frac{2\delta v v}{c^2} - \frac{\delta v^2}{c^2}) \right)^{1/2} \right] \\ &= -f_a \frac{\delta v\ v/c^2}{1 - v^2/c^2} = -f_o \frac{\delta v\ v/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned} \tag{10}$$

If it takes a time $\tau = \ell_o / \delta v$ to separate the clocks, the time difference between them is

$$\begin{aligned} \Delta t &= \frac{f_A - f_B}{f_A} \tau = \left(\sqrt{1 - v^2/c^2} - \sqrt{1 - (v - \delta v)^2/c^2} \right) \tau \\ &= -\frac{\delta v\ v/c^2}{\sqrt{1 - v^2/c^2}} \times \frac{\ell_o}{\delta v} = -\frac{\ell_0\ v/c^2}{\sqrt{1 - v^2/c^2}} = -\ell' v/c^2 \end{aligned} \tag{11}$$

Note that the speed with which the clock moves drops out and the change in reading is proportional only to the distance displaced and the velocity of the moving system.

Clocks get out of synchronization (phase) by an amount proportional to their separation ℓ_o and v. If brought back together, the clocks will go into synchronization. **The clock that is farther behind in space is further ahead in time.**

Note that in the frame S$'$ the difference in rate of time kept between the clock at the origin and the one being moved back to its place is second order in v/c rather than first order:

$$f'_B = \frac{f'_A}{\sqrt{1 - (\delta v)^2/c^2}} \simeq f'_A \times \left(1 - \frac{1}{2} \frac{v^2}{c^2} \right)$$

So that by moving with a very, very slow velocity the integrated effect in the S$'$ frame can be made arbitrarily small while the effect as observed in the S frame is always $-\ell' v/c^2$ independent of δv. That is because the effect in frame S is first order in $\delta v/c$ and integrated over time equals the displacement.

The final Lorentz transformations are:

$$
\begin{aligned}
t' &= t\sqrt{1-v^2/c^2} - \frac{x'v}{c^2} = t\sqrt{1-v^2/c^2} - \frac{v^2}{c^2}\frac{1}{\sqrt{1-v^2/c^2}}(x-vt) \\
&= \frac{1}{\sqrt{1-v^2/c^2}}\left[t - \frac{vx}{c^2}\right] = \gamma\left[t - \frac{vx}{c^2}\right] \qquad (12)
\end{aligned}
$$

Notice for $v << c$ get Galilean tranforms and there is also a symmetry between the transformation equations.

$$
\begin{array}{cc}
t' = \gamma(t - vx/c^2) & t = \gamma(t' + vx'/c^2) \\
x' = \gamma(x - vt) & x = \gamma(x' + vt') \\
y' = y & y = y' \\
z' = z & z = z'
\end{array} \qquad (13)
$$

4.1.1 Remarks on Lorentz Transformation

History: Lorentz transformation was derived by Lorentz before Einstiein's work. Lorentz obtained them by considering invariance of Maxwell's Equations.

Significance: They define the mathematical specification required to discus a kinematic occurrence – a sequence of space-time events.

Agreement with First Postulate: If one does the inversion, one obtains the same equations. Try replacing v by $-v$.

.....

Agreement with Second Postulate:

In S, light is described by

$$
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2 \qquad (14)
$$

This is equivalent to

$$
dx^2 + dy^2 + dz^2 - c^2 dt^2 = 0 \qquad (15)
$$

Substitute in the Lorentz transformations

$$
\frac{(dx' + v dt')^2}{\left(\sqrt{1-v^2/c^2}\right)^2} + dy'^2 + dz'^2 - c^2 \frac{\left(dt' + \frac{v}{c}dx'\right)^2}{\left(\sqrt{1-v^2/c^2}\right)^2}
$$

$$
= \frac{1}{\left(\sqrt{1-v^2/c^2}\right)^2}\left[dx'^2 + 2v dx' dt' + v^2 dt'^2 - \frac{v^2}{c^2}dx'^2 - 2v dx' dt' - c^2 dt'^2\right] + dy'^2 + dz'^2
$$

$$
= dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = 0
$$

Group Property of Lorentz Transformations in a Line

Idenity exists: $v = 0$
Inverse exists: $v \to -v$
Transitive: $\mathrm{S} \to \mathrm{S}' \to \mathrm{S}'' \equiv \mathrm{S} \to \mathrm{S}''$
Exhibit as an Exercise?
It is true that all Lorentz Transformations also form a group.

4.2 Composition of Velocities

In Galilean relativity one simply adds velocities when changing frames of reference. Velocity composition is slightly more complicated in Special Relativity. We can readily derive the velocity compostion formulae from the Lorentz transformation.

$$dt = \frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1 - v^2/c^2}}; \qquad \frac{dt}{dt'} = \frac{1 + \frac{v}{c^2}\frac{dx'}{dt'}}{\sqrt{1 - v^2/c^2}} = \frac{1 + vu_x/c^2}{\sqrt{1 - v^2/c^2}}$$

$$\frac{dt}{dt'} = \frac{1 + u'_x \frac{v}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$u_x = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt'\sqrt{1 - v^2/c^2}}\frac{dt'}{dt} = \frac{\frac{dx'}{dt'} + v}{\sqrt{1 - v^2/c^2}}\frac{dt'}{dt}$$

$$u_x = \frac{u'_x + v}{\sqrt{1 - v^2/c^2}}\frac{\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}$$

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \tag{16}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{dt} = \frac{dy'}{dt'}\frac{dt'}{dt}$$

$$u_y = u'_y \frac{\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}} \tag{17}$$

$$u_z = u'_z \frac{\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}} \tag{18}$$

These three equations are the **Einstein Velocity Addition Law**.
Velocities do not add like vectors!

There are other important quantities for which transformation equations are needed. That is to say that they do not transform like vectors. Can work them out as exercises, e.g. transformation of acceleration and force. We will discuss these more later.

$$a_x - \frac{du_x}{dt}; \qquad a'_x - \frac{du'_x}{dt'}; \qquad \text{etc.}$$

Answers:

$$a_x = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x \tag{19}$$

$$a_y = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^2} a'_y - \frac{\frac{u'_y v}{c^2}\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x \tag{20}$$

$$a_z = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^2} a'_z - \frac{\frac{u'_z v}{c^2}\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x \tag{21}$$

Note as hint that

$$\sqrt{1 - u^2/c^2} = \sqrt{1 - (u')^2/c^2}\,\frac{\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}$$

and

$$u^2 = u_x^2 + u_y^2 + u_z^2$$

4.2.1 Transformation of the Lorentz Factor γ

Now that we have the composition of velocities or Lorentz transformation of velocities, we can find the transformation of the Lorentz Factor γ and/or $\sqrt{1 - v^2/c^2}$.

First consider $\sqrt{1 - u^2/c^2}$ where u is the speed of the particle in frame S and u' is the speed of the particle in frame S$'$ and the frames have relative velocity V.

$$u^2 = u_x^2 + u_y^2 + u_z^2 = \left(\frac{u'_x + V}{1 + u'_x V/c^2}\right)^2 + \left(\frac{u'_y\sqrt{1 - V^2/c^2}}{1 + u'_x V/c^2}\right)^2 + \left(\frac{u'_z\sqrt{1 - V^2/c^2}}{1 + u'_x V/c^2}\right)^2$$

so that

$$1 - \frac{u^2}{c^2} = 1 - \frac{(u'_x + V)^2 + \left(u'^2_y + u'^2_z\right)(1 - V^2/c^2)}{c^2\left(1 + u'_x V/c^2\right)^2}$$

$$= \frac{c^2 + 2u'_x V + u'^2_x V^2/c^2 - u'^2_x - 2u'_x V - V^2 - \left(u'^2_y + u'^2_z\right)(1 - V^2/c^2)}{c^2\left(1 + u'_x V/c^2\right)^2}$$

$$= \frac{c^2 - V^2 - \left(u'^2_x + u'^2_y + u'^2_z\right)(1 - V^2/c^2)}{c^2\left(1 + u'_x V/c^2\right)^2}$$

$$= \frac{(1 - V^2/c^2)\,(1 - u'^2)}{\left(1 + u'_x V/c^2\right)^2}$$

Taking the square root yields

$$\sqrt{1 - u^2/c^2} = \frac{\sqrt{1 - V^2/c^2}\sqrt{1 - u'^2/c^2}}{1 + u'_x V/c^2} \tag{22}$$

And since $\gamma = 1/\sqrt{1 - u^2/c^2}$ we have

$$\gamma_p = \left(1 + u'_x V/c^2\right) \gamma_f \gamma'_p \tag{23}$$

where γ_p and γ'_p are the Lorentz γ of the particle in the S and S′ frames, respectively, and $\gamma_f = 1/\sqrt{1 - V^2/c^2}$ is the Lorentz γ of one frame relative to the other.

We will use these transformations again later.

4.2.2 Velocity of Light as Maximum

The velocity addition law indicates that the velocity of light is the maximum velocity attainable by a material object. (Hence the origin of the t-shirt with Einstein in policeman's cap saying Speed Limit: 186,000 miles/sec! It's not just a good idea; it's the law.)

The velocity addition law for motion in the x-direction is

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$$

If $v = u'_x = c/2$,

$$u_x = \frac{c/2 + c/2}{1 + 1/4} = \frac{4}{5}.$$

If $v = u'_x = c$,

$$u_x = \frac{c + c}{1 + 1} = c!!$$

That is adding together two velocities that are very near the speed of light only gets one closer to the speed of light; one cannot keep adding velocities and exceed the speed of light.

Exercise: Show that if one has a particle moving at ϵc slower than c ($u' = (1 - \epsilon)c$ in the fram S′ moving at speed $v = (1 - \delta)c$ just less than the speed of light in the same direction, the velocity observed in frame S is just a little less than c.

Solution: Once can use the compostion of velocities formula

$$u = \frac{v + u'_x}{1 + u_x v/c^2} = \frac{(1 - \epsilon + 1 - \delta)c}{1 + (1 - \epsilon)(1 - \delta)} = \frac{2 - \epsilon - \delta}{2 - \epsilon - \delta + \epsilon\delta} c = \frac{c}{1 + \frac{\epsilon\delta}{2 - \epsilon - \delta}} \simeq (1 - \epsilon\delta/2)c$$

4.2.3 Velocity of a Causal Impulse

In Frame S: (From the point of view of observer O in frame S)

$$\Delta t = t_2 - t_1 = \frac{x_2 - x_1}{u}; \qquad u = \frac{x_2 - x_1}{t_2 - t_1}$$

In Frame S′: (From the point of view of observer O′ in frame S′)

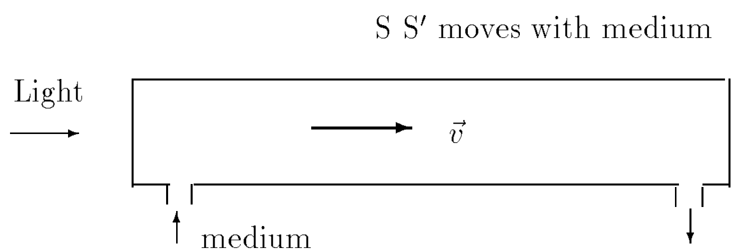
$$\Delta t' = t_2' - t_1' = \frac{1}{\sqrt{1 - v^2/c^2}}\left[t_2 - \frac{x_2 v}{c^2} - t_1 + \frac{x_1 v}{c^2}\right]$$

$$= \frac{t_2 - t_1}{\sqrt{1 - v^2/c^2}}\left[1 - \frac{v}{c^2}\left(\frac{x_2 - x_1}{t_2 - t_1}\right)\right] = \frac{1 - uv/c^2}{\sqrt{1 - v^2/c^2}}\Delta t.$$

Now, if the causal impulse velocity u is greater than c (the speed of light), one can choose v to make $\Delta t'$ negative! Effect precedes cause! This is impossible, if we are to keep causality, so the maximum velocity of a causal impulse is c.

(This is of course the group velocity - with which signals can be sent. Phase velocities may have any value!)

4.2.4 Velocity of Light in a Moving Medium



$$u' = \frac{c}{n}; \qquad n = \text{index of refraction} \tag{24}$$

$$u = \frac{u' + v}{1 + u'v/c^2} = \frac{c/n + v}{1 + cv/(nc^2)} \simeq \left(\frac{c}{n} + v\right)\left(1 - \frac{v}{nc}\right).$$

$$u \simeq \frac{c}{n} + v - \frac{c}{n^2}\frac{v}{c} - \frac{v^2}{nc}$$

$$\frac{u}{c} \simeq \frac{1}{n} + \frac{v}{c} - \frac{v}{n^2 c} - \frac{1}{n}\frac{v^2}{c^2} \simeq \frac{1}{n} + \left(1 - \frac{1}{n^2}\right)\frac{v}{c}$$

$$u = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right) v \tag{25}$$

This is exactly Fresnel's drag coefficient from 1818.

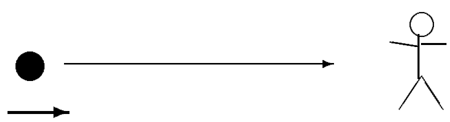
Note that the effect is a little more complicated when disersion (index of refraction n depends on wavelength/frequency) is taken into account because of the Doppler shift (see next section). The speed c_m of light in a moving medium is equal to

$$c_m = \frac{c}{n} + k v_m \tag{26}$$

where v_m is the speed of the medium and

$$k = 1 - \frac{1}{n(\lambda)^2} - \frac{\lambda}{n(\lambda)}\frac{dn(\lambda)}{d\lambda}$$

4.2.5 Doppler Effect



A stationary observer sees light from a distant source, e.g. a star, The observer sees the light with period P

$$P = P_o \frac{(1 \mp u/c)}{\sqrt{1 - u^2/c^2}} \tag{27}$$

And wavelength λ:

$$\lambda = \lambda_o \frac{(1 \pm u/c)}{\sqrt{1 - u^2/c^2}}. \tag{28}$$

One approach to this result is

$$\lambda = \lambda_o \frac{(1 \pm u/c)}{\sqrt{(1 - u/c)(1 + u/c)}} = \sqrt{\frac{1 + u/c}{1 - u/c}}$$

Remember that $\lambda f = c$ or $\lambda / P = c$.

For the Ether Theory:

$$\lambda = \lambda_o \{ \begin{array}{ll} (1 - u/c) & \text{for moving source} \\ 1/(1 + u/c) & \text{for moving observer} \end{array}$$

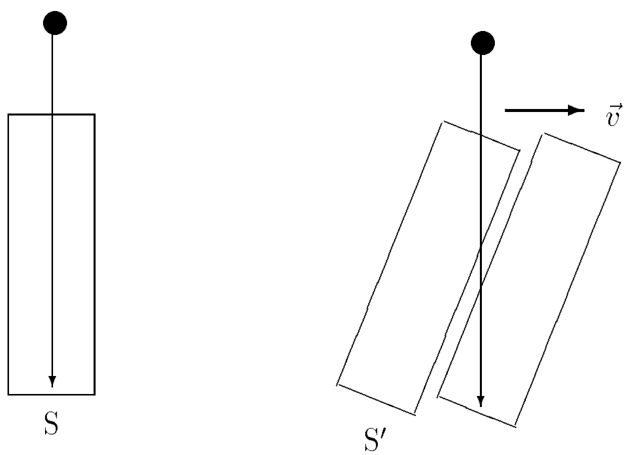
Where u is positive for approach.

The Special Relativity result is the geometric mean of these:

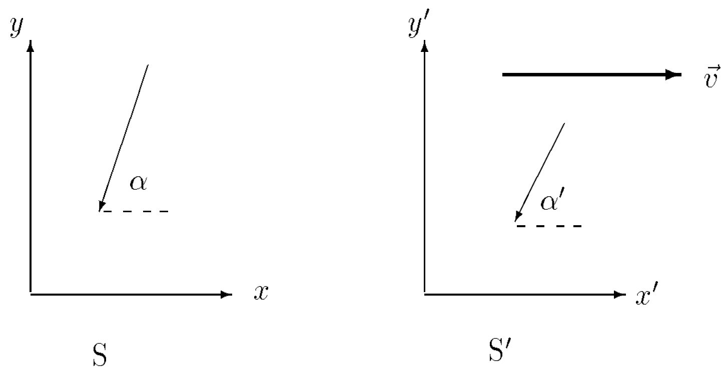
$$\lambda = \lambda_0 \sqrt{\frac{1 - u/c}{1 + u/c}} \tag{29}$$

4.2.6 Aberration of Starlight

First consider light coming from a star perpendicular to the direction of motion of the telescope.



Then consider more general directions.



$$u_x = -c cos\alpha \quad u'_x = -c cos\alpha'$$
$$u_y = -c sin\alpha \quad u'_y = -c sin\alpha'$$

Now apply Einstein velocity addition:

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}$$

$$cos\alpha' = \frac{cos\alpha + v/c}{1 + \frac{v}{c} cos\alpha}$$

$$v'_y = u_y \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$$

$$sin\alpha' = sin\alpha \frac{\sqrt{1 - v^2/c^2}}{1 + \frac{v}{c} cos\alpha}$$

It is easy to check that $sin^2\alpha' + cos^2\alpha' = 1$.

In the simple case $\alpha = \pi/2$ so $cos\alpha = 0$,

$$cos\alpha' = \frac{v}{c}$$

which is the Bradley result.

In the general case, use the trigometric identity

$$tan\frac{\theta}{2} = \frac{sin\theta}{1 + cos\theta}$$

$$tan\frac{\alpha'}{2} = \frac{sin\alpha'}{1 + cos\alpha'} = sin\alpha \frac{\sqrt{1 - v^2/c^2}}{\left(1 + \frac{v}{c} cos\alpha\right)\left[1 + \frac{cos\alpha + v/c}{1 + (v/c)cos\alpha}\right]}$$

$$tan\frac{\alpha'}{2} = \sqrt{\frac{1 - v/c}{1 + v/c}} tan\frac{\alpha}{2} \tag{30}$$

For outgoing rays, $c \rightarrow -c$.

5 Einstein's Special Relativity

It is straight-forward to show that from Einstein's postulates one also obtains the Lorentz transformations.

Two Postulates

1. No physical experiment (without reference to outside) can determine the absolute speed of the frame of reference.

2. The speed of light is independent of the speed of the source (or observer).

Consider an expanding sphere of light

$$c^2t^2 - x^2 + y^2 + z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

viewed by two inertial frames of reference (S and S′) by observers O and O′ respectively with origins coinciding – at $t = t'^2 = 0$, $x = x'^2 = 0$, $y = y'^2 = 0$, $z = z'^2 = 0$.

By simple argument one can see that lengths transverse to the direction of motion must be unchanged only x and t will be modified. One argument is the one made before about considering two identical cyclinders aligned with each other and their axes parallel to the direction of motion $\vec{v}$. If the dimension perpendicular to the direction of motion changes, then one cyclinder will grow or shrink relative to the other and could pass through the other. If one then swithes to the other frame, the opposite should happen or one can determine the absolute velocity. One would be able to tell which one went inside and which outside.

Or consider the earlier discussion of two meter sticks aligned perpendicular to the direction of motion. When the two meter sticks pass by each other one can use them to measure each other and tell which is longer and thus establish the absolute velocity.

Thus by symmetry and logic using postulate (1) we have

$$y = y', \qquad z = z'$$

so that the equation of expanding light sphere reduces to

$$c^2t^2 = c^2t'^2 - x'^2$$

If we accept the second postulate and assume coordinate transformations are linear and homogeneous we have

$$x' = Ax + Bt$$

$$t' = Cx + Dt$$

Now consider special cases:
(1) O′ origin has $x' = 0$, which implies $x = -\frac{B}{A}t$. Since velocity of O′ relative to O is v, so that $v = -\frac{B}{A}$ which yields $B = -Av$.
(2) The origin of O has $x = 0$, which gives $x' = Bt$, $t' = Dt$ implying $x' = \frac{B}{D}t$ or $B = -Dv$.

Combining (1) and (2) yields $D = A$. The linear transformation simplifies to

$$x' = A(x - vt)$$

$$t' = Cx + At$$

(3) Putting this back into the expanding light sphere formula

$$c^2t^2 = c^2t'^2 - x'^2 c^2 [Cx + At]^2 - [A(x - vt)]^2$$

$$= c^2C^2x^2 + 2c^2CAxt + c^2A^2t^2 - A^2x^2 + 2A^2vx - A^2c^2t^2$$

$$= A^2\left(1 - v^2/c^2\right)c^2t^2 + 2c^2A\left(C + \frac{v}{c^2}A\right)xt - \left(A^2 - c^2C^2\right)x^2$$

We can conclude that $A^2(1 - v^2/c^2) = 1$ or $A = 1/\sqrt{1 - v^2/c^2}$ and $A^2 - c^2C^2 = 0$ so that

$$C = -vA/c^2 = -\frac{v}{c^2}\frac{1}{\sqrt{1 - v^2/c^2}}$$

and thus

$$A^2 - c^2C^2 = \frac{1}{1-v^2/c^2} - \frac{v^2}{c^2}\frac{1}{1-v^2/c^2} = 1$$

This gives us the Lorentz transformation

$$\begin{array}{cc} t' = \gamma(t - vx/c^2) & t = \gamma(t' + vx'/c^2) \\ x' = \gamma(x - vt) & x = \gamma(x' + vt') \\ y' = y & y = y' \\ z' = z & z = z' \end{array} \tag{31}$$

Thus we have the indentical Lorentz transformations from simple logical deduction. We can construct the full theory of Special Relativity by using these postulates and a series of thought ("gendanken") experimental. This approach is quite elegant and intellectually pleasing and makes a very nice and tight exposition and thus coherent little books. However, here we are emphasizing both the experimental basis and applications and the importance of understanding relativity from more than one point of view.

In the next section we rederive the Lorentz transformations using the Minkowski geometrical view and the Poincare relativity principle (Einstein's postulate (1) but with a wider implication).

6 Minkowski Space-Time

The Minkowski (1908-1909) geometrical interpretation of Special Relativity is quite a technically powerful approach. The primary step is to assume that our world is described by a 3+1 dimension space-time continuum. There are four dimensions and space is Euclidean but the addition of time to be the fourth dimension makes space pseudo-Euclidean because the metric which defines distance has a different sign between time and space: There are two possible signatures for the signs: $\pm, \mp, \mp, \mp$ yielding the two possible metric equations:
The proper time convention:

$$(cd\tau)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{32}$$

The proper distance convention:

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \tag{33}$$

For most of this course and notes I use the proper time (first) convention since it has a postive value for the physical objects we consider.

Note that such a space is intrinsically different from a 4-D space with signature $+, +, +, +$. It is conceptually confusing to smooth this over by replacing ct by ict or just $i\tau = x_4$. Even if this is done for the stated reason that most people know the sine and cosine better than sinh and cosh.

6.1 Comments on 4-D Geometry for S.R.

The Minkowski metric and 4-D geometry makes quite an inpact on how one can approach problems in Special Relativity.

Importance

1) Assists in developing the needed space-time intuitions
2) Avoids always singling out a particular axis ($x \parallel v_{\text{relative}}$).
3) 4-D language is suggestive and seldom misleading. e.g. ict is avoided! and it is more likely to account for all coordinates in appropriate frame.
4) 4-D vectors and invariants are powerful tools.
5) Is an essential approach of geometrical General Relativity.

With 4 axes one needs 4 numbers to specify an "event" in space-time. But directions are not equivalent. A meter stick can be rotated to measure y or z instead of x, but it cannot be rotated into a clock.

6.2 Invariant Interval

The (Minkowski) geometry of space-time is constructed so that the interval: $dx^2 + dy^2 + dz^2 - c^2dt^2$ is **invariant** under a Lorentz transformation. And the signature is invariant under all real transformations of coordinates.

In more general form the signature is written as a bilinear transfomation or a matrix:

$$(ds)^2 = \sum_{\mu\nu} \eta_{\mu\nu} (dx_\mu)(dx_\nu) \tag{34}$$

where the Minkowski metric term $\eta_{\mu\nu}$ can be expressly written as

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{35}$$

Because the determinant of the signature is not equal to one but is -1, there are three different kinds of intervals:

(1) Space-like: Two space-time events separated such that

$$\Delta x^2 + \Delta y^2 + \Delta z^2 > c^2 \Delta t^2$$

$$\Delta s^2 > 0 \qquad \Delta \tau^2 < 0 \tag{36}$$

One can always find a Lorentz transformation to proper coordinates in which $\Delta t^2 = 0$. That means that one can find an inertial coordinate system in which two events which have a space-like interval happen simultaneously.

(2) Time-like: In this case events are separated such that $\Delta\tau^2 > 0$ (or $\Delta s^2 < 0$) because $c^2\Delta t^2 > \Delta x^2 + \Delta y^2 + \Delta z^2$. One can always find a Lorentz transformation to proper coordinates in which $\Delta x^2 + \Delta y^2 + \Delta z^2 = 0$

(3) Singular: In this case events separated such that $\Delta s^2 = \Delta\tau^2 = 0$ These events lie on the light cone, such as a light ray in vacuum.

The result is that space-liek intervals can always be measured with a meter stick and time-like with a clock.

6.3 What leaves Δs^2 invariant?

(1) Moving origin in space. (translation in space) E.g.

$$\begin{aligned} x' &= x + x_o \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{37}$$

(2) Re-setting zero time (translation in time) x, y, z remain the same and $t' = t + t_o$.
(3) Rotation of spatial axes, E.g.

$$\begin{aligned} x' &= xcos\theta + ysin\theta \\ y' &= -xsin\theta + ycos\theta \\ z' &= z \\ t' &= t \end{aligned} \tag{38}$$

(4) Lorentz Transformation:

$$\begin{aligned} x' &= \gamma\,(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - xv/c^2\right) \end{aligned} \tag{39}$$

Which is equivalent to

$$\begin{aligned} x' &= xcosh(\phi) + ctsinh(\phi) \\ y' &= y \\ z' &= z \\ ct' &= -xsinh(\phi) + ctcosh(\phi) \end{aligned} \tag{40}$$

where $cosh(\phi) = \gamma \equiv 1/\sqrt{1 - v^2/c^2}$.

This is a Lorentz rotation of axes. It can be considerred an imaginary rotation in the $x - t$ plane. Remember the hyperbolic trigonetry idenity/defintion.

$$cosh^2(\phi) - sinh^2(\phi) = 1$$

Consider $cosh(\phi) = cosh(i\phi), \quad isinh(\phi) = sin(i\phi)$ which gives

$$x' = xcos(i\phi) + ictsinh(\phi)$$

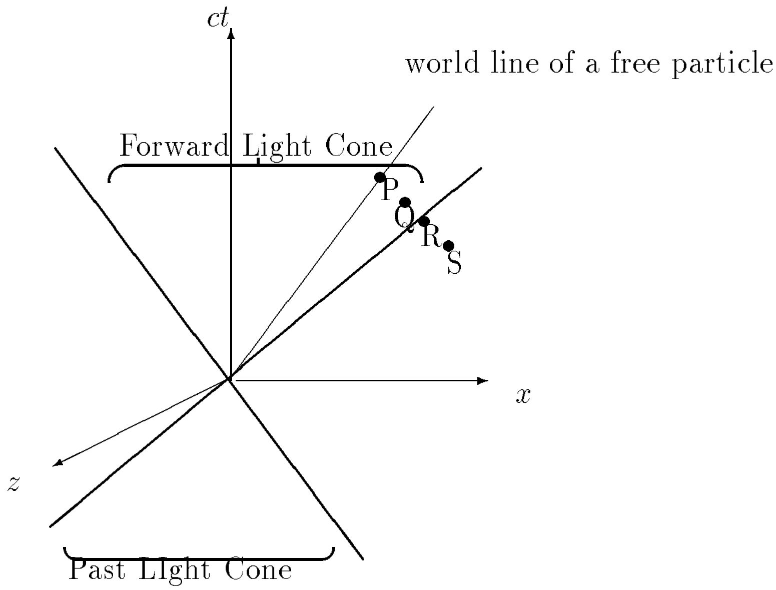
$$
\begin{aligned}
y' &= y \\
z' &= z \\
ict' &= -x sin(i\phi) + ict cos(i\phi)
\end{aligned}
\tag{41}
$$

Consider a space-like interval

$$|dx| > |cdt|$$

Then $dx' = \gamma\,(dx - vdt)\,, \qquad cdt' = \gamma\left(cdt - \frac{v}{c}dx\right)$ and one can always find a value of v/c with $|v/c| < 1$ for which $cdt' = 0$. Its magnitude is $|v/c| = |cdt/dx| < 1$. A similar argument works for time-like intervals.

In the following picture, OQ is time-like, OS is space-like, and OR is singular.



By a Lorentz transformation we may:

(1) Move Q to the t' axis.

or

(2) Move S to the x' axis.

But R will always be on a line of slope 1 in any S′

Q may be on the particle's world line, then we may find a frame in which x' stays zero, which is called the rest frame of the particle. In this frame clocks at reast measure the particle's proper time, $d\tau$. In other frames $dt = d\tau/\sqrt{1 - v^2/c^2} = \gamma\ d\tau$. $\overline{OQ}$ may be ameter stick. One can find S′ so that it lies on the x' axis and $x' = 0$.

Consider particles to be pieces of the stick. They are laid out in S′ to measure proper length λ. For other frames, $\ell' = \lambda\sqrt{1 - v^2/c^2} = \lambda/\gamma$. with the x direction of v and stick on the x axis.

For particles not free, that is with forces on them, we have the instantaneous rest frame.

6.4 Derivation of Lorentz Transformations

The Lorentz transformations result automatically from the metric and the assumption that in all inertial systems proper distances or times are invariants. That is any observer in any inertial system will calculate the same proper distance between two space-time events.

We start with two postulates:

(1) **Poincare' Relativity:** The Laws of Physics are the same in all inertial frames.

(2) **Minkowski Geometry/Metric** Space-time is a continuum in 3+1 dimensions with metric

$$(cd\tau)^2 = -(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

where τ is the proper time and s is the proper distance. Proper time is invariant for all intertial systems.

Immediately we get time dilation

$$(cd\tau)^2 = (dt)^2 \left[c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 \right]$$

$$(d\tau)^2 = (dt)^2 \left[1 - \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2} \right] = (dt)^2 \left[1 - \frac{v^2}{c^2} \right]$$

$$d\tau = dt\ \sqrt{1 - v^2/c^2}$$

If proper time is invariant,the we can show Lorentz transformation is linear.

$$(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

$$= (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$$

the second equation defines a numbering system for coordinates. But this same sum in the primed coordinate system must give the same proper time.

$$= (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$

The conversion from one coordinate system to another

$$dx'_\alpha = \sum_\beta \frac{\partial x'_\alpha}{\partial x_b eta} dx_\beta \equiv \frac{\partial x'_\alpha}{\partial x_b eta} dx_\beta$$

where the second right hand side defines the Einstein summation convention that a repeated index (in this case β) mean summation on that index. The Greek symbol index sums over four (4-D) going 0, 1, 2, 3 and Roman letters sum over three spatial coordinates going 1, 2, 3.

$$c^2 d\tau^2 = \sum_\alpha dx_\alpha^2 = \sum_\beta \sum_\alpha \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial x'_\alpha}{\partial x_\delta} dx_\beta dx_\delta$$

$$= \sum_\alpha dx_\alpha^2 = \sum_\beta \sum_\alpha \delta_{\beta\alpha} dx_\beta dx_\delta$$

Therefore

$$\frac{\partial x'_\alpha}{\partial x_\beta}\frac{\partial x'_\alpha}{\partial x_\delta} = \delta_{\beta\alpha}$$

implying if one takes the derivative: $\frac{\partial}{\partial x_\epsilon}$ one finds

$$\frac{\partial^2 x'_\alpha}{\partial x_\beta \partial x_\epsilon}\frac{\partial x'_\alpha}{\partial x_\delta} + \frac{\partial x'_\alpha}{\partial x_\beta}\frac{\partial^2 x'_\alpha}{\partial x_\delta \partial x_\epsilon} = 0$$

Now one can then shift through the indicies: $\epsilon \to \beta \to \delta \to epsilon$ and get generically

$$\frac{\partial^2 x'}{\partial x \partial x}\frac{\partial x'}{\partial x} = 0$$

and the determinant of $\partial x'/\partial x = \pm 1$ which implies

$$\frac{\partial^2 x'}{\partial x \partial x} = 0$$

and

$$x'_\alpha = A_\alpha + \sum_\beta A_{\alpha\beta} x_\beta$$

Which shows that the coordinate (Lorentz transformation) must be linear to preserve invariant the proper distance and time. Thus

$$dx'_\alpha = \sum_\beta A_{\alpha\beta} x_\beta$$

and

$$\sum_\alpha A_{\alpha\beta} A_{\alpha\delta} = \delta_{\beta\delta}$$

The solution to these equations is

$$A = \begin{bmatrix} cosh\psi & -sinh\psi & 0 & 0 \\ -sinh\psi & cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or equivalently

$$[ct', x', y', z',] = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$