Does COBE rule out a toroidal Universe?

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The cosmic microwave background (CMB) is a unique probe of cosmological parameters and conditions. There is a connection between anisotropy in the CMB and the topology of the Universe. Adopting a universe with the topology of a 3-Torus, or a universe where only harmonics of the fundamental mode are allowed, and using 2-years of COBE/DMR data, we obtain constraints on the topology of the Universe. We obtain more accurate results than previous work by using all multipole moments, avoiding approximations by computing their full covariance matrix. The best fit for a cubic toroidal universe is obtained at a scale of $7200h^{-1}$Mpc for $n = 1$. The data set a lower limit on the cell size of $4320h^{-1}$Mpc at 95% confidence and $5880h^{-1}$Mpc at 68% confidence. These results show that the most probable cell size would be around 1.2 times larger than the horizon scale, implying that the 3-Torus topology is no longer an interesting cosmological model.

1. Introduction

In the past few years, the study of the topology of the Universe has become an important problem for cosmologists and some hypotheses, such as a Universe composed of spatial sections that are topologically connected, the "small universe" model [1], have received considerable attention. From the theoretical point of view, it is possible to have quantum creation of the Universe with a multiply-connected topology [2]. From the observational side, this model has been used to explain "observed" periodicity in the distributions of quasars [3] and galaxies [4]. The topology of these spatial sections can be quite complicated. However, for simplicity, we limit our analysis to the case where these spatial sections form a rectangular basic cell with sides $L \equiv L_x = L_y = L_z$ and with opposite faces topologicaly connected, the topology known as T^3.

We use the CMB to constrain the parameter L/y, the ratio between the cell size L and radius of the decoupling sphere y, where $y = 2cH_o^{-1}$. By constraining the parameter L/y, we expect to obtain some information about the topology of the Universe and, at least, confirm or not the possibility that we live in a Universe with topology as described by the T^3 model.

2. The T^3 universe model

As pointed out by Zel'dovich [5], the power spectrum of density perturbations is continuous (i.e., all wave numbers are possible) if the Universe has a Euclidean topology, and discrete (i.e., only some wave numbers are possible) if the topology has finite space sections. In other words, in a Euclidean topology the Sachs-Wolfe spectrum C_l is an integral over the power spectrum; however, in the T^3 universe this is not the case. In this model, only wave numbers that are harmonics of the cell size are allowed. We have a discrete **k** spectrum [6]

$$\mathbf{k}^2 = \sum_{i=1}^{3} \left(\frac{2\pi}{L_i}\right)^2 p_i^2, \qquad (1)$$

where L_1, L_2 and L_3 are the dimensions of the cell and p_i are integers. For simplicity, assuming $L = L_x = L_y = L_z$ and a power-law power spectrum with shape $P(k) = |\delta_{\mathbf{k}}|^2 = Ak^n$, where A is the amplitude of scalar perturbations and n the spectral index, the power spectrum of the T^3 is given by [7,8]

$$C_l = \frac{16\pi A}{y^n} \sum_{p_x,p_y,p_z} \left(\frac{L}{2\pi y p}\right)^{4-n} j_l^2 \left(\frac{2\pi y p}{L}\right), \quad (2)$$

where $p^2 = p_x^2 + p_y^2 + p_z^2$ and j_l are spherical Bessel functions of order l. According to (2), the lth multipole of the CMB temperature is function of the ratio L/y. This shows that the more multipole components we use in our fit, the stronger our constraints on the cell size will be.

Using eq. (2), we calculated the expected power spectrum for a T^3 universe with different cell sizes L/y from 0.1 to 3.0, $n = 1$ and $l_{max} = 30$, where $l_{max} = 30$ is the limit at which we truncate our data power spectrum. In Figure 1, we plot $l(l+1)C_l$ versus l and normalize all values to the last multipole component $l = 30$. Note that for very small cells ($L \ll y$), the low order multipoles are suppressed. The power spectrum for small cells (as $L/y = 0.1$, 0.5 or 1.0) shows the presence of "bumps" that disappear as the cell size increases ($L/y \gtrsim 1.5$). The power spectrum finally becomes flat for large cell sizes ($L/y \gtrsim 3.0$). These "bumps" can be explained if we remember that only the harmonics of the cell size are allowed to be part of the sum in (2). When the cell size is small there are fewer modes of resonance, and no modes larger than the cell size appear in the sum in (2). As the cell size increases, the sum approaches an integral and the T^3 power spectrum becomes flat.

We restrict our analysis to $n = 1$. This assumption does not weaken our results, since the T^3 model with other n-values tends to fit the data as poorly as with $n = 1$. For instance, we obtain the minimum χ^2 at the same ratio L/y for $n = 1$ and $n = 1.5$.

3. Data Analysis

The method that we use to constrain the parameter L/y is quite different from previous work [7,9]: we compute the full covariance matrix for all multipole components and use this covariance matrix to make a χ^2 fit of the power spectrum extracted from the 2 years of COBE/DMR data to the power spectrum expected for a small universe with different cell sizes L.

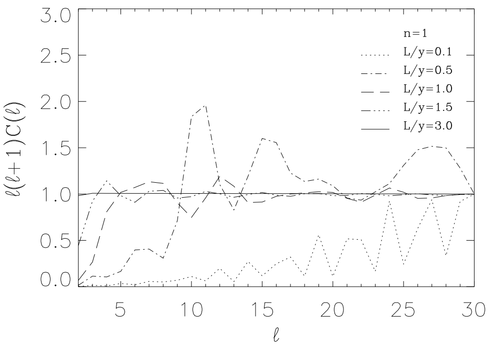

Figure 1. Expected power spectrum for the T^3 universe model with $n = 1$ for different cell sizes with L/y from 0.1 to 3.0, and all values normalized to $l = 30$

Each DMR sky map is composed of 6144 pixels and each pixel i contains a measurement of the sky temperature at position $\widehat{\mathbf{x}}_i$. Considering that the temperatures are smoothed by the DMR beam B_l [10] and contaminated with noise n_i, the sky temperatures are described by

$$\left(\frac{\delta T}{T}\right)_i = \sum_{lm} a_{lm} B_l Y_{lm}(\widehat{\mathbf{x}}_i) + n_i, \quad (3)$$

where Y_{lm} are the spherical harmonics in the direction $\widehat{\mathbf{x}}_i$ and a_{lm} their coefficients. We model the quantities n_i as Gaussian random variables with mean $\langle n_i \rangle = 0$ and variance $\langle \quad n_j \rangle = \sigma_i^2 \delta_{ij}$, assuming uncorrelated pixel noise [11].

Because of the uncertainty in Galactic emission, we are forced to remove all pixels between 20° below and above the Galactic plane. This cut represents a loss of almost 34% of all sky pixels and destroys the orthogonality of the spherical harmonics. We define "new" coefficients

$$b_{lm} \equiv w \sum_{i=1}^{N_{pix}} \left(\frac{\delta T}{T}\right)_i Y_{lm}^*(\widehat{\mathbf{x}}_i), \quad (4)$$

where the normalization is chosen to be $w \equiv 4\pi/N_{pix}$ and N_{pix} is the number of pixels that remain in the sky map after the Galaxy cut has taken place. Substituting (3) into (4), we obtain

$$
\begin{aligned}
b_{lm} &= \sum_{l_1 m_1} a_{l_1 m_1} B_{l_1} W_{l l_1 m m_1} \\
&+ w \sum_{i=1}^{N_{pix}} n_i Y^*_{lm}(\widehat{\mathbf{x}}_i), \qquad (5)
\end{aligned}
$$

with covariance

$$
\begin{aligned}
\langle b_{lm} b^*_{l'm'} \rangle &= \sum_{l_1 m_1} W_{l l_1 m m_1} W_{l' l_1 m' m_1} C_{l_1} B^2_{l_1} \\
&+ w^2 \sum_{i=1}^{N_{pix}} \sigma_i^2 Y^*_{lm}(\widehat{\mathbf{x}}_i) Y_{l'm'}(\widehat{\mathbf{x}}_i), \qquad (6)
\end{aligned}
$$

where

$$
W_{l l_1 m m_1} \equiv w \sum_{i=1}^{N_{pix}} Y^*_{lm}(\widehat{\mathbf{x}}_i) Y_{l_1 m_1}(\widehat{\mathbf{x}}_i). \qquad (7)
$$

Defining our multipole estimates as

$$
C_l^{DMR} \equiv \frac{1}{2l+1} \sum_m b_{lm} b^*_{lm}, \qquad (8)
$$

their expectation values are simply

$$
\langle C_l^{DMR} \rangle \equiv \frac{1}{2l+1} \sum_m \langle b_{lm} b^*_{lm} \rangle \qquad (9)
$$

and their covariance matrix M is given by

$$
M_{ll'} \equiv \frac{2}{(2l+1)(2l'+1)} \sum_{mm'} \langle b_{lm} b^*_{l'm'} \rangle^2. \qquad (10)
$$

The C_l^{DMR} coefficients are not good estimates of the true multipole moments C_l. However, they are useful for constraining our cosmological parameters.

The χ^2 is defined by

$$
\chi^2 \equiv \mathbf{C}^T \mathbf{M}^{-1} \mathbf{C}, \qquad (11)
$$

where $\mathbf{C}^T$ and $\mathbf{C}$ are l_{max}-dimensional row and column vectors with entries $C_l = \widehat{C}_l^{DMR} - \langle C_l^{DMR} \rangle$ and $\mathbf{M}$ is the covariance matrix as described in (10) with dimensions l_{max} x l_{max}. Here $\widehat{C}_l^{DMR}$ denotes the C_l^{DMR}-coefficients actually extracted from the data.

4. Results

We now compute the chi squared function $\chi^2(L/y, \sigma_{7^\circ})$ and use it to constrain the ratio L/y and the normalization σ_{7°, where σ_{7° is the rms variance at 7°. In Figure 2, we plot the probability that the T^3 model is consistent with the data as a function of the ratio L/y and the normalization σ_{7° (bottom). Confidence limits of 68%, 95% and 99.7% are shown in the contour plot (top). We found the highest consistency probability (minimum χ^2) at $(L/y, \sigma_{7^\circ}) = (1.2, 49.7\mu K)$, represented by a cross in the contour plot. Removing the quadrupole, we obtained similar results; see Table 1 for the lower limits on cell sizes. We obtain the constraint $L/y = 1.2^{+\infty}_{-0.48}$ at 95% confidence. We cannot place an upper limit on the cell size: all large cells are equally probable.

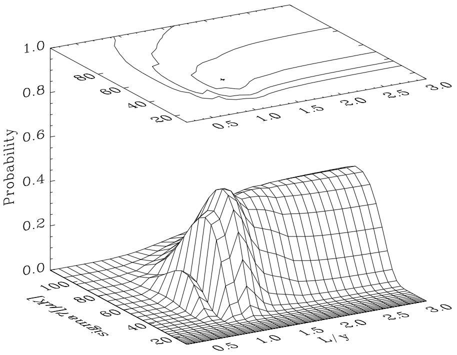

Figure 2. The probability that the T^3 model is consistent with the data is plotted as a function of the ratio L/y and the normalization σ_{7° (bottom). Confidence limits of 68%, 95% and 99.7% are shown in the contour plot (top). We found the highest consistency probability (minimum χ^2) at $L/y = 1.2$, represented by a cross in the contour plot.

5. Conclusions

The strong constraint from our analysis comes from the predicted power spectrum of the T^3 uni-

Table 1: Lower limits on L/y

Confidence Level	L/y	
	with C_2	without C_2
68 %	0.98	0.97
90 %	0.75	0.68
95 %	0.72	0.65
99.7%	0.61	0.60

verse; see Figure 1. According to this plot, a reduction in the cell size to values below the horizon scale should suppress the quadrupole and low multipole anisotropies, while the suppression is negligible if the cell is very large, at least, larger than the horizon. It is possible to notice these properties in Figure 2: it favors large cell sizes. The observed presence of the quadrupole and other low order anisotropies automatically constrains our cell to be very large. In other words, even before making the χ^2 fit, we expect to obtain very large cells.

We remind the reader that our analysis is for $n = 1$. We made this assumption because the results of fitting the T^3 model seem to be relatively insensitive to changes in n and the "bumps", not the overall slope, are responsible for the poor fit between the model and the data. In other words, our results are independent of any particular inflationary model.

From the COBE/DMR data, we obtain the best χ^2 fit for a toroidal universe with $L/y = 1.2$, which corresponds to a cell size of $L = 7200h^{-1}$Mpc. A cell size below 72% of the size of the horizon ($L/y < 0.72$) is incompatible with the COBE measurements at 95% confidence, and a cell size below roughly the size of the horizon ($L/y < 0.98$) is ruled out at 68% confidence. Since the T^3 topology is interesting if the cell size is considerably smaller than the horizon, this model loses most of its appeal.

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