IDS160B - Relativistic Astrophysics and Cosmology
Problem Set 7 1995 April

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1 Angular Diameters

This problem is to calculate or estimate the angular diameter of the light horizon and sound horizon at the surface of last scattering (or decoupling) of the cosmic background radiation. It is useful for estimating the angular diameter of galaxies and other known yard sticks.

1.1 Review/Reminder - Particle/Light Horizon

For a matter dominated universe ($w = 0$) the result of the integral over the expansion.

$$d_H = \{ \begin{array}{cc} \frac{1}{H_0(1+z)\sqrt{\Omega_0 - 1}} arccos(1 - \frac{2(\Omega_0-1)}{\Omega_0(1+z)}) & \Omega_0 > 1 \quad or \quad k = 1 \\ \frac{2}{H_0(1+z)^{3/2}} & \Omega_0 = 1 \quad or \quad k = 0 \\ \frac{1}{H_0(1+z)\sqrt{1-\Omega_0}} arccosh(1 - \frac{2(1-\Omega_0)}{\Omega_0(1+z)}) & \Omega_0 < 1 \quad or \quad k = -1 \end{array} \tag{1}$$

We expect this to be approximately correct at $z \sim 1100$, as we expect that the universe is matter dominated (horizon growing as $3ct$) since $z_{eq} > 10^4$.

1.2 Some handy formulas for getting by in curved space

This section contains a collection of useful formulas, many of which are used in matter dominated universe. It is intended as more of a review and reference section expanding the material just above.

1.2.1 Friedmann solutions, conformal time

Since $a_0/a = 1+z$, the Friedmann equation (neglecting radiation) can also be written

$$H = H_0\sqrt{\lambda_0 + (1+z)^2(1 - \lambda_0 + \Omega_0 z)}$$

or, alternatively,

$$-\frac{dz}{dt} = H_0(1+z)\sqrt{\lambda_0 + (1+z)^2(1 - \lambda_0 + \Omega_0 z)} \tag{2}$$

where λ_0 is the effective cosmological constant energy density and Ω_0 is the conventional energy density. In the rest of this discussion, we will set $\lambda_0 = 0$ unless the contrary is explicitly stated. This means that the Friedmann equation can be solved analytically, the $\Omega > 1$ solution being the ubiquitous cycloid.
It is often convenient to introduce a new time coordinate η called *conformal time*, defined by

$$\eta(t) \equiv \int_0^t \frac{dt'}{a(t')}.$$

In terms of the conformal time, the solutions $a(t)$ to the Friedmann equation can be written in the parametric form

$$H_0 t = \{ \begin{array}{ll} \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}(\sinh\eta - \eta) & \text{for } \Omega_0 < 1, \\ \frac{2}{3}\left(\frac{\eta}{\eta_0}\right)^3 & \text{for } \Omega_0 = 1, \\ \frac{\Omega_0}{2(\Omega_0-1)^{3/2}}(\eta - \sin\eta) & \text{for } \Omega_0 > 1, \end{array} \tag{3}$$

$$\frac{a}{a_0} = \frac{1}{1+z} = \{ \begin{array}{lll} \frac{\Omega_0(\cosh\eta-1)}{2(1-\Omega_0)} & \text{for} & \Omega_0 < 1, \\ \left(\frac{\eta}{\eta_0}\right)^2 & \text{for} & \Omega_0 = 1, \\ \frac{\Omega_0(1-\cos\eta)}{2(\Omega_0-1)} & \text{for} & \Omega_0 > 1. \end{array} \tag{4}$$

When $\Omega_0 \neq 1$, η is often called the *development angle*. When $\Omega_0 = 1$, $\eta_0 = 2/(H_0 a_0)$, where the constant a_0 can be chosen arbitrarily. Again, this is because the radius of curvature, which fixed a_0 when $\Omega_0 \neq 1$, is infinite when the universe is flat. In all cases, $\eta = 0$ corresponds to the Big Bang. Inverting these last expressions, we obtain

$$\eta \equiv \{ \begin{array}{ll} 2\text{arcsinh}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0(1+z)}}\right) & \text{for } \Omega_0 < 1, \\ \frac{\eta_0}{\sqrt{1+z}} & \text{for } \Omega_0 = 1, \\ 2\text{arcsin}\left(\sqrt{\frac{\Omega_0-1}{\Omega_0(1+z)}}\right) & \text{for } \Omega_0 > 1. \end{array}$$

Thus the conformal time today is

$$\eta_0 = \{ \begin{array}{ll} 2\text{arcsinh}\left(\sqrt{\Omega_0^{-1}-1}\right) & \text{for } \Omega_0 < 1, \\ 2\text{arcsin}\left(\sqrt{1-\Omega_0^{-1}}\right) & \text{for } \Omega_0 > 1. \end{array}$$

Although one cannot give an explicit expression for a in terms of t in general,

$$\frac{a}{a_0} \approx \Omega_0^{1/3}\left(\frac{3}{2}H_0 t\right)^{2/3}$$

for $t \ll H_0^{-1}$.

1.2.2 Causality, horizons, angles

A useful property of η is that during a conformal time interval $d\eta$, a light ray always travels the same comoving distance $d\eta$. If $\Omega = 1$ so that space is flat, the situation becomes especially simple when viewed in $(\eta, \overline{\mathbf{x}})$-space: all light rays (null geodesics) simply move in straight lines, and always make 45 degree angles with the η axis. Thus after changing variables to comoving spatial coordinates and conformal time, we can use all our intuition about light propagation and causality in good old non-expanding Euclidean space.

Parts of the universe receding from us faster than the speed of light are not in causal contact with us. No signals have ever been able to reach a comoving object in the universe from objects that are outside of its backward light cone. Such objects are said to be outside of its *horizon*. Thus defining the *horizon radius* as $a\eta$ and combining some of the previous expressions, we obtain

$$r_h \equiv a\eta = \left\{ \begin{array}{ll} \frac{H_0^{-1}}{(1+z)\sqrt{1-\Omega_0}}\cosh^{-1}\left[1+\frac{2(1-\Omega_0)}{\Omega_0(1+z)}\right] & \text{if } \Omega_0 < 1, \\ \frac{2H_0^{-1}}{(1+z)^{3/2}} & \text{if } \Omega_0 = 1, \\ \frac{H_0^{-1}}{(1+z)\sqrt{\Omega_0-1}}\cos^{-1}\left[1-\frac{2(\Omega_0-1)}{\Omega_0(1+z)}\right] & \text{if } \Omega_0 > 1. \end{array} \right. \tag{5}$$

If an object at redshift z has a proper diameter D (perpendicular to us), then one can show that it will subtend an angle on the sky given by

$$2\tan\frac{\theta}{2} = \left\{ \begin{array}{ll} \frac{\Omega_0^2(1+z)\sinh\left[\frac{1}{2}(1+z)\sqrt{1-\Omega_0}H_0D\right]}{\sqrt{1-\Omega_0}\left[\Omega_0 z-(2-\Omega_0)(\sqrt{1+\Omega_0 z}-1)\right]} & \text{if } \Omega_0 < 1, \\ \frac{(1+z)^{3/2}H_0D}{2(\sqrt{1+z}-1)} & \text{if } \Omega_0 = 1, \\ \frac{\Omega_0^2(1+z)\sin\left[\frac{1}{2}(1+z)\sqrt{\Omega_0-1}H_0D\right]}{\sqrt{\Omega_0-1}\left[\Omega_0 z-(2-\Omega_0)(\sqrt{1+\Omega_0 z}-1)\right]} & \text{if } \Omega_0 > 1. \end{array} \right.$$

Thus if we look at an object at a redshift z, its horizon radius subtends an angle θ in the sky that is given by

$$2\tan\frac{\theta}{2} = \frac{\Omega_0^{3/2}\sqrt{1+z}}{\Omega_0 z-(2-\Omega_0)(\sqrt{1+\Omega_0 z}-1)},$$

for all values of Ω_0. For $\Omega_0 = 1$, this takes the particularly simple form

$$2\tan\frac{\theta}{2} = \frac{1}{\sqrt{1+z}-1},$$

whereas for large z, it reduces to

$$\theta \approx \sqrt{\frac{\Omega_0}{z}}.$$

This last expression is a rather poor approximation for for typical values of Ω_0, unless $z \gg 100$.

1.2.3 Age of the universe

The current age of the universe, t_0, is given by

$$t_0 = \int dt = \int_0^\infty (aH)^{-1}(z) \left|\frac{da}{dz}\right| dz = \int_0^\infty \frac{dz}{(1+z)H(z)}.$$

Performing this integral for $\lambda_0 = 0$ gives the age of the universe as

$$H_0 t_0 = \{ \begin{array}{cl} \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\cosh^{-1}\left(\frac{2}{\Omega_0} - 1\right) & \text{for } \Omega_0 < 1, \\ \frac{2}{3} & \text{for } \Omega_0 = 1, \\ \frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\cos^{-1}\left(\frac{2}{\Omega_0} - 1\right) - \frac{1}{\Omega_0 - 1} & \text{for } \Omega_0 > 1. \end{array}$$

For the flat case where $k = 0$ and $\Omega_0 = 1 - \lambda_0$, the result is

$$H_0 t_0 = \frac{1}{\lambda_0^{1/2}} \ln \left[\frac{1+\lambda_0^{1/2}}{(1-\lambda_0)^{1/2}}\right].$$

1.3 Angle of Light & Sound Horizon

Use the formulae above to get an approximate answer for the angular diamter of the light and sound horizons at the surface of last scattering. Hint: Put your answer in the form

$$\theta_H \approx \frac{1}{2}\sqrt{\frac{\Omega_0}{z_{dec}}} \cong ?\sqrt{\frac{\Omega_0}{(z_{dec}/1100)}} \tag{6}$$

where the approximation is valid for $\frac{\Omega_0}{(z_{dec}/1100)} << 1$. Make the rough approximation that $v_s = c/\sqrt{3}$ which is fairly accurate up until the decoupling epoch.

2 Earth angular diameter from a Rocket

A space ship accelerates away from the Earth at a constant acceleration g in the space ship frame.
(a) Calculate the angular diameter of the Earth as viewed from the space ship as a function of proper time on the space ship.
(b) Show that the angular size approaches a limit as the space ship proper time gets large. What is that size?

3 Anisotropy from a Collapsing Structure

Consider a CBR photon passing through a large massive structure that is collapsing (going non-linear). During the time taken for the photon to cross the structure, the gravitational potential well changes by a non-negligible amount. Estimate the value of $\delta T/T$ induced by such a process.

4 Anistropy from a Cosmic String

Again we consider a cosmic string with mass per unit length μ. For a straight cosmic string on the z-axis, the metric line element, in cylindrical coordinates, is

$$\begin{aligned} ds^2 &= dt^2 - dr^2 - (1-4G\mu)^2\, r^2\, d\theta^2 - dz^2 \\ &\approx dt^2 - dr^2 - (1-8G\mu)\, r^2\, d\theta^2 - dz^2, \end{aligned}$$

where the mass per unit length is $\mu = \pi r_0^2 \rho_0$ provided that $G\mu \ll 1$.

In a previous problem set we saw that space-time is flat around such a cosmic string but it is like a flat piece of paper with missing wedge of angle $\phi = 8\pi G\mu$. As a result the geometry is like a cone.

As a result of the conical geometry and the associated deficit wedge angle, strings can act as lenses. In the previous problem set we considered a source, an observer, and a cosmic string, all at rest in an otherwise empty flat space-time, with the observer and the source on opposite sides of the string. Let $r_{\rm s}$ denote the proper distance from the observer to the string, and $r_{\rm q}$ denote the proper distance from the string to the source. Orient the coordinate system so that the string passes through $z = 0$, and so that the observer is at $\theta' = \pi - (8\pi G\mu/2)$ on the surface of a cone at that point where the base of the cone would have had radius $r_{\rm s}$. Also, assume that the source is at $\theta' = 0$ on the surface of the same cone, but at that point where the base would have radius $r_{\rm q}$. Now imagine cutting through the cone along the line that passes through the point that contains the source and the apex of the cone, and laying the cone down flat. In this flat space, the cone appears to have a missing wedge (see Figure 1).

Arrange the coordinate system on the flat space to be oriented similarly to that on the cone; the string is at the origin and the observer is at $(r_{\rm s}, \pi - [8\pi G\mu/2])$. However, the source appears to be at two positions, $(r_{\rm q}, 0)$ and $(r_{\rm q}, 2\pi - 8\pi G\mu)$. (Recall that the wedge deficit geometry means that all points with $\theta = 0$ and $\theta = 2\pi(1-4G\mu)$ are to be identified.) Since, in this space-time, geodesics (the paths travelled by light) are straight lines, just as they are with the Minkowski metric, the angle between the two images of the source can be calculated using simple trigonometry. Each image will be at an angle ϕ on either side of the string, where

$$\begin{aligned} \sin\phi &= r_{\rm q}\sin(4\mathrm{G}\mu) \times \left(\mathrm{r_s^2 + r_q^2 + 2r_sr_q}\cos(4\mathrm{G}\mu)\right)^{-1/2} \\ &\approx r_{\rm q}\sin(4\mathrm{G}\mu) \times \left(\mathrm{r_s^2 + r_q^2 + 2r_sr_q}\left(1 - \frac{(4\mathrm{G}\mu)^2}{2}\right)\right)^{-1/2} \\ &\approx r_{\rm q}\, 4G\mu \times \left((r_{\rm s}+r_{\rm q})^2\left(1 - \frac{r_{\rm s}r_{\rm q}\,(4G\mu)^2}{(r_{\rm s}+r_{\rm q})^2}\right)\right)^{-1/2} \\ &\approx \frac{r_{\rm q}}{r_{\rm s}+r_{\rm q}}\, 4G\mu. \end{aligned}$$

where the final expressions assume that $4G\mu \ll 1$. In this limit, and since $r_{\rm q}/[r_{\rm s}+r_{\rm q}]$ is also less than unity, $\sin\phi \approx \phi$. Thus, the angle between the two images is

$$2\phi \approx 8G\mu\, r_{\rm q}\,(r_{\rm s}+r_{\rm q})^{-1}.$$

The conical metric also leads to discontinuities in the temperature of the cosmic background radiation. Consider the source to be a point on the last scattering surface (decoupling of matter and radiation). An observer at rest with respect to the string will see two images of the same point on the last scattering surface, separated by an angle $\approx 8\pi G\mu$ as long as the string is significantly closer to the observer than the last scattering surface. The brightness and temperature will be the same and there will be no anisotropy.

However, if there is relative motion between the string and the observer perpendicular to the line of sight, the momentum vector of one image will have a small component parallel to the direction of the velocity and the other a small component antiparallel. The net effect is a small Doppler shift of the radiation temperature of the two images. Show that this is of the order

$$\frac{\delta T}{T} \cong 8\pi G\mu v \qquad (7)$$

where v is the relative transverse velocity. This should be a line step-change of temperature.

Estimate the size of the anisotropy for a string with mass per unit length μ which is of the GUT scale so that $G\mu \sim 10^{-6}$ using what you have learned about typical velocities for cosmic strings.

If the measured CBR anisotropies are $\frac{\delta T}{T} \leq 10^{-5}$, then what is the limit on μ.

5 Anisotropies from Perturbations

What is the typical CBR anisotropy expected from adiabatic potential fluctuations, if the observed typical kinetic galactic velocities in a galactic cluster are of order 500 km/sec?