IDS160B - Relativistic Astrophysics and Cosmology
Problem Set 1 1995 February 15 Due:

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Show your development of the solution by equations and diagrams before the evaluation to indicate your understanding of the problem.

$$G = 6.67 \times 10^{-8}\ \mathrm{cm^2 gm^{-1} sec^{-2}} \qquad \text{Newton gravitational constant}$$

$$g = 9.81 \times 10^{2}\ \mathrm{cm\ sec^{-2}} \qquad \text{surface gravity of the Earth}$$

1 The Curvature of Space at the Earth's Surface

1.1 Curvature of Space-Time

Calculate the curvature of space at the surface of the Earth using the equivalence principle. Here for this calculation, we extend Galileo's (Newton's) principle of equivalence of inertial and gravitational mass to the equivalence of the local gravitational potential to acceleration. Assume that the curvature of space-time at the surface of the Earth is spherical The radius of the Earth is 6.38×10^8 cm and the surface gravity is 9.81×10^2 cm sec^{-2}. Express the answer both in cm and in light years.

Suggestion: Calculate the path of a photon traveling between two horizontal points assuming that it undergoes acceleration according to the Earth's surface gravity. An alternate way to approach the problem is to assume that light in free fall travels a straight path and determine how the photon's path would appear to an observer in an elevator accelerating upward from the Earth's surface. Then calculate the radius of curvature that matches the photon's path. Would you get the same answer doing the calculation for a ball or a bullet going through the same two points?

2 Black Hole Radius

Calculate the black hole radius of the Earth.

a) What is the radius for which the escape velocity would be the speed of light, if all the mass (5.98×10^{27} gm) associated directly with the Earth were concentrated in an infinitesimally small region (that is, smaller than the black hole radius)?

b) What is the radius for which the gravitational potential energy just matches the rest mass (mc^2) of a test particle?

c) What would be the speed at these radii of a test particle that came from rest at infinity? Calculate this using Newtonian physics. What would be the velocity in the case of a treatment of special relativity? Remember that $E = m(1-(v/c)^2)^{-1/2}$.

2.1 Surveying the Curvature

If Gauss's surveyors could have measured with ideal, instead of the 0.7", accuracy the sum of the angles in the approximately 100 km equilateral triangle, then what answer would they get in the cases: i.e. How different would the sum of the angles be from 180 degrees?

a) The surface of the Earth was a perfect sphere and they put out measureing tape (or line) along the surface of the Earth?

b) They used light and so you can now use the radius of curvature of space at the surface of the Earth you calculated earlier in this problem.

What sign would the difference be? I.e. is space here positively or negatively curved.

Hint: For calculating the angle difference from 180 degrees: Consider a triangle where two vertices on the equator and the third on the north pole. The two angles at the vertices will be right angles adding to 180 degrees so that the included angle at the pole will be the excess angle over 180 degrees. If it is an equilateral triangle, then the angle at the pole will be 90 degrees and the triangle's area will be one eight of the spheres. If the angle at the top is as large as possible 180 degrees then the area of the triangle will be one fourth of the sphere. For this configuration there is a clear relation between the excess angle is proportional to the area of the triangle divided by the area of the sphere: Excess angle (in radians) = $4\pi A_{triangle}/A_{sphere}$ = solid angle of triangle You can use this relation to estimate the difference in the sum of the angles from 180 degrees?

2.2 Sign of Effect

How would you make this calculation for a negatively curved space-time?

2.3 Extra Credit/Thought Provoking

How could Gauss and his surveyors (or we) tell if space was actually curved or was the angular difference from a plane simply caused by the curving of a light beam being accelerated by gravity? (the equivalence principle)

3 Minkowski Metric as the limit of the Robertson-Walker Metric

One form of the Robertson-Walker metric is:

$$ds^2 = \left\{ \begin{array}{ll} c^2dt^2 - a(t)^2[dr^2/(1-kr^2) + r^2d\Omega^2] & \quad positive \\ c^2dt^2 - a(t)^2dl^2 = c^2dt^2 - a(t)^2[dr^2 + r^2d\Omega^2] & \quad flat \\ c^2dt^2 - a(t)^2[dr^2/(1+kr^2) + r^2d\Omega^2] & \quad negative \end{array} \right.$$

where $k = 1/R^2$ is the Gaussian curvature which is positive or negative times one over the radius of curvature of space. For small differences in time dt and distance dr that the Robertson-Walker metric is well approximated by the Minkowski metric:

$$ds^2 = c^2dt^2 - [dr^2 + r^2d\Omega^2] = c^2dt^2 - [dx^2 + dy^2 + dz^2])$$

3.1

What is necessary for that to be true?

3.2

Expand the Robertson-Walker metric to first order in dt and dr and evaluate the deviation from the Minkowski metric to show how accurate the Minkowski metric is for laboratory experiments in our current universe.

Use the relation H(t) = 1/a(t) da(t)/dt which defines the Hubble expansion rate. At the present the Hubble "constant" Ho is roughly 50 km/sec/Mpc. This corresponds to an inverse time of about 1010 years. This gives us a rough scale for both the time rate of change of a(t) at the present and a curvature radius of 1010 light years (or higher).

3.3

How large are the deviations on the scale of the solar system? On the scale of the Galaxy (i.e. 30,000 light years)? Could this affect our estimate of how much dark matter is necessary to cause the rotation curve of the stars orbiting Galaxy? The total mass inside of a stars orbit is determined by using Newton's theory of Universial Gravitation and the star's orbital velocity.

4 The Bending of Light by the Sun

The successful prediction of the bending of light by the Sun was the triumph that elevated Einstein's General Relativity to the point that it got the world's attention. In this problem you will calculate the bending of light by the Sun and see if you can get the correct answer by using Newtonian physics and the principle of equivalence - so that GR was not needed - and by dimensional analysis. The mass of the Sun is $M_\odot = 1.99 \times 10^{33}$ gm. The radius of the Sun is $R_\odot = 6.96 \times 10^{10}$ cm.

4.1 Bending According to Equivalence Principle

Use the principle of equivalence - gravity = acceleration - and Newtonian physics to calculate the angle by which a light from a distant star (effectively infinity) nearly graxing the Sun's surface is bent by the time it passes us on Earth (also treat this as effectively infinity). If you need it in this approach, because of the spherical symmetry, angular momentum is conserved relative to the center of ths Sun.

4.2 Dimensional Analysis

What two numbers do you have? Make two distances by calculating the black hole radius of the Sun and compute an angle by taking the ratio of two distances. How close is your answer to what you got in 3.1?

4.3 Use the Schwarzschild metric

Use the Schwarzschild metric, i.e for a spherically symmetic mass outside of the mass:

$$ds^2 = (1 - 2GM/rc^2)c^2dt^2 - (1 - 2GM/rc^2)^{-1}dr^2 + r^2d\Omega^2$$

and calculate the angle for the bending of starlight grazing the Sun.

5 The Gravitational Redshift

This problem will give you some familiarity with the gravitational redshift.

5.1 Gravitational Redshift in a Tower

Suppose you are Pound and Rebka (1960) and you have a source of photons (57Fe)in the basement and you detect the frequency of the emitted photons with a detector on the same level in the basement and with another detector on a tower up stairs some 22.6 meters above the emitter. Using the equivalence principle calculate the fractional difference in detected frequencies. 4.2 Give a formula for the fractional change in frequency (energy) of photons when the gravitational potential changes by an amount df.

5.2 Redshift from Sun

What is the fractional difference (redshift) for photons leaving the surface of the Sun?

5.3 Redshift from Center of the Earth

Suppose the Earth were transparent, so we could see the light emitted by the hot, glowing center of the Earth. What would be the redshift of this light received at the surface of the Earth? For this calculation, pretend that the Earth is a sphere of uniform density, and ignore the rotation of the Earth.

6 Gravitational Lenses and Time Delay

Suppose a galaxy acting as a "gravitational lens" produces two "images" of a distant quasar. Suppose that the galaxy is at a distance r, midway between the quasar and observer (as illustrated). Suppose that the impact parameters of the two light rays passing by the galaxy are b1 and b2. Treating the galaxy as a point-like mass M and using the linear approximation for the relativistic gravitational field, find an expression for the difference in travel times along the two light rays. Evaluate the time delay numerically for $b_1 = 5 \times 10^5$ light years, $b_2 = 1 \times 10^5$ light-years, and $M = 10^{11}$ solar masses.

Hint: Take into account the difference in path length between the two rays and the difference in the effective speed of light in the gravitational field; in lowest order of approximation, these two contributions can be combined additively. Ignore cosmological effects, that is, pretend that the background space-time is flat and static.