P139 - Relativity
General Relativity Introduction

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1 GRAVITATIONAL PHYSICS

1.1 Introduction Relativistic Gravity

Our approach to gravity is to treat it as an effect of geometry. One could treat gravity as a field theory and, in fact, we will have to make reference to the field theory from time to time. This is specifically true since we will have to demonstrate that our geometric treatment reduces to the Newtonian field theory – Newton's Theory of Universal Gravitation – in the appropriate limit.

Newton's theory is that each and every mass, M (and ultimately every form of energy) produce a gravitational field in such a way as to produce a gravitational force, F_N, on a test mass m

$$F_N = -\frac{GMm}{r} \tag{1}$$

where r is the difference between the two masses and the force is negative to indicate it is attractive. The corresponding potential for this force is

$$\phi = \frac{GM}{r} \quad \phi = \int\int\int \frac{G\rho}{r} d^3x' \tag{2}$$

Few theories compare in the accuracy of their predictions with Newton's theor of universal gravitation. The predictions of the celestial mechanics for the orbits of the major planets agree with observations to within a few seconds of arc over many years. Neptune was discovered as a consequence of an anomaly in these predictions and is regarded as a spectacular success testifying to the accuracy of Newton's theory.

However, Newton's theory is not perfect and needs some correction to be relativistically correct. There are five areas one can point out:
(1) The predicted motions of the perhelion of Mercury is in disagreement by 43 arcseconds per century. This small deviation was discovered by LeVerrier in 1845, and it was recalculated by Newcomb in 1882.

$$\begin{aligned} \phi_N^{100} &= 5557.62'' \pm 0.20'' \\ \phi_{Obs}^{100} &= 5600.73'' \pm 0.40'' \\ \Delta\phi &= 43.11'' \pm 0.45'' \end{aligned} \tag{3}$$

Many explantions were suggested such as the existence of a new planet closer to the Sun than Mercury. Named Vulcan because it would be in such a hot location.

A quadrupole moment for the Sun could cause such an advance. The explanation of this perihelion precession was one of the early successes of Einstein's relativistic theory of gravitation.
(2) Newton's theory did not explain the equality of inertial and gravitational mass but postulates it so that there is a second equation

$$m_I = m_G \tag{4}$$

called Newton's Principle of Equivalence Our geometric interpretation of gravitation will have the equivalence of passive (that acted on) gravitational mass and inertial mass as an automatic consequence.
(3) The Equivlance Principle is an extension of Galilean and special relativity to accelerated frames and to situations that include gravitation. One expression of the Equivalence Principle is: "No experiment can distinguish between the inside of an elevator that is uniformly accelerating by an amount g and one that is stationary in a gravitational field having uniform strength g over the dimensions of the elevator."
(4) Bending of Light by the Sun

1.2 Geometric Interpretation

What path does a body follow, if no non-gravitational forces act upon it? The answer is a geodesic.

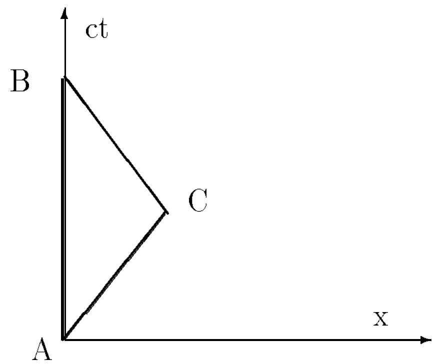
First consider the solution to this problem in Special Relativity. The body moves with uniform speed along a straight path. The world lines as measured from an inertial frame is given by four equations

$$\frac{d^2t}{d\tau^2} = 0; \qquad \frac{d^2\vec{r}}{d\tau^2} = 0. \tag{5}$$

i.e. non acceleration and Newton's first law.

But these equations are not in generally covariant form. That is they depend upon the coordiante system.

In 4-D Minkowki space this can be expressed in a coordinate independent way. A particle acted on by no forces travels the path of maximum proper time. We are used to the 3-space path being a straight line and thus the minimum distance between the two points. However, it is straight-forward to demonstrate in Minkowski 3+1-D space, the path is one of maximum proper distance. Consider the following diagram where we choose a frame colocated on the test body at $t = 0$. In that frame $\tau^2_{AB} = t^2_{AB} - 0 - 0 - 0$ for the path of the body remaining at rest so $\tau_{AB} = t_{AB}$.



$$
\begin{aligned}
\tau_{ACB} &= \tau_{AC} + \tau_{CB} \\
&= \sqrt{t_{AC}^2 - x_C^2/c^2} + \sqrt{t_{CB}^2 - x_C^2/c^2} \\
&\leq t_{AB}
\end{aligned} \tag{6}
$$

Note that this is a clear and straightforward solution to the 'twin paradox'.

1.2.1 The Newtonian Approximation

First consider a static gravitational field and a corresponding static coordinate system. This meas that $g_{\mu\nu}$ are constant in time and thus $g_{\mu\nu,0} = 0$

Further, we must have $g_{m0} = 0$, where $m = 1, 2, 3$ our usual notation. Thus

$$g^{m0} = 0, \qquad g^{00} = (g_{00})^{-1}$$

and thus

$$,_{m0n} = 0, \quad \text{and} \quad ,^{m}_{0n}$$

Secondly consider a slowly moving particle, $v \ll c$. Then u^m is small, that is of first order and we will be neglecting terms in second order. This leads to the following relationship

$$c^2 = g_{\mu\nu} u^\mu u^\nu \rightarrow g_{00} \left(u^0\right)^2$$

Thirdly, we now assume that the particle will move along a geodesic:

$$\frac{dv^\sigma}{ds} + ,^{\sigma}_{\mu\nu} u^\mu u^\nu = 0$$

If we now neglect second order quantities, the geodesic equation reduces to

$$\frac{dv^\sigma}{ds} \simeq -,^{m}_{00} \left(u^0\right)^2 = -g^{mn} ,_{n0m} \left(u^0\right)^2 = \frac{1}{2} g^{mn} g_{00,n} \left(u^0\right)^2$$

Recalling the derivative chain rule

$$\frac{du^m}{ds} = \frac{du^m}{dx^\mu} \frac{dx^\mu}{ds} \simeq \frac{du^m}{dx^0} u^0$$

Thus

$$\frac{du^m}{dx^0} = \frac{1}{2} g^{mn} g_{00,n} u^0 = g^{mn} \left(g_{00}^{1/2}\right)_{,n} c$$

where the last equality comes form $g_{00,n} \left(u^0\right)^2 = c^2$.

Since the $g_{\mu\nu}$ are independent of x^0, we may lower the suffix m and get

$$\frac{du_m}{dx^0} = \left(g_{00}^{1/2}\right)_{,n} c; \quad a_m = \left(g_{00}^{1/2}\right)_{,n} c^2 \tag{7}$$

The particle fallowing the geodesic moves as though it were under the influence of a potential whose gradient is equal to the gradient of $g_{00}^{1/2} c^2$.

Suppose that the gravitational field is weak so that the curvature of the coordinate lines (assumed to be well-chosen) is small. Then to first order

$$,^{\alpha}_{\mu\alpha,\nu} - ,^{\alpha}_{\mu\nu,\alpha} = 0$$

($g_{\mu\nu}$ is approximately constant, $g_{\mu\nu,\sigma}$ and , are small.)

$$g^{\rho\sigma} \left(g_{\rho\sigma,\mu\nu} - g_{\nu\sigma,\mu\rho} + g_{\mu\nu,\rho\sigma}\right) = 0$$

Consider the $\mu = 0$, $\nu = 0$ case:

$$\begin{aligned} g^{mn} g_{00,mn} &= 0 \\ g^{\mu\nu} \phi_{,\mu\nu} &= 0 \\ g_{00} &= 1 + 2\phi/c^2 \\ g_{00}^{1/2} &= 1 + \phi/c^2 \end{aligned} \tag{8}$$

Sample Gravitational Potential Strengths

At Surface of	ϕ/c^2
proton	10^{-39}
Earth	10^{-9}
Sun	10^{-6}
White Dwarf	10^{-4}

Another way to see this is the following approach that also starts with the geodesic equation and makes the Newtonian approximation outlined above.

$$,^{\mu}_{00} = g^{\mu\nu} ,_{\nu 00} = \frac{1}{2} g^{\mu\nu} \left(g_{\nu 0,0} + g_{\nu 0,0} - g_{00,\nu}\right)$$

The first two terms in the parenthesis are zero since the metric is stationary in time – static mass distribution. Thus

$$\frac{du^\mu}{dt} = \frac{d^2 x^\mu}{dt^2} = -,_{\nu 00} = \frac{1}{2} g^{\mu\nu} g_{00,\nu} \simeq -\frac{1}{2} g_{00,\nu}$$

The Newtonian equations of motion are

$$a^m = \frac{dv^m}{dt} = \frac{d^2x^m}{dt^2} = -\frac{\partial\phi}{\partial x^m}$$

Therefore $g_{00,m} = 2\frac{\partial\phi}{\partial x^m}\frac{1}{c^2} = (2\phi)_{,m}/c^2$ and

$$g_{00} = 1 + 2\phi/c^2.$$

1.3 Linearized Relativistic Gravitation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{9}$$

Where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is the weak field perturbation.

$$dp_\mu = -\left(h_{\mu\alpha,\beta} - \frac{1}{2}h_{\alpha\beta,\mu}\right)p^\alpha dx^\beta \tag{10}$$

1.4 Geodesic Deviation and Tidal Forces

Consider a curve in space-time, filled with a vector field A^μ. Then consider the displacement along the curve in terms of "length" s or proper time τ – the parameter for the curve. The rate of change of A^μ along the curve is $dA^\mu/d\tau$. However, how much is A^μ "really" changing? As opposed to the part of the change due to the curvilinear coordinates? We know that to determine that we use the covariant derivative.

$$\frac{DA^\mu}{D\tau} \equiv A^\mu_{;\beta}$$

$$\frac{DA^\mu}{D\tau} = \frac{dA^\mu}{d\tau} + ,^{\mu}_{\alpha\beta} A^\alpha \frac{dx^\beta}{d\tau}$$

which is the derivative along the curve.

$$\frac{D^2A^\mu}{D\tau^2} = \frac{D}{D\tau}\left(\frac{DA^\mu}{D\tau}\right)$$

So we can now write

$$\frac{D^2s^\mu}{Ds^2} = -R^\mu_{\alpha\sigma\beta}s^\sigma\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

The non-relativisitic limit has

$$\frac{dx^{\alpha\prime}}{d\tau} \simeq (1,0,0,0)$$

$$\frac{d^2s^{\mu\prime}}{dt^{\prime 2}} \simeq -R^{k\prime}_{0\ell 0}s^{\ell\prime}$$

The tidal force is then

$$f^k \simeq m_o R^{k\prime}_{0\ell 0}s^{\ell\prime}$$

1.5 The Schwarzschild Metric

1.6 The Deflection of Light

1.7 The Retardation of Light

In relativistic gravitation light closer to a mass (energy density) move more slowly. In a weak gravitational field

$$c^2 d\tau^2 = -ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) \quad (11)$$

Light will follow a null-geodesic, i.e. $d\tau = 0$. Thus

$$c = \left[\frac{v_r}{\left(1 - \frac{2GM}{c^2 r}\right)^2} + \frac{v_t}{\left(1 - \frac{2GM}{c^2 r}\right)} \right]^{1/2} \quad (12)$$

where v_r is the radial velocity and v_t is the transverse component of velocity.

$$v_r = \left(1 - \frac{2GM}{c^2 r}\right) c$$

$$v_t = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} c$$

Thus light closer to the mass appears to move more slowly. There is no frequency dependence and thus the group velocity is equal to the phase velocity and the speed of light is truly slower. Note that it is also anisotropic. This will also be true for other particles, objects, etc.

The time lag to travel along a radial path is

$$\Delta t = \int \frac{dr}{v_r} = \int \frac{dr}{c\left(1 - \frac{2GM}{c^2 r}\right)}$$

The difference in relativistic time and naive non-relativitistic is

$$\delta\Delta t = \int \frac{dr}{c} \left(1 - \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)}\right) \simeq \frac{\Delta r}{c} \frac{2GM}{c^2 r}$$

where the last near equality hold for the weak field regime.

1.8 The Gravitational Redshift

1.9 The Geometric Interpretation Revisited

1.10 Einstein's Theory of Gravitation

1.11 The Schwarzschild Metric

1.12 The Binary Pulsar

The binary pulsar, PSR 1913 + 16, was discovered by Hulse and Taylor (1975) in 1974. They quickly realized that apparent changes in the pulsar frequency could be explained by the Doppler effect due to the orbital motion about an unseen companion. The orbit had a period of about 8 hours. The presence of a very high-precision clock (the pulsar) moving with a velocity of about 300 kilometers per second through the gravitational field of its campanion provides a powerful test bed for relativity of gravitation. Nature had provided us with a system for testing various relativistic effects by studying the arrival times of pulsar pulses – effectively the period of the pulsar.

The binary pulsar checks the advance of the periastron, $4.2°$ per day, the gravitational (Shapiro) time delay, power of emitted gravitational radiation, and the combined second-order Doppler effect and the gravitational redshift.

One can work out the expected pulse arrival time by determining the orbital and dynamical parameters of the binary pulsar pair in the usual manner of treating spectroscopic binary star pairs.

Denote the pulsar mass as M_1 and the companion mass as M_2. If they can be treated as spherical, then to first order they move in elliptical obits around their common center of mass. Define the orbit plane as the $x-y$-plane and then the inclination to the line of sight is angle i. The axis is set along the line of nodes created by the intersection of the plane normal to the line of sight and the orbital plane. Let ω be the angular distance of the periastron from the node, measured in the orbital plane. The the position of the pulsar is given by

$$x = r_1 cos\psi, \qquad y = r_1 sin\psi \tag{13}$$

where

$$\psi = \omega + \phi, \qquad r_1 = \frac{a_1\left(1-e^2\right)}{1+1+ecos\phi}. \tag{14}$$

The angle ϕ, the polar coordinate measured from periastron, is called the "true anomaly" in celestial mechanics.

The ratio of received to emitted period of the pulsar can be written

$$\frac{\delta t_{rec}}{\delta t_{em}} = \frac{\delta t_{rec}}{\delta t_{stat}}\frac{\delta t_{stat}}{\delta t_{em}} \tag{15}$$

where "stat" denotes the period of a pulsar at the position of the pulsar but is stationary (not moving to take out Doppler effect) with respect to the center of mass. Assume for the moment that the receiver (rec) at the Earth is also static with respect tot he center of mass. The if r is the distance between M_1 and M_2,

$$\frac{\delta t_{rec}}{\delta t_{stat}} = \left(1 - \frac{GM_2}{c^2 r}\right)^{-1} \tag{16}$$

because of the gravitational redshift. The Doppler formula gives

$$\frac{\delta t_{stat}}{\delta t_{em}} = \frac{(1 + \vec{v}_1 \cdot \hat{n}/c)}{(1 - v_1^2/c^2)^{1/2}} \tag{17}$$

where $\hat{n}$ is the unit vector pointing along the line of sight from the Earth to the emitting pulsar. To first order in v_1/c and $GM_2/c^2 r$,

$$\frac{\delta t_{rec}}{\delta t_{em}} = 1 + \vec{v}_1 \cdot \hat{n}/c + \frac{1}{2}\frac{GM_2}{c^2 r} \tag{18}$$

$$\hat{n} = \vec{e}_{z'} = cos(i)\hat{e}_z + sin(i)\hat{e}_y \tag{19}$$

So that

$$\vec{v}_1 \cdot \hat{n} = \left(\dot{r}_1 sin\psi + r_1\dot{\psi}\right) sin(i) \tag{20}$$

From Kepler's second law and the formula relating r_1 and ϕ

$$\dot{\phi} = \frac{2\pi}{P\left(1 - e^2\right)^{3/2}} \left(1 + ecos\phi\right)^2, \tag{21}$$

So that

$$\vec{v}_1 \cdot \hat{n} = K\left[cos(\omega + \phi) + ecos\omega\right] \tag{22}$$

where

$$K \equiv \frac{2\pi a_1 sin(i)}{P\left(1 - e^2\right)^{1/2}} \tag{23}$$

Thus far the analysis is exactly the analysis for a single-line spectroscopic binary, with the important difference that δ_{em}, not being from a spectral line, is not known. Thus any constant term on the right-hand side of the equation is not measureable observationally; it must simply be absorbed in the emitted period. In particular, a uniform velocity between the solar system center of mass and the binary pulsar center of mass is not measurable. The Earth's orbital motion around the Sun leads to a Doppler effect that must be accounted for using the known velocity of the Earth in the solar system.

From the first-order Doppler term the following parameters can be determined: e and P using the equation for $\dot{\phi}$ which gives $\phi(t)$ when integrated. K and ω from the two independent from the two independent time-varying terms proportional to cosϕ and sinϕ in the euqtion for $\vec{v}_1 \cdot \hat{n}$. From the equation for K gets $a_1 sin(i)$, and from P and $a_1 sin(i)$ one gets the mass function

$$f \equiv \frac{(M_2 sin(i))^3}{(M_1 + M_2)^2} = \frac{(a_1 sin(i))^3}{G} \left(\frac{2\pi}{P}\right)^2 \tag{24}$$

in the usual way.

Because of the high precision of pulsar timing, the transverse Doppler shift and gravitational redshift terms can also be measured. One finds

$$v_1^2 = \dot{r}_1^2 + r_1^2 \dot{\psi}^2 = \left(\frac{2\pi}{P}\right) \frac{a_1^2}{1 - e^2} \left(1 + 2e cos\phi + e^2\right) \tag{25}$$

and

$$\frac{GM_2}{r} = \frac{GM_2^2}{(M_1 + M_2)\, r_1}. \tag{26}$$

From Kepler's third law gives

$$\left(\frac{2\pi}{P}\right)^2 = \frac{GM_2^2}{(M_1 + M - 2)^2\, a_1^3}, \tag{27}$$

we get

$$\frac{1}{2} v_1^2 + \frac{GM_2}{r} = \beta cos\phi + \text{constant} \tag{28}$$

where

$$\beta \equiv \frac{GM_2^2\, (M_1 + 2M_2)\, e}{(M_1 + M_2)^2\, a_1\, (1 - e^2)}. \tag{29}$$

That only one new measurable quantity would arise from the second-order Doppler shift and gravitational redshift could have been foreseen from the virial theorem. However, notice that the time dependence is exactly the same as that of the first-order term $K cos\omega cos\phi$. For pure elliptic motion, β is not measurable.

Fortunately, General Relativity shows us that the orbit is not exactly and ellipse. There is a periastron advance given by

$$\dot{\omega} = \frac{6\pi G M_2}{a_1\, (1 - e^2)\, P c^2} \tag{30}$$

The measured value of $\dot{\omega}$ is about $4.2°$ per year for the binary pulsar. This is compared to the 43 seconds of arc per century for the planet Mercury. Thus if we let $\omega \to \omega_0 + \dot{\omega} t$, there are clearly four independent time-varying trignometric combinations of ϕ and $\dot{\omega} t$. Thus on a time scale of years one can separate K, ω_0, $\dot{\omega}$, and β. In particular, $\dot{\omega}$ and β involve two *different* combinations of the four parameters: M_1, M_2, a_1, and $sin(i)$ from the mass function and $a_1 sin(i)$. Thus the measurement of $\dot{\omega}$ and β allows a complete solution for the parameters of the binary system.

We can predict $\dot{P}$ for a binary system given all the system parameters. If it agrees that provides strong indirect evidence for gravitational waves since the power from gravitational waves is significant.

As the timing accuracy improves, it becomes possible to measure further relativistic effects. One of these is the time-delay of signals as they cross the orbit on the way to the Earth. In addition there are various post-Newtonian periodic deviations from elliptic motion, not yet verified in the solar system. For General Relativity each of these terms contains a known different combination of M_1, M_2, a_1, and $sin(i)$. As more of these terms are measured and if they agree with the General Relativistic predictions, then the case for General Relativity and gravitational waves becomes stronger.