Physics 139 Relativity
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1 The Geometry of the Universe

The usual assumption, assigned the name *The Cosmological Principle*, is that on sufficiently large scales the Universe is isotropic and homogeneous. The Cosmological Principle holds that any observer in the Universe will when averaging upon sufficiently large scales see the same mean density etc. in all directions as any other observer.

This assumption leads one to use the generic metric for 3+1 dimensional isotropic, homogeneous spaces called the Robertson-Walker metric for the two people who first derived it in cosmology.

In 1967 J. Ehlers, P. Geren, and R.K. Sachs (J. Math. Phys., 9, 1344) proved that if all observers in the Universe saw an isotropic background radiation, then the correct metric was Robertson-Walker metric. I and many others assumed that if the exact theorem was true, then the almost theorem would be also. With a little encouragement W. Stoeger, R. Maartens, & G.F.R. Ellis (author of curved space-time book) proved the almost theorem, which indicates that if all observers in the Universe see a nearly isotropic background radiation (or more usefully, if one observer sees a nearly isotropic background radiation for all times), the the appropriate metric is nearly the Robertson-Walker. When can then approximate the metric for the Universe as

$$g^{\mu\nu} \cong \eta^{\mu\nu} + h^{\mu\nu} \tag{1}$$

where $\eta^{\mu\nu}$ is the overall background metric (Robertson-Walker which you will easily be able to show reduces to the Minkowski metric for small distances and times) and $h^{\mu\nu}$ is our usual weak field linearization. This is useful when we are treating structure formation, small changes from homogeneity or isotropy.

Large angular scale observations of the cosmic background radiation indicate it is isotropic to about one part in 10^5 so that we can anticipate that the large scale metric will be the Robertson-Walker metric to roughly the 10^{-5} level so that a linear expansion is quite adequate for most calculations.

1.1 Robertson-Walker Metric

Rather than solve the Einstein General-Relativity equations which have both dynamics of gravity and geometry built into them, we will first derive the Robertson-Walker metric by simple geometry arguments and then the equation of motion of the

Universe (dynamics) using Newtonian gravity and then see that the GR equations give the same result.

The goal is to find the metric for a space that is isotropic and homogeneous. That will mean that the large-scale curvature is the same every location. Consider the analogous *two dimensional* problem: What two-dimensional space has uniform (homogeneous and isotropic) curvature. The obvious answer is the surface of an ordinary sphere, usually called the two-sphere.

The three-dimensional space of uniform curvature solution should be the three-sphere which we can find by embedding our three-dimensional space into a four-dimensional space and finding the three-sphere equation and eliminate the four coordinate from the metric just as we do in the two-sphere case. The equation of the three-sphere surface in four-dimensional hyperspace in rectangular coordinates is:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = R^2 \tag{2}$$

where R is the "radius" of the sphere. The distance between any two nearby points on the surface is given by the metric:

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \tag{3}$$

The extra coordinate x^4 is simply an extra unphysical coordinate that is introduced to help us find the metric for the three-sphere as a solution to the uniformly curved three-dimensional space.

By using the equation for the three-sphere we can eliminate x^4 from the metric:

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \frac{(x^1dx^1 + x^2dx^2 + x^3dx^3)^2}{R^2 - (x^1)^2 - (x^2)^2 - (x^3)^2} \tag{4}$$

In this version of the metric there appear only the physical coordinates x^1, x^2, and x^3, and is the metric for an isotropic, homogeneous three-dimensional space. It is easy to check that the curvature tensor for the metric has $K = 1/R^2$ for its diagonal in rectangular coordinates. A geometry with a positive value of K is said to have a *positive* curvature.

It is easy to go to polar coordinates in the usual way:

$$\begin{aligned} x^1 &= rsin\theta cos\phi \\ x^2 &= rsin\theta sin\phi \\ x^3 &= rcos\theta \end{aligned} \tag{5}$$

The space interval metric is then

$$dl^2 = \frac{dr^2}{1 - r^2/R^2} + r^2d\theta^2 + r^2sin^2\theta d\phi^2 \tag{6}$$

Note that the "rectangular" coordinates x^k and the "radial" coordinate r are actually periodic coordinates. If one starts at the "top" of the three sphere ($x^4 = R$), move off in the positive x^1 direction, and continue straight ahead, one will reach the equator $x^4 = 0$, then the bottom of the sphere ($x^4 = -R$); continuing straight ahead one comes back to the equator and then back to the top. The values of x^1 and r for the top, the first equatorial crossing, the bottom, and the second equatorial crossing are, $x^1 = 0, R, 0, -R$ and $r = 0, R, 0, R$, respectively. That there are two distinct points with the same value of x^1 and r of x^1 and r can be handled by dividing them into upper and lower hemispheres.

For a little exploration compare the radius and circumference of a circle in this positively curved three-space. For convenience, take the circle defined by $r = b =$ constant around the origin. This circle has for its radius the distance between $r = 0$ and $r = b$,

$$\text{radius} = \int_0^b \frac{dr}{\sqrt{1 - r^2/R^2}} = Rsin^{-1}(b/R) \tag{7}$$

The circumference of the circle is obtained in the usual way. Take, for example, the circle in the plane $\theta = \pi/2$.

$$\text{circumference} = \int_0^{2\pi} bsin\theta d\phi = 2\pi b \tag{8}$$

The ratio of circumference to radius is therefore larger than 2π, which is a property of spaces with positive curvature. Note that for a radius larger than $\pi R/2$, the circumference decreases as the radius increases.

The surface area of the sphere $r = b =$ constant surrounding the origin is

$$\text{area} = \int_0^{2\pi} \int_0^{\pi} b^2 sin\theta d\theta d\phi = 4\pi b^2 \tag{9}$$

Hence the ratio of the radius squared to the area is larger than $1/4\pi$. The volume inside the sphere $r = b$ is

$$\text{volume} = \int_0^{2\pi} \int_0^{\pi} \int_0^{b} \frac{r^2}{\sqrt{1 - r^2/R^2}} dr\ sin\theta d\theta d\phi = \frac{4\pi}{2} \left(R^3 sin^{-1} \frac{b}{R} - bR^2 \sqrt{1 - b^2/R^2} \right) \tag{10}$$

One can find the total volume of the three-sphere (all of space) by letting $b = 0$ but at the "bottom" of the sphere $sin^{-1} b/R = \pi$. The total volume is then $2\pi R^3$. The three-sphere is a closed space; it has finite volume even through it has no boundaries.

Since r is a periodic coordinate, it is convenient to introduce a new angular coordinate χ such that

$$r = Rsin\chi, \quad 0 < \chi < \pi \tag{11}$$

The coordinate χ has the advantage over the coordinate r in that it is single valued. The metric becomes:

$$dl^2 = R^2 \left[d\chi^2 + sin^2\chi \left(d\theta^2 + sin^2\theta d\phi^2 \right) \right] \tag{12}$$

Now the radial distance from the origin is simply: l = radial distance = $R\chi$.

Now there is also another possibility which is a negatively curved space. It is easy to show that one finds the solution by replacing R^2 by $-R^2$ in all the equations, including for the curvature $K = -1/R^2$. The space interval metric is then

$$dl^2 = \frac{dr^2}{1 + r^2/R^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{13}$$

The radius of the circle $r = b$ around the origin becomes

$$\text{radius} = \int_0^b \frac{dr}{\sqrt{1 + r^2/R^2}} = R sinh^{-1}(b/R) \tag{14}$$

so that the sine is replaced by the hyperbolic sine function. We can now calculate the circumference Take, for example, the circle in the plane $\theta = \pi/2$.

$$\text{circumference} = \int_0^{2\pi} b sin\theta d\phi = 2\pi b \tag{15}$$

The ratio of circumference to radius is therefore smaller than 2π, which is a property of spaces with negative curvature. Likewise the ratio of the area to the radius squared is less than 4π The volume of the sphere with $r = b$ is then

$$\text{volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{b} \frac{r^2}{\sqrt{1 + R^2/r^2}} dr\ sin\theta d\theta d\phi = \frac{4\pi}{2}\left(-R^3 sinh^{-1}\frac{b}{R} + bR^2\sqrt{1 + b^2/R^2}\right) \tag{16}$$

As $b \to \infty$, this volume diverges. The space of negative curvature is open and infinite.

again we can introduce a change to angular coordinates

$$r = R sinh\chi \tag{17}$$

and get the metric

$$dl^2 = R^2\left[d\chi^2 + sinh^2\chi\left(d\theta^2 + sin^2\theta d\phi^2\right)\right] \tag{18}$$

There is only the boundary left: the case of zero curvature. This has a trivial solution of flat Euclidean space.

2 The Dynamics of the Universe

2.1 The Friedmann Models

Although the three-dimensional geometry, of positive or negative curvature, can be described equally well by the various coordinates, (x^1, x^2, x^3), (r, θ, ϕ), and $(\chi, \theta,$

ϕ), these coordinates are not equal convenient for describing the expanding Universe. Only the coordinates (χ, θ, ϕ) may be regarded as comoving with the typical galaxies. The reason is that isotropy and homogeneity of the expansion require that if $dl(t)$ is the distance between two galaxies at a time t, then at a time $t + \Delta t$ must be proportional to the initial distance: $dl(t + \Delta t) = A(t)dl(t)$ where the proportionality factor A depends on time and *not on the position in space*. That is when the distance between two tracer galaxies increases by a factor A, then the distance between any other pair also increases by the same factor.

In the expressions for the metric, the only quantity that can change with time is R. An expanding universe will have a time-dependent value of R. Only the metric expressed in terms of (χ, θ, ϕ) can have a simple time dependence on the scale factor R. Thus the coordinates (χ, θ, ϕ) remained fixed for fundamental observers (typical galaxy with no peculiar velocity relative to the mean of all other galaxies).

The time dependence of R is determined by the Einstein equations. In order to write down the Einstein equations, one must also make some assumptions about the energy-momentum tensor of the constituents of the Universe. First consider a universe in which the energy content can be treated as a uniform gas with density and pressure. The galaxies can be regarded as particles out of which this gas is made. Since the galaxies do not have significant velocities ($\beta \lesssim 10^{-3}$) with respect to the uniform expansion (fundamental observers), one can actually neglect the pressure of the gas. HOwever, in the very early, very hot stages of the Universe, the pressure due to radiation and relativistic particles was significant.

The general homogeneous, isotropic models of the universe with mass density, pressure, and, possibly, a cosmological term are the *Lemaître models*. The special models with zero pressure and zero cosmological constant are usually called *Friedmann models*. This section discusses Friedmann models

2.1.1 Positive Curvature

The spacetime interval for the closed isotropic universe is

$$ds^2 = c^2dt^2 - dl^2 = c^2dt^2 - R(t)^2\left[d\chi^2 + sin^2\chi\left(d\theta^2 + sin^2\theta d\phi^2\right)\right] \qquad (19)$$

The solution of the Einstein General Relativity equations is simplified, if we replace the time coordinate t by a time parameter η defined by

$$dt = R(t)d\eta \qquad (20)$$

Then the metric becomes

$$ds^2 = R(t)^2\left[d\eta^2 - d\chi^2 - sin^2\chi\left(d\theta^2 + sin^2\theta d\phi^2\right)\right] \qquad (21)$$

where $c = 1$ and η is called the conformal time. In this form, you will note that light travels on a 45° slant. Explicitly the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} R^2 & & & 0 \\ & -R^2 & & \\ & & -R^2 sin^2\chi & \\ 0 & & & -R^2 sin^2\chi sin^2\theta \end{pmatrix} \tag{22}$$

Some of the Christoffel symbols are

$$\begin{array}{lll} \Gamma^0_{00} = \dot{R}/R & \Gamma^0_{kn} = -(\dot{R}/R)g_{kn} & \Gamma^k_{0n} = (\dot{R}/R)\delta^k_n \\ \Gamma^0_{k0} = 0 & \Gamma^k_{00} = 0 & \end{array} \tag{23}$$

where $\dot{R} \equiv dR/d\eta$. Note there is a different definition here for an over dot. The relation between the the derivative with respect to proper time is

$$\frac{dR}{d\tau} = \frac{dR}{dt} = \frac{1}{R}\frac{dR}{d\eta} = \frac{1}{R}\dot{R} \tag{24}$$

The R_{00} component of the Ricci tensor determined from the Christoffel symbols above is

$$R_{00} = \frac{3}{a^2}\left(a\ddot{a} - \dot{a}^2\right) \tag{25}$$

The R_{kn} components can easily be evaluated by relating them to the three dimensional curvature tensor components $^(3)R_{kn}$. By definition

$$R_{kn} = R^\alpha_{kn\alpha} = R^0_{kn0} + R^l_{knl} \tag{26}$$

The term R^0_{kn0} only involves the Christoffel symbols above. The R^l_{knl} is

$$R^l_{knl} = -\Gamma^l_{kn,l} + \Gamma^l_{kl,n} + (\Gamma^0_{kl}\Gamma^l_{0n} + \Gamma^s_{kl}\Gamma^l_{sn}) - (\Gamma^0_{kn}\Gamma^l_{0l} + \Gamma^s_{kn}\Gamma^l_{sl}) \tag{27}$$

$$R^l_{knl} =^{(3)} R^l_{knl} + \Gamma^0_{kl}\Gamma^l_{0n} - \Gamma^0_{kn}\Gamma^l_{0l} \tag{28}$$

The three space Ricci tensor is

$$^{(3)}R^l_{knl} = \frac{1}{R^2}{}^{(3)}g^{lm}\left[^{(3)}g^{(3)}_{mn}g_{kl} -^{(3)} g^{(3)}_{ml}g_{kn}\right] = \frac{2}{R^2}g_{kn} \tag{29}$$

Thus

$$R_{kn} = \frac{1}{a^4}\left(2a^2 + \dot{a}^2 + a\ddot{a}\right)g_{kn} \tag{30}$$

The scalar curvature is then given by

$$R = R^0_0 + R^k_k = g^{00}R_00 + g^{kn}R_{kn} = \frac{6}{a^3}(a + \ddot{a}) \tag{31}$$

The Einstein equation, including the cosmological constant, is

$$R^\nu_\mu - \frac{1}{2}\delta_m u^\nu R = -8\pi G T_m u^\nu - \Lambda \delta_m u^\nu \tag{32}$$

Combining equations

$$-\frac{3}{a^4}(a^2 + \dot{a}^2) = -8\pi T^0_0 - \Lambda \tag{33}$$

In this equation the $\ddot{a}$ has canceled. This equation is sufficient to determine $a(t)$, if T_{00} is known.

We now need to decide what to use for the energy density T^0_0 for the contents of the Universe.

$$T^\nu_\mu = \rho u_\mu u^\nu \tag{34}$$

where ρ is the proper mass density. In comoving coordinates, the matter is at rest and hence

$$T^0_0 = \rho \tag{35}$$

Since a volume element of the Universe varies with the scale factor a as a^3, it follows that the density of conserved objects varies as a^{-3}:

$$\rho(t) = \frac{M}{2\pi^2 a^3} \tag{36}$$

where M is a constant. Since $2\pi^2 a^3$ is he total volume of the universe for a closed universe, M would be the total "mass of the universe". M is really the summ of all the proper masses of the particles in the universe; special relativity says that this sum will not give the total mass since it fails to take into account kinetic and binding energies.

The general dynamical equation becomes

$$\begin{aligned} \frac{3}{a^4}(\dot{a}^2 + a^2) &= 8\pi G\rho + \Lambda \\ &= \frac{4GM}{\pi a^3} + \Lambda \end{aligned} \tag{37}$$

In the Friedmann models, it is assumed that the cosmological constant is either exactly zero, or else that it is so small that it can neglected comapred to ρ. Under this assumption the equation reduces to:

$$\frac{3}{a^4}(\dot{a}^2 + a^2) = \frac{4GM}{\pi a^3} \tag{38}$$

This is the differential equation that describes the *closed Friedmann model.*

The solution of this equation is

$$a = a_*(1 - cosh\eta) \quad \text{where} \quad a_* = \frac{2GM}{3\pi} \tag{39}$$

So that the equation for time is

$$t = a_*(\eta - sinh\eta) \tag{40}$$

For a negatively curved, the equation is

$$\frac{3}{a^4}(\dot{a}^2 - a^2) = \frac{4GM}{\pi a^3} + \Lambda \tag{41}$$

If we assume that Λ is zero,the equation is

$$\frac{3}{a^4}(\dot{a}^2 - a^2) = \frac{4GM}{\pi a^3} \tag{42}$$

The model of the universe described by this equation is called the *open Friedmann model* or *negatively curved Friedmann model.*

The solution of this equation is

$$a = a_*(cosh\eta - 1) \tag{43}$$

So that the equation for time is

$$t = a_*(sinh\eta - \eta) \tag{44}$$

For zero curvature or a flat universe and $\Lambda = 0$

$$\frac{3}{a^4}\dot{a}^2 = 8\pi G\rho + \Lambda \tag{45}$$

which has the solution

$$a(t) = \left(\frac{3GM}{\pi}\right)^{1/3} t^{2/3} \tag{46}$$

where $\rho = M/2\pi^2 a^3$.

2.2 Newtonian Derivation

Consider a test particle at a distance R from the observer in an infinite, uniform Universe. We know that we can ignore the gravitational affect of material outside side of a spherical cavity of radius equal to R and only consider the mass contained within that radius. The Newtonian acceleration of the test particle due to the gravitational attraction of the matter is

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} = -\frac{4\pi}{3}G\rho R$$

where M is the mass included in radius R and the last equality holds when the mass is sufficiently uniformly spread that we can treat it as a constant density ρ. Multiply through by $\dot{R} \equiv dR/dt$ to get

$$\dot{R}\frac{d^2R}{dt^2} = -\frac{GM}{R^2}\dot{R}$$

$$\frac{d}{dt}(\frac{1}{2}\dot{R}^2) = -\frac{d}{dt}(\frac{GM}{R})$$

or equivalently

$$\frac{d}{dt}[\frac{1}{2}(\dot{R}^2 + K)] = \frac{d}{dt}(\frac{GM}{R})$$

where K is a constant of integration. (Later we will see that K is the curvature of the universe and it is also equal $-2E/m =$ minus twice the fractional binding energy of a particle.) Integrating this equation we obtain

$$\frac{1}{2}(\dot{R}^2 + K) = \frac{GM}{R}$$

or

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = -\frac{K}{2}$$

We could have found this same formula by writing down the equation for the total energy and dividing through by the mass.

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = \frac{E_{total}}{m}$$

So that the constant of integration is $K = -2E_{total}/m$.

Now we convert included mass to mean density to get an interesting formula:

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi}{3}G\rho R^2 = -\frac{K}{2}$$

Divide this through by R^2 and multiply by 2 to find that

$$(\frac{\dot{R}}{R})^2 - \frac{8\pi}{3}G\rho = -\frac{K}{R^2}$$

We can also express this in terms of our comoving coordinates $R = a(t)r$ where for uniform expansion a la the Hubble law r is a constant. $\dot{R} = \dot{a}r$ so that

$$\frac{\dot{R}}{R} = \frac{\dot{a}}{a(t)}$$

Thus we readily get the equation

$$\mathbf{(\frac{\dot{a}}{a})^2 - \frac{8\pi}{3}G\rho = -\frac{K}{a^2}}$$

This is the first Friedmann equation, if we use the Equivalence Principle and replace the mass density with the energy density T_{00}. We could also use the Equivalence Principal to get the other full relativistic Friedmann equation but let us continue on using the Newtonian approach because it gives a 'better 'feel" for what is going on.

We can find the value for K by using the present epoch values where $H_0 \equiv \frac{\dot{R}}{R}$ is the current value for Hubble constant in the Hubble's law $v = \dot{R} = H_0 R$ and the present mean density ρ_0 of the Universe.

$$K = (H_0^2 - \frac{8\pi}{3} G\rho) R_0^2$$

which we can put back into the original equation

$$(\frac{\dot{R}}{R})^2 = \frac{8\pi}{3} G\rho - \frac{K}{R^2} = \frac{8\pi}{3} G\rho - \frac{8\pi}{3} G (\frac{R_0}{R})^2 [\rho_0 - \frac{3}{8\pi} \frac{H_0^2}{G}]$$

If we define the current critical density as

$$\rho_{c0} = \frac{3}{8\pi} \frac{H_0^2}{G}$$

which is often called just ρ_c dropping the zero subscript and is roughly

$$\rho_{c0} \cong 2 \times 10^{-29} h_{100}^2 \text{ g cm}^{-3}$$

where $h_{100} = H_0/100 kms^{-1} Mpc^{-1}$. Notice that ρ_c is the density that is the dividing line between having the test particle escape or be bound to the central mass. Thus it marks the division between a universe that will expand forever and on that will slow down enough to turn around and collapse.

We also define the parameter $\Omega = \rho/\rho_c$ which is the ratio of the density of the Universe to the critical density. Now we can recast the formula as one for the expansion rate

$$(\frac{\dot{R}}{R})^2 = H^2 = H_0^2 \Omega - H_0^2 (\Omega_0 - 1) (\frac{R_0}{R})^2 = H_0^2 \Omega_0 a^{-3} - H_0^2 (\Omega_0 - 1) a^{-2}$$

The last equality holds for conserved massive particles where $\Omega R^3 = \Omega_0 R_0^3$ and we have set $a(0) = 1$, that is the present scale factor to comoving coordinates is one. Note that $R/R_0 = a/a_0$. In this case the equation reduces to the simple form

$$\dot{a}^2 = H_0^2 \Omega a^2 - H_0^2 (\Omega_0 - 1) = H_0^2 \Omega_0 / a - H_0^2 (\Omega_0 - 1)$$

(where the last equality holds for the matter dominated case).

We pause a moment to reflect that if $\Omega_0 < 1$ then $\dot{a}^2 \to H_0^2 (\Omega_0 - 1)$ and $a \propto t$. The universe will keep expanding forever.

If $\Omega_0 > 1$ then $\dot{a}^2 \to 0$ and then later and < 0. This means that the universe will stop expanding and collapse.

The rate at which the expansion is slowing down $d\dot{a}/dt$ is proportional to the density of the universe so that the time back to $a \cong 0$ is going to be a function of Ω_0 and the scale is set by H_0^{-1}, i.e.

$$t_u = f(\Omega_0) H_0^{-1}$$

It is clear that if $\Omega_0 = 0$, then there is no decelleration and $f(0) = 1$. We can integrate the equations easily, if $\Omega_0 = 1$, since for a matter dominated case reduces to

$$\dot{a}^2 = H_0^2/a$$

or taking the square root

$$a^{1/2}\dot{a} = H_0$$

or

$$\frac{2}{3}a^{3/2} = H_0 t \qquad or \qquad t = \frac{2}{3}a^{3/2}H_0^{-1}$$

where the constant of integration is taken care of by defining the origin of time $t = 0$ as the time $a = 0$. Since we have defined $a = 1$ as the present, we are at $t_u = 3/2\ H_0^{-1}$. Thus $f(\Omega_0 = 1) = 2/3$ and thus $t_u(\Omega_0 = 1) \approx 6.7 \times 10^9 h_{100}^{-1}$ years. Note also that the scale factor for a matter dominated universe is proportional to $a \propto t^{2/3}$. (Actually $a = (3/2\ H_0 t)^{2/3}$.)

We return to deriving the Friedmann equations via Newtonian physics. We know that the internal energy U of a gas of particles is $U = \rho V$. Thus

$$dU = -PdV = \rho dV + V d\rho$$

so that

$$V d\rho = -(\rho + P)dV$$

so we get the continuity equation

$$\dot{\rho} = -(\rho + P)\frac{\dot{V}}{V} = -3(\rho + P)\frac{\dot{a}}{a}$$

where the last equality comes from the relationship $V \propto a^3$. Now the Newtonian equation of motion has an additional term coming from the pressure which we can see from differentiating the energy conservation equation and using the continuity equation. First recall the equation

$$(\frac{\dot{a}}{a})^2 - \frac{8\pi}{3}G\rho = -\frac{K}{a^2}$$

multiply by a^2 and differentiate.

$$2\dot{a}\ddot{a} = 2\frac{8\pi}{3}G\rho a\dot{a} + \frac{8\pi}{3}G\dot{\rho}a^2$$

where K is a constant so that the derivative is zero. Dividing through by $2\dot{a}$

$$\ddot{a} = \frac{8\pi}{3}G\rho a + \frac{4\pi}{3}G\dot{\rho}\frac{a^2}{\dot{a}}$$

The continuity equation $\dot{\rho} = -3(\rho + P)\frac{\dot{a}}{a}$ can be used to eliminate $\dot{\rho}$.

$$\ddot{a} = \frac{8\pi}{3}G\rho a - \frac{4\pi}{3}G3(\rho + P)a$$

gathering terms we get the other Friedmann equation

$$\mathbf{\ddot{a} = -\frac{4\pi}{3}G(\rho + 3P)a}$$

Note that if $\rho + 3P > 0$ is positive then the universe is decellerating - that is the rate of expansion is slowing.

2.3 Lemaître Models

The Lemaître models differ from the Friedmann models in that the cosmological constant is not zero. Consider first cases where the universe is effectively empty of matter and energy except for the cosmological constant ($\Lambda \gg 8\pi G T_0^0$). Negelecting T_0^0 and regarding the universe as empty and dominated by the cosmological term, which has an effective uniform mass density of $\rho_{eff} = -\Lambda/4\pi G$. A positive value of Λ corresponds to a negative effective mass density (repulsion), and a negative value of Λ corresponds to a positive mass density (attraction). Hence, a universe with a positive value of Λ will have accelerating expansion; whereas a universe with a negative value of Λ will have a slowing, stopping, and then reversing expansion (oscillatory behavior).

2.3.1 Positive Curvature and $\Lambda \neq 0$

The equation of motion is

$$-\frac{3}{a^4}(a^2 + \dot{a}^2) = -\Lambda \tag{47}$$

With $\dot{a} = ada/dt$, this becomes

$$\left[\frac{da}{dt}\right]^2 = -1 + \frac{\Lambda a^2}{3} \tag{48}$$

It is clear that $-1 + \Lambda a^2/3$ cannot be negative. This implies that $\lambda > 0$, and that the value of a can never be less than $\sqrt{3/\Lambda}$. Thus, the empty Lemaître model with positive curvature requires a positive cosmological constant, and its radius of curvature can never be zero.

Integrating the equation of motion yields the solution

$$a(t) = \sqrt{\frac{3}{\Lambda}} cosh \sqrt{\frac{3}{\Lambda}} t \tag{49}$$

where t is zero at the time for minimum a.

2.3.2 Negative Curvaure and $\Lambda \neq 0$

In this case the equation of motion is

$$\left[\frac{da}{dt}\right]^2 = 1 + \frac{\Lambda a^2}{3} \tag{50}$$

which can be integrated to give

$$a(t) = \sqrt{\frac{3}{\Lambda}} sinh \sqrt{\frac{3}{\Lambda}} t \quad \text{for} \quad \Lambda > 0 \tag{51}$$

and

$$a(t) = \sqrt{\frac{3}{-\Lambda}} sinh \sqrt{\frac{-\Lambda}{3}} t \quad \text{for} \quad \Lambda < 0 \tag{52}$$

Both of these solutions have a point where $a = 0$ at $t = 0$. The first expands monotonically, whereas the second oscillates with a period $2\pi\sqrt{3/(-\Lambda)}$.

2.3.3 Zero Curvature and $\Lambda \neq 0$

The equation of motion is then

$$\left[\frac{da}{dt}\right]^2 = \frac{\Lambda a^2}{3} \tag{53}$$

which can be integrated to give

$$a(t) = a(0) e^{\sqrt{\frac{3}{\Lambda}} t} \quad \text{for} \quad \Lambda > 0 \tag{54}$$

The univese expands exponentially, with a characteristic time of $\sqrt{3/\Lambda}$ (doubling time of $0.693\sqrt{3/\Lambda}$. This model is usually called the *de Sitter model* since W. de Sitter found this solution shortly after Einstein published his theory of General Relativity cosmological constant addition.

2.4 Scale factor Expansion Rates

Earlier we derived the time dependence of the scale factor a for a matter dominated universe. Let us now return to this issue. Consider a simple equation of state: $P = w\rho$. If $w =$ constant, i.e. independent of time, then we can use the internal energy relation

$$dU = d(\rho V) = -PdV \propto -Pd(a^3)$$

which implies

$$d[a^3(\rho + P)] = a^3 dP$$

putting in the equation of state

$$d[a^3\rho(1+w)] = a^3 w d\rho$$

from which we can conclude that

$$\rho \propto a^{-3(1+w)}$$

So we can now consider the equation of state for three simple cases

$$\begin{array}{cccc} Radiation & P = 1/3\ \rho & \rightarrow & \rho \propto a^{-4} \\ Matter & P = 0 & \rightarrow & \rho \propto a^{-3} \\ Vacuum\ Energy & P = -\rho & \rightarrow & \rho = constant \end{array}$$

To turn these relations into ones between a and t,

Now we get back on track again. Converting to Hubble expansion rate $H = \dot{a}/a$ and utilizing

$$H^2 = \frac{8\pi}{3}G\rho - \frac{K}{a^2} = \frac{8\pi}{3}G\rho_0\ a^{-3(1+w)} - \frac{K}{a^2}$$

We note that for a small enough the ρ_0 term will dominate the K curvature term as long a $3(1+w) > 2$ which happens for matter and radiation. Neglecting the curvature K term we have the relation

$$H^2 \propto \rho, \quad \text{and thus } \frac{\dot{a}}{a} \propto a^{-3(1+w)/2}$$

This gives the relation

$$a^{(1+3w)/2} da \propto dt \quad \text{or} \quad a^{3(1+w)/2} \propto t \quad (\text{or if } w = -1, \quad ln(a) \propto t)$$

Solving for a we have $a \propto t^{2/[3(1+w)]}$ or for $w = 1$, $a \propto e^{Ht}$.

We can go back to our old table and add another column

$$\begin{array}{ccccc} Stuff & P = w\rho & \rightarrow & \rho \propto a^{-3(1+w)} & a \propto t^{2/[3(1+w)]} \\ Radiation & P = 1/3\ \rho & \rightarrow & \rho \propto a^{-4} & a \propto t^{1/2} \\ Matter & P = 0 & \rightarrow & \rho \propto a^{-3} & a \propto t^{2/3} \\ Vacuum\ Energy & P = -\rho & \rightarrow & \rho = constant & a(t) \propto e^{Ht} \end{array}$$

We need to check our assumption about neglecting the K term. Returning to the equations of motion.

$$(\frac{\dot{a}}{a})^2 - \frac{8\pi}{3}G\rho = -\frac{K}{a^2}$$

Note that this implies

$$\frac{K}{H^2a^2} = \frac{8\pi G}{3H^2}\rho - 1 = \Omega - 1$$

since $H^2a^2 \geq 0$, there is a sign correspondence between the sign of K and the sign of $\Omega - 1$. This equation can be expressed as

$$\frac{K}{\dot{a}^2} = \Omega - 1$$

Since K is a constant and the expansion rate $\dot{a}$ is larger in the past $\Omega - 1 \rightarrow 0$ as $a \rightarrow 0$. We find that

$$|\Omega - 1| \cong (\frac{a}{a_o})^{3(1+w)/2}$$

Thus for matter and radiation dominated universes $|\Omega - 1|$ is proportional to $(1+z)^{-1}$ and $(1+z)^{-2}$, respectively. Since the likely range for Ω_0 based upon observation is $0.01 < \Omega_0 < 2$, only at late times can the curvature term K be important. (You will note later that we have a strong philosophical bent that $\Omega = 1$.)

It is also possible that particles decay. The general energy density of particles of species σ is

$$\rho_\sigma = E_\sigma N_\sigma / V \sim E_\sigma N_\sigma a^{-3} \tag{55}$$

where E_σ is the energy fper particle or quantum (or region) and N_σ is the number of particles or whatever and V is the volume. The the equation of state parameter w is

$$w = -\frac{1}{3}\left[\frac{dlnE_\sigma}{dlna} + \frac{dlnN_\sigma}{dlna}\right] \tag{56}$$

So for example nonrelativistic matter does not always have $w = 0$. If the number of particles evolves, then so does the equation of state. This is the case for decaying particles. Consider an unstable nonrelativistic particle X decaying into relativistic offspring.

$$\begin{aligned} \dot{\rho}_X &= -3H\rho_X - \Gamma\rho_X \\ \dot{\rho}_{RO} &= -4H\rho_{RO} + \Gamma\rho_X \end{aligned} \tag{57}$$

where $H = \odot a/a$ and Γ is the decay width (1/lifetime). Find E, N, and w for the parent and relativistic offspring.

Recall from the General Relativity derivation of the Friedmann equations that the 3-D space curvature ${}^3R = 6k/a^2$ We now have the relation

$${}^3R = \frac{6k}{a^2} = 6H^2(\Omega - 1)$$

The Gaussian curvature ${}^3R = 1/R^2_{curvature}$, where $R_{curvature}$ is the 3-space radius of curvature. In problem set 1 we found this to be 0.97 light years at the surface of the earth. For the Universe

$$R_{curvature} = \frac{cH^{-1}}{|\Omega - 1|^{1/2}}; \qquad R_{curvature} = \frac{cH^{-1}}{(\Omega - 1 + \Lambda/3H^2)^{1/2}}$$

Clearly $\Omega = 1$ means space is flat.

Show that

$$q_0 \equiv \frac{\ddot{a}}{a}(\frac{a}{\dot{a}})^2 = \Omega_0[1 + 3P/\rho]/2 = \Omega_0(1 + 3w)/2$$

and expand the Robertson-Walker metric to second order, that is including not only $H = \dot{a}/a$ but also q, in a Taylor series expansion and derive to that order the relations we will find now for the age of the universe or look back time for a given redshift.

3 The Expansion Age of the Universe

Starting from the relation $\dot{a} = da/dt$ and the starting time $t = 0$ when $a = 0$ we can compute the age of the Universe by doing an integration

$$t_u = \int_0^{t_u} = \int_0^{a(t_u)} \frac{da}{\dot{a}} = \int_0^{\infty} \frac{dz}{(1+z)H(z)} \tag{58}$$

We can solve this from the equations of motion *first for no cosmological constant or dark energy*

$$(\frac{\dot{a}}{a})^2 - \frac{8\pi}{3}G\rho = -\frac{K}{a^2} \tag{59}$$

or

$$\dot{a}^2 = \frac{8\pi}{3}G\rho a^2 - K \tag{60}$$

and we set the density equal to the sum of the matter and radiation densities

$$\Omega = \Omega_r + \Omega_m = \Omega_{ro}(1+z)^4 + \Omega_{mo}(1+z)^3 \tag{61}$$

remembering $k/a^2 = H_0(\Omega_0 - 1)$ we find

$$t_u = H_0^{-1} \int_0^{a(t_u)} \frac{dx}{[1 - \Omega_0 + \Omega_{mo}x^{-1} + \Omega_{ro}x^{-2}]^{1/2}} \tag{62}$$

Note we have the formula $t_u = f(\Omega_0)H_0^{-1}$ and we have an equation for calculating $f(\Omega)$ which is reasonably solved. Consider the case where $\Omega_0 = 1$

$$t_u = \frac{1}{H_0\Omega^{1/2}} \int_0^{a(t_u)=1} \frac{dx}{[\Omega_{mo}x^{-1} + \Omega_{ro}x^{-2}]^{1/2}} \tag{63}$$

which can be integrated in closed form to be

$$t_u = \frac{1}{H_0\Omega^{1/2}}(\frac{a_{eq}}{a_0})^{3/2}[\frac{2}{3}(1 + \frac{a}{a_{eq}})^{3/2} - 2(1 + \frac{a}{a_{eq}})^{1/2} + \frac{4}{3}] \tag{64}$$

where a_{eq} is the scale size of the universe where $\Omega_m = \Omega_r$, that is the energy density in radiation is equal to the energy density in matter. Note that this must happen since they have a different dependence on the scale factor. For the present universe a reasonable estimate can be made by assuming that the radiation is dominated by the cosmic background radiation of photons and neutrinos which we will discuss later. This gives

$$1 + z_{eq} = \frac{a_0}{a_{eq}} = \frac{\Omega}{\Omega_r} = 2.4 \times 10^4 \Omega h_{100}^2 \tag{65}$$

now include material with a more complicated equation of state

Assume that 'Dark Energy' is smooth and does not clump at all (or only on the very, very large ($k \sim h_0$) scales. All of its consequences (if there is no decay) arise from the modification it induces in the expansion rate:

$$H(z)^2 = H_0^2 \left[\Omega_m (1+z)^3 + \Omega_X exp3 \int_0^z (1+w(x)) dln(1+x) \right]. \tag{66}$$

For a given rate of expansion today H_0, the expansion rate in the past $H(z)$, was smaller in the presence of dark energy. Therefore, dark energy increases the age of the Universe. The comoving distance $r(z) = \int dz/H(z)$ also increases in the presence of dark energy. The same follows for the comoving volume element $dV/d\Omega dz = r(z)^2/H(z)$.

Note that for an equation of state with $w(z) =$ constant the equation for $H(z)$ becomes

$$H(z)^2 = H_0^2 \left[\Omega_m (1+z)^3 + \Omega_X (1+w)^{3(1+w)} \right]. \tag{67}$$