Physics 139 Relativity
Relativity Notes 2002

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http://aether.lbl.gov/www/classes/p139/homework/homework.html

5 Four Vectors

A natural extension of the Minkowski geometrical interpretation of Special Relativity is the concept of four dimensional vectors. One could also arrive at the concept by looking at the transformation properties of vectors and noticing they do not transform as vectors unless another component is added. We define a four-dimensional vector (or four-vector for short) as a collection of four components that transforms according to the Lorentz transformation. The vector magnitude is invariant under the Lorentz transform.

5.1 Coordinate Transformations in 3+1-D Space

One can consider coordinate transformations many ways: If $x_1, x_2, x_3, x_4 = x, y, z, ict$, then ordinary rotations (in $x_1 - x_2$ plane around x_3)

$$\begin{aligned} x_1' &= x_1 cos\theta + x_2 sin\theta \\ x_2' &= -x_1 sin\theta + x_2 cos\theta \end{aligned} \qquad \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

But in $x_1 - x_4$ plane:

$$\begin{aligned} x_1' &= x_1 cos\alpha + x_4 sin\alpha \\ x_4' &= -x_1 sin\alpha + x_4 cos\alpha \end{aligned} \qquad \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix}$$

where the angle α is defined by

$$cos\alpha = 1/\sqrt{1 - v^2/c^2} = 1/\sqrt{1 + tan^2\alpha} = \gamma$$

$$sin\alpha = i\frac{v/c}{\sqrt{1 - v^2/c^2}} = \frac{tan\alpha}{\sqrt{1 + tan^2\alpha}}$$

$$tan\alpha = iv/c = i\beta.$$

And thus one has the trignometic identity:

$$cos^2\alpha + sin^2\alpha = \gamma^2\left(1 - \beta^2\right) = 1$$

$$x_1' = \gamma\left[x_1 + (ict)(i\beta)\right] = \gamma\left[x_1 - \beta ct\right]$$
$$x_4' = \gamma\left[x_4 - i\beta x_1\right]$$
$$ict' = \gamma\left[ict - i\beta x_1\right]$$
$$ct' = \gamma\left[ct - \beta x_1\right]$$
$$t' = \gamma\left[t - \beta x_1/c\right]$$

So the extension to 3+1-D includes Lorentz transformations, if angles are imaginary.

Really, we are considering the set of all 4×4 orthogonal transformations matrices in which one angle may be pure imaginary.

In general all angles may be complex, combining real rotations in 2-space with imaginary rotations relative to t.

An alternate way of writing this is

$$x' = xcosh\phi - ctsinh\phi$$
$$ct' = -xsinh\phi + ctcosh\phi$$

where $\phi = cosh^{-1}\gamma$.

$$x' = xcos(i\phi) + ictsin(i\phi)$$
$$ict' = -xsin(i\phi) + ictcos(i\phi)$$

and

$$\alpha = i\phi = icosh^{-1}\gamma, \qquad tan\alpha = i\beta = iv/c$$

Still another notation is (with $x_4 = ict$)

$$x_1' = \gamma\left(x_1 + i\beta x_4\right)$$
$$x_4' = \gamma\left(x_4 - i\beta x_1\right)$$

The transformation matrix is then

$$\begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

Still yet another notation is with $x_0 = ct$

$$x_0' = \gamma\left(x_0 - i\beta x_1\right)$$
$$x_1' = \gamma\left(x_1 + i\beta x_0\right)$$

The transformation matrix is then

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

5.1.1 Generalized Lorentz Transformation

For spatial coordinates the Lorentz transform fits the linear form

$$(x^{\mu})' = \sum_{\nu=1}^{4} \Lambda^{\mu}_{\nu} x^{\nu} \tag{1}$$

subject to the condition that the proper length

$$(cd\tau)^2 = -(ds)^2 = \sum_{\mu}(x^{\mu})' = \sum_{\mu} x^{\mu} = (ct)^2 - |\vec{x}|^2 \tag{2}$$

is an invariant. This condition requires that the coefficients Λ^{μ}_{ν} form an orthogonal matrix:

$$\begin{aligned} \sum_{\alpha} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\alpha} &= \delta^{\mu\nu} \\ \sum_{\alpha} \Lambda^{\alpha}_{\mu} \Lambda^{\alpha}_{\nu} &= \delta_{\mu\nu} \\ \sum_{\alpha} \Lambda^{\alpha}_{\mu} \Lambda^{\nu}_{\alpha} &= \delta^{\nu}_{\mu} \end{aligned} \tag{3}$$

where the Kronecker delta is defined by $\delta_{\mu\alpha} = \delta_{\mu\nu} = \delta^{\nu}_{\mu} = 1$ when $\mu = \nu$ and 0 otherwise.

The invariance group can be enlarged to be the *Poincaré group* by the addition of translations:

$$(x^{\mu})' = \sum_{\nu=1}^{4} \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu} \tag{4}$$

The full group includes: translations, 3-D space rotations, and the Lorentz boosts.

5.2 The Inner Product of 3+1-D Vectors

The definition of the inner product (dot product) must be modified in 3+1 dimensions.

$$\tilde{A} \cdot \tilde{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_4 B_4$$

if $x_4 = ict$. But with our usual convention

$$\tilde{A} \cdot \tilde{B} = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3$$

or with the opposite signature metric one has

$$\tilde{A} \cdot \tilde{B} = -A_0 B_0 + A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$\tilde{A} \cdot \tilde{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 - A_4 B_4$$

if $x_4 = ct$ which is often the convention for the opposite sign convention. It is an exercise to show that the inner product is unchanged under a Lorentz transformation. Can be done simply by substitution. This can be extended to the general class of Lorentz transformations.

5.3 Four Velocity

So we have the position 4-vector $\tilde{x} = (x_0, x_1, x_2, x_3)$ and the displacement 4-vector $\tilde{dx} = (dx_0, dx_1, dx_2, dx_3)$. What other 4-vectors are there? That is what other 4-vectors are natural to construct? What we mean by a four-vector is a four-dimensional quantity that transforms from one inertial frame to another by the Lorentz transform which will then leave its length (norm) invariant.

Consider generalizing the 3-vector velocity $(v_x, v_y, v_z) = (dx/dt, dy/dt, dz/dt)$ what can we do to make this into a 4-vector naturally? One clear problem is that we are dividing by a component dt of a vector so that the ratio is clearly going to Lorentz transform in a complicated way. We need to take the derivative with respect to a quantity that will be the same in all reference frames, e.g. $d\tau$ the differential of the proper time, and add a fourth component to make the 4-vector. It is clear that the derivative of the 4-vector position (ct, x, y, z) with respect to the proper time τ will be a 4-vector for Lorentz transformations since (ct, x, y, z) transform properly and $d\tau$ is an invariant. So we can define the 4-velocity as

$$u_\alpha = \frac{dx_\alpha}{d\tau}; \qquad \tilde{u} = \left(\frac{dct}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) \tag{5}$$

Note that

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = dt^2 \left(c^2 - \frac{dx}{dt}^2 - \frac{dy}{dt}^2 - \frac{dy}{dt}^2\right)$$

$$= dt^2 \left(c^2 - v_x^2 - v_y^2 - v_z^2\right) = dt^2 \left(c^2 - v^2\right)$$

or the time dilation formula we got before

$$\frac{d\tau}{dt} = \sqrt{1 - v^2/c^2}; \quad \text{and} \quad \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

So we can now explicitly write out the 4-velocity using the chain derivative rule:

$$u_\alpha = \frac{dx_\alpha}{d\tau} = \frac{dx_\alpha}{dt}\frac{dt}{d\tau}$$

$$\tilde{u} = (u_0, u_1, u_2, u_3) = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z) = \gamma\left(c, v_x, v_y, v_z\right)$$

Thus three components of the 4-velocity are the three components of the 3-vector velocity times γ.

$$\frac{d}{d\tau}\left(\tilde{u} \cdot \tilde{u}\right) = \frac{d\tilde{u}}{d\tau} \cdot \tilde{u} + \tilde{u} \cdot \frac{d\tilde{u}}{d\tau} = 2\tilde{u} \cdot \frac{d\tilde{u}}{d\tau} = \frac{dc^2}{d\tau} = 0$$

Thus

$$\tilde{u} \cdot \tilde{a} = 0 \tag{6}$$

Note also that the norm - the magnitude or vector invariant length - of the four-velocity is not only unchanged but it is the same for all physical objects (matter plus energy). For 3+1 dimensions the norm or magnitude is found from the inner product or dot product which has the same signature as the metric (see just above) so that

$$\tilde{u} \cdot \tilde{u} = u_0^2 - u_1^2 - u_2^2 - u_3^2 = \gamma^2 \left(c^2 - v_x^2 - v_z^2 - v_z^2 \right) = c^2 \frac{1 - v^2/c^2}{1 - v^2/c^2} = c^2$$

Thus every physical thing, including light, moves with a 4-velocity magnitude of c and the only thing that Lorentz transformations do is change the direction of motion. A particle at rest is moving down its time axis at speed c. When it is boosted to a fixed velocity, it still travels through space-time at speed c but more slowly down the time axis as it is also moving in the spatial directions.

One should also note that as the spatial speed (three-velocity) approaches c, all components of the 4-velocity u_α are unbounded as $\gamma \to \infty$. One cannot then define a Lorentz transformation that moves to the rest frame. Thus all massless particles will have no rest frame.

5.3.1 Law of Transformation of a 4-Vector

We can write the transformation in our standard algebraic Lorentz notation

$$A'_0 = \gamma \left(A_0 - \beta A_1 \right) \qquad \gamma = 1/\sqrt{1 - \beta^2}$$

$$A'_1 = \gamma \left(A_1 - \beta A_0 \right) \qquad \beta \equiv \frac{V}{c}$$

$$A'_2 = A_2; \qquad A'_3 = A_3$$

where β and γ refer to the relative velocity V of the frames.

5.3.2 Law of Transformation of a 4-Velocity

$$u'_1 = \gamma \left(u_1 - \beta u_0 \right)$$

where β and γ are for the relative velocity of the frames and not of the particle. But in the formula for the 4-velocity

$$\tilde{u} = (u_0, u_1, u_2, u_3) = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z) = \gamma \left(c, v_x, v_y, v_z \right)$$

The γ is for the particle! So we should have labeled it γ_p and the β and γ for the frame transform β_f and γ_f. Then we have

$$\gamma'_p v'_x = \gamma_f \left(\gamma_p v_x - \beta_f \gamma_p c \right)$$

So we can get out a formula for v_x'

$$v_x' = \frac{\gamma_f \gamma_p}{\gamma_p'} (v_x - V) = \frac{\sqrt{1-\beta_p'^2}}{\sqrt{1-\beta_p^2}\sqrt{1-\beta_f^2}} (v_x - V)$$

This is our old friend on the law of transformation of $\sqrt{1-u^2/c^2}$

$$\sqrt{1-u^2/c^2} = \frac{\sqrt{1-(u')^2/c^2}\sqrt{1-V^2/c^2}}{1+u_x'V/c^2}$$

and

$$\sqrt{1-(u')^2/c^2} = \frac{\sqrt{1-u^2/c^2}\sqrt{1-V^2/c^2}}{1+u_xV/c^2}$$

which is simply

$$\frac{1}{\gamma_p'} = \frac{1}{\gamma_p \gamma_f (1 - u_x V/c^2)}$$

So

$$v_x' = \frac{v_x - V}{1 - u_x V/c^2}$$

as derived earlier by the differential route.

Continuing onward

$$u_2' = u_2; \quad \text{or} \quad \gamma_p' v_y' = \gamma_p v_y$$

so that

$$v_y' = \frac{\gamma_p}{\gamma_p'} v_y$$

$$\frac{\gamma_p}{\gamma_p'} = \frac{1}{\gamma_f (1 - u_x V/c^2)}$$

and

$$v_y' = \frac{\sqrt{1-V^2/c^2}}{1-u_xV/c^2} v_y$$

which is the same relationship as before from the differential Lorentz transform. Similarly for v_z' and v_t':

$$u_o' = \gamma_f (u_0 - \beta_f u_1)$$

Explicitly this is

$$\gamma_p' c = \gamma_f (\gamma_p c - \gamma_p u_x V/c) = \gamma_p \gamma_f c \left(1 - u_x V/c^2\right)$$

So

$$\gamma_p' = \gamma_p \gamma_f \left(1 - u_x V/c^2\right)$$

which is our relation from the transformation of γ's and its reciprocal used above.

5.4 Four Momentum

What is the natural extension of the 3-vector momentum to 4-momentum. The answer is clear from dimensional/transform analysis and from our experimental approach on how masses transformed. The 4-momentum is simply:

$$p_\alpha = m_o u_\alpha ;\ \tilde{p} = (p_0, p_1, p_2, p_3) = \gamma m_o \left(c, v_x, v_y, v_z\right) \tag{7}$$

The three spatial components are just the Newtonian 3-momentum with the mass of the particle replaced by γm_o.

We can see that the 4-momentum also has an invariant norm by making use of our results for the 4-velocity:

$$\tilde{p}\cdot\tilde{p} \equiv p_0^2 - p_1^2 - p_2^2 - p_3^2 = E^2/c^2 - p^2 = m_o^2 \tilde{u}\cdot\tilde{u} = m_o^2 c^2$$

Thus the invariant length of the 4-momentum vector is just the rest mass of the particle times c.

$$\tilde{p} = m_0\tilde{u} = m_0\gamma(c, v_x, v_y, v_z) = (m_0\gamma c, \gamma m_0 v_x, \gamma m_0 v_y, \gamma m_0 v_z) = (mc, mv_x, mv_y, mv_z) = (E/c, \vec{p})$$

Thus the magnitude squared of the four-momentum is

$$\tilde{p}\cdot\tilde{p} = (m_0\gamma)^2(c^2 - v_x^2 - v_y^2 - v_z^2) = m_0^2\gamma^2(1 - v^2/c^2)c^2 = (m_0 c)^2$$

or we can write this equation as

$$\left(\frac{E}{c}\right)^2 - p^2 = (m_0 c)^2$$

$$E^2 = p^2c^2 + \left(m_0 c^2\right)^2 \tag{8}$$

This is particularly simple to use when the units of mass are $Energy/c^2$ and the units of momnetum are $Energy/c$. Fore example, a proton has a rest mass $m_p = 938$ MeV/c^2 and an electron $m_e = 0.511$ MeV/c^2.

5.5 The Acceleration Four-Vector

In a similar way one may derive the acceleration four-vector. Again we differentiate with respect to the proper time τ.

$$a_\alpha = \frac{du_\alpha}{d\tau} \tag{9}$$

The four-vector acceleration will have a part parallel to the acceleration three-vector and a part parallel to the velocity three-vector.

Exercise: Prove that the inner product of the 4-acceleration and the 4-velocity are zero; $\tilde{a} \cdot \tilde{u} = 0$ as they must be if the norm of the four-velocity is to remain constant c.

We have also constructed the 4-acceleration to be a 4-vector so that $\tilde{a} \cdot \tilde{a}$ is an invariant. Evaluate it in the rest frame $\tilde{a} \cdot \tilde{a} = |\tilde{a}|^2$

$$\tilde{a} \cdot \tilde{a} = |\tilde{a}_{\text{rest frame}}|^2$$

in any frame. This can be very useful in various calculations and we will use it later to treat radiation from and accelerating charged particle.

Acceleration 4-vector transforms by the relations:

$$a_0' = \gamma_f \left(a_0 - \beta_f a_1\right), \qquad a_2' = a_2,$$

$$a_1' = \gamma_f \left(a_1 - \beta_f a_0\right), \qquad a_3' = a_3,$$

This is the best starting place from which to derive the detailed Lorentz transformation equations for acceleration.

5.6 The Four Vector Force

We now consider the four-vector force, which we define the following way:

$$\tilde{F} \equiv \frac{d\tilde{p}}{d\tau} \tag{10}$$

$$\tilde{F} \equiv \frac{d\tilde{p}}{d\tau} = \frac{d\tilde{p}}{dt}\frac{dt}{d\tau} = \gamma \frac{d\tilde{p}}{dt}$$

$$\tilde{F} \equiv (F_0, F_1, F_2, F_3) = \gamma\left(W/c, F_{N1}, F_{N2}, F_{N3}\right) \rightarrow \gamma\left(\vec{F}_N \cdot \vec{\beta}, F_{N1}, F_{N2}, F_{N3}\right) \tag{11}$$

where $\vec{F}_N$ is the three-dimensional Newtonian force, e.g. $\vec{F}_N = (F_{N1}, F_{N2}, F_{N3})$

Note that the four force can be space-like, time-like and null. If a frame can be found where the three-force on an object is zero but the object is exchanging internal energy with the environment, then the four-force is time-like. The converse is space-like.

Then the 4-vector force $\tilde{F}$ has the same transformation law as all 4-vectors:

$$\tilde{F}_0' = \gamma_f \left(F_0 - \beta_f \tilde{F}_1\right)$$

$$\tilde{F}_1' = \gamma_f \left(F_1 - \beta_f \tilde{F}_0\right)$$

$$\tilde{F}_2' = \tilde{F}_2; \qquad \tilde{F}_3' = \tilde{F}_3$$

So we can now conveniently transform any of the familiar vectors used in mechanics, but not electric and magnetic fields, and pseudovectors obtained from cross-products, such as angular momentum and angular velocity. We will treat these later.

The 4-vector force transforms are much easier than the 3-D fource transforms which involve a γ^3. See the homework problem for the transformaiton of acceleration to grasp how much more complicated it is.

5.7 4-D Potential

It is convenient to do physics in terms of potential and find the resulting force as the derivative, e.g. the gradient, of the potential. Classical physics examples are:

$$\begin{aligned} F_G &= -m\vec{\nabla}\Phi_G && \text{Newtonian Gravitation} \\ F_E &= -q\vec{\nabla}\Phi_E && \text{Electrostatics} \end{aligned} \tag{12}$$

Once we have a 4-D potential, then we need to learn how to take derivatives in 4-D spaces.

One approach is to make the simplest possible frame-independent (scalar) estimate of the interaction of two particles. This manner of thinking eventually leads one to the interaction Lagrangian as a the product of the two currents (electrical, matter, strong, weak, gravitational).

$$L = \alpha\tilde{j}_1 \cdot \tilde{j}_2 \tag{13}$$

where α is the coupling constant and the next term is the inner (4-D dot) product of the current of particle 1 and the current of particle 2. When the two currents are in contact (zero proper distance separation), there is an interaction. When they are not in proper distance contact, there is no interaction. This means that all interaction is on the proper distance null (the light cone). Thus there is no action at a proper distance. It is manifestly invariant as the inner product of two 4-D vectors.

From this Lagrangian we can generate the 4-D potential of the effect of all other currents (or a single current) $\tilde{j}_2$ on our test particle which has current $\tilde{j}_1$.

$$\tilde{A}(\tilde{x}_1) = \int\int\int\int \alpha f(s_{12}^2)\tilde{j}_2(\tilde{x}_2)dV_2dt_2 = \int\int\int\int \alpha f(c^2(t_1-t_2)^2 - r_{12}^2)\tilde{j}dVdt \tag{14}$$

where $s_{12}^2 \equiv |\tilde{x}_1 - \tilde{x}_2|^2 = c^2(t_1-t_2)^2 - r_{12}^2$ is the invariant separation between $\tilde{x}_1$ and $\tilde{x}_2$ dV is the 3-D spatial volume and dt is the time. $f(s_{12}^2)$ is a function which is zero every where but peaks when the square of the 4-vector distance s_{12}^2 between the source (2) and the point of interest (1) is very small. The integral over $f(s_{12}^2)$ is also normalized to unity. The Dirac delta function is the limiting case for $f(s_{12}^2)$. Thus $f(s_{12}^2)$ is finite only for

$$s_{12}^2 = c^2(t_1-t_2)^2 - r_{12}^2 \approx \pm\epsilon^2 \tag{15}$$

Rearranging and taking the square root

$$c(t_1-t_2) \approx \sqrt{r_{12}^2 \pm \epsilon^2} \approx r_{12}\sqrt{1 \pm \frac{\epsilon^2}{r_{12}^2}} \approx r_{12}(1 \pm \frac{\epsilon^2}{2r_{12}^2}) \tag{16}$$

So

$$(t_1-t_2) \approx \frac{r_{12}}{c} \pm \frac{\epsilon^2}{2cr_{12}} \tag{17}$$

which says that the only times t_2 that are important in the integral of $\tilde{A}$ are those which differ from the time t_1, for which one is calculating the 4-potential, by the delay r_{12}/c ! – with negligible correction as long as $r_{12} \gg \epsilon$. Thus the Bopp theory approaches the Maxwell theory as long as one is far away from any particular charge.

By performing the integral over time one can find the approximate 3-D volume integral by noting that $f(s_{12}^2)$ has a finite value only for $\Delta t_2 = 2 \times \epsilon^2/2r_{12}c$, centered at $t_1 - r_{12}/c$. Assume that $f(s_{12}^2 = 0) = K$, then

$$\tilde{A}(\vec{x}_1) = \int \tilde{j}(t_2, \vec{x}_2) f(s_{12}^2) dV_2 dt_2 \approx \frac{K\epsilon^2}{c} \int \frac{\tilde{j}(t - r_{12}/c, \vec{x}_2)}{r_{12}} dV_2 \tag{18}$$

which is exactly the 3-D version, if we pick K so that $K\epsilon^2 = 1$.

5.8 Derivative in 4-Space

The 3-D vector gradient operator is DEL:

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \tag{19}$$

which behaves as a 3-D vector.

This can be generalized to 4-D:

$$\Box = (\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) \tag{20}$$

How does it transform?

$$\Box' = (\frac{\partial}{\partial x'_0}, \frac{\partial}{\partial x'_1}, \frac{\partial}{\partial x'_2}, \frac{\partial}{\partial x'_3}) \tag{21}$$

Operate first on a scalar function $\phi(x_0, x_1, x_2, x_3)$

$$\frac{\partial \phi(x_0, x_1, x_2, x_3)}{\partial x'_\nu} = \sum_\mu \frac{\partial \phi}{\partial x_\mu} \frac{\partial x_\mu}{\partial x'_\nu} = \sum_\mu \frac{\partial \phi}{\partial x_\mu} R_{\nu\mu} \tag{22}$$

where $R_{\nu\mu}$ is the rotation matrix/tensor defined by

$$\begin{aligned} x'_\mu &= \sum_\nu a_{\mu\nu} x_\nu \\ x_\nu &= \sum_\mu (a^{-1})_{\nu\mu} x'_\mu \end{aligned} \tag{23}$$

$A^{-1} = a^\dagger$ ($\dagger$ means transpose), if a is orthogonal.

$$x_\nu = \sum_\mu a_{\mu\nu} x'_\mu \tag{24}$$

$$\frac{\partial \phi}{\partial x'_\mu} = \sum_\nu a_{\mu\nu} \frac{\partial \phi}{\partial x_\nu}$$

$$x'_\mu = \sum_\nu a_{\mu\nu} x_\nu \tag{25}$$

so that

$$\Box'_\mu = \sum_\nu a_{\mu\nu} \Box_\nu \tag{26}$$

and $\Box$ is a Lorentz 4-vector.

5.9 Operate with $\Box$

Operate with $\Box$ on a Lorentz 4-vector, to get the dot (inner) product:

$$\begin{aligned} \Box \cdot \tilde{x} &= \frac{\partial ct}{\partial ct} + \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \\ &= 1 + 1 + 1 + 1 = 4 = \textbf{invariant} \end{aligned} \tag{27}$$

Now operate on velocity 4-vector $\tilde{u}$:

$$\begin{aligned} \Box \cdot \tilde{u} &= \frac{\partial \gamma c}{\partial ct} + \frac{\partial \gamma v_x}{\partial x} + \frac{\partial \gamma v_y}{\partial y} + \frac{\partial \gamma v_z}{\partial z} \\ &= \frac{\partial}{\partial t}\frac{1}{\sqrt{1-\beta^2}} + \frac{\partial}{\partial x}\frac{v_x}{\sqrt{1-\beta^2}} + \frac{\partial}{\partial y}\frac{v_y}{\sqrt{1-\beta^2}} + \frac{\partial}{\partial z}\frac{v_z}{\sqrt{1-\beta^2}} \\ &= \frac{\partial}{\partial t}(\frac{1}{\sqrt{1-\beta^2}}) + \vec{\nabla} \cdot (\frac{\vec{v}}{\sqrt{1-\beta^2}}) \end{aligned} \tag{28}$$

This equation is an expression related to continuity.

5.9.1 Hydrodynamics

Conservation of fluid matter is expressed by the equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{29}$$

If one integrates this equation over a fixed volume containing mass M

$$\frac{\partial}{\partial t}\int_{vol} \rho dxdydz + \int_{vol} \vec{\nabla} \cdot (\rho \vec{v}) dxdydz = M \tag{30}$$

The first term is the mass contained in the volume and the second part is the divergence theorem and yields:

$$\frac{\partial M}{\partial t} + \int_{surface} \rho \vec{v} \cdot \hat{n} dS = 0 \tag{31}$$

$\frac{\partial M}{\partial t}$ = - outward transport of mass and equals the inward transport of mass.

Since our expression for $\Box \cdot \tilde{u}$ is

$$\Box \cdot \tilde{u} = \frac{\partial}{\partial t}(\frac{1}{\sqrt{1-\beta^2}}) + \vec{\nabla} \cdot (\frac{\vec{v}}{\sqrt{1-\beta^2}}) \tag{32}$$

the role of density is played by $\gamma = 1/\sqrt{1-\beta^2}$.

5.9.2 Wavevector $\tilde{k}$

The 3-D wave vector $\vec{k}$ is simply $\vec{k} = \hat{v}/\lambda$ The natural 4-vector is

$$\tilde{k} = \left(\omega, \frac{\hat{n}}{\lambda}\right)$$

Note that

$$\tilde{p} = \hbar\,(\omega, k_x, k_y, k_z)$$

$$\tilde{p} \cdot \tilde{x} = \phi = \hbar(\omega t - \vec{k} \cdot \vec{x})$$

5.10 The Metric Tensor

Now before moving to make electromagnetism consistent with our relativistic mechanics, we need to generalize the concepts of the distance, vectors, vector algebra and tensors as they work in 3+1 D space.

The metric tensor defines the measurement properties of space-time. (Metric means measure – Greek: metron = a measure.)

Riemannian space in general, flat space in particular

$$(ds)^2 = \sum_{i,j} g^{ij} dx_i dx_j \tag{33}$$

by definition $g_{ij} = g_{ji}$ since the measure must be symmetric under interchange of coordinate multiplication order.

In the general case: Riemannian space

$$(ds)^2 = \sum_{i,j} g_{ij} dx^i dx^j = \text{scalar invariant} \tag{34}$$

(Note the superscripts. Section of covariant and contravariant vectors explains this.)

If g_{ij} is **diagonal**, the coordinates are **orthogonal**.

Physical interpretation: $g_{ii} = h_i^2$, where h_i is defined by the components of the vector line element, $ds_i = h_i dx_i$. An example of this is spherical polar coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{35}$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{36}$$

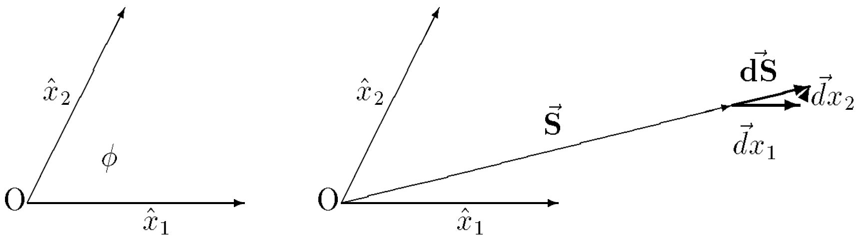
For the 3 + 1 dimension Minkowski space-time

$$ds^2 = d(c\tau)^2 = d(ct)^2 - dx^2 - dy^2 - dz^2 \tag{37}$$

$$g_{\nu\mu} \equiv \eta_{\nu\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{38}$$

In general the symbol $\eta_{\nu\mu}$ is used to denote the Minkowski metric. Usually it is displayed in rectangular coordinates (ct, x, y, z) or (x_0, x_1, x_2, x_3) but could be expressed in spherical (ct, r, θ, ϕ) or cylindrical (ct, r, θ, z) equally well.

The off-diagonal $g_{ij} = \sqrt{h_i h_j}(\vec{ds}_i \cdot \vec{ds}_j)$ for $i \neq j$. An example is skew coordinates in two dimensions.



By the law of cosines

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + 2dx_1dx_2cos\phi \\ &= g_{11}dx_1^2 + g_{22}dx_2^2 + g_{12}dx_1dx_2 + g_{21}dx_1dx_2 \end{aligned} \tag{39}$$

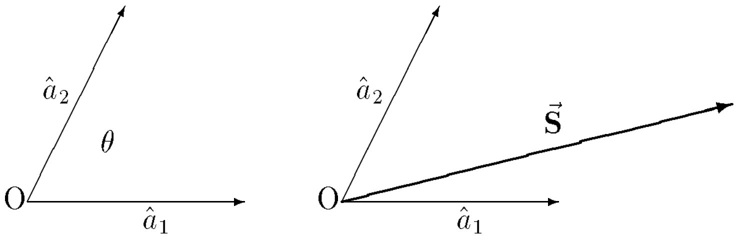
$$\begin{gathered} ds_1 = dx_1, \quad ds_2 = dx_2 \\ g_{11} = h_1^2 = 1, \quad g_{22}h_2^2 = 1 \end{gathered} \tag{40}$$

$$g_{12} = g_{21} = \sqrt{h_1 h_2} cos\phi = cos\phi \tag{41}$$

$$g_{ij} = \begin{bmatrix} 1 & cos\phi \\ cos\phi & 1 \end{bmatrix} \tag{42}$$

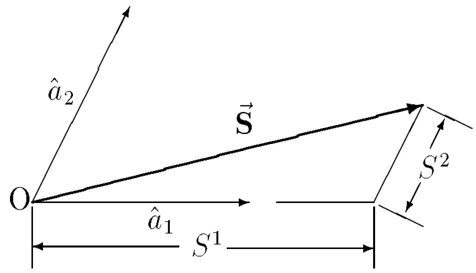
5.11 Contra & Covariant Vectors

First we consider a simple example to illustrate the significance of contravariant and covariant vectors. Consider two non-parallel unit vectors $\hat{a}_1$ and $\hat{a}_2$ in a plane with $\hat{a}_1 \cdot \hat{a}_2 = cos\theta \neq 1$.



A displacement from O to P can be represented by a vector, $\vec{\mathbf{S}}$. Its components in the directions of $\hat{a}_1$ and $\hat{a}_2$ can be denoted S^1 and S^2:

$$\vec{\mathbf{S}} = S^1\hat{a}_1 + S^2\hat{a}_2 \tag{43}$$

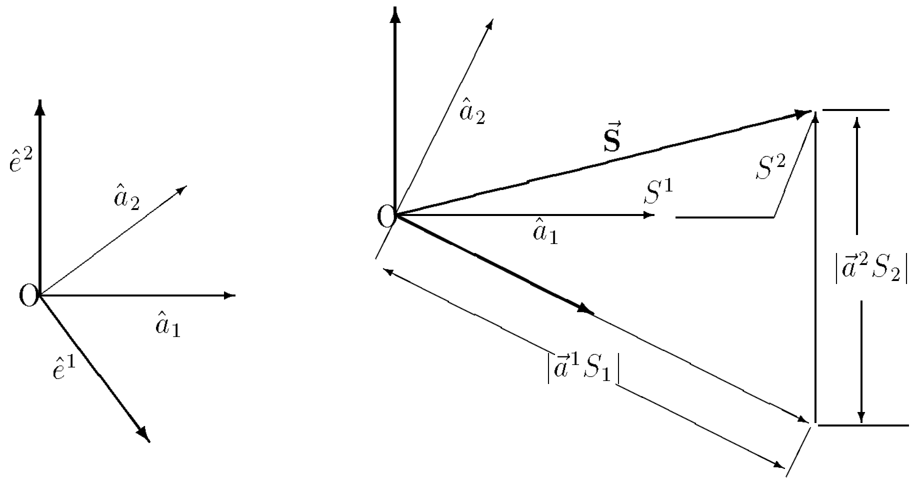


Another set of basis vectors $\vec{a}^1$ and $\vec{a}^2$, respectively, may be defined, being perpendicular to $\hat{a}_1$ and $\hat{a}_2$ and having lengths found the following way: Let $\hat{a}_3$ be a unit vector normal to the plane, proportional to $\hat{a}_1 \times \hat{a}_2$. Then

$$\vec{a}^1 = \frac{\hat{a}_2 \times \hat{a}_3}{\hat{a}_1 \times \hat{a}_2 \cdot \hat{a}_3} = \frac{\hat{e}^1}{sin\theta} \tag{44}$$

$$\vec{a}^2 = \frac{\hat{a}_3 \times \hat{a}_1}{\hat{a}_1 \times \hat{a}_2 \cdot \hat{a}_3} = \frac{\hat{e}^2}{sin\theta} \tag{45}$$

We denote the triple scalar product by $[\ \ ]_{123}$.



The displacement vector, $\vec{\mathbf{S}}$ may also be expressed by its components S_1 and S_2 as follows:

$$\vec{\mathbf{S}} = S_1\vec{a}^1 + S_2\vec{a}^2. \tag{46}$$

The relations among S^1, S^2, S_1, and S_2 may be found by elementary geometry: They are:

$$v_1 = v^1 + v^2 cos\theta \tag{47}$$

$$v_1 = v^1 cos\theta + v^2 \tag{48}$$

$$v^1 = (v_1 - v_2 cos\theta)/sin^2\theta \tag{49}$$

$$v^2 = (-v_1 cos\theta + v_2)/sin^2\theta. \tag{50}$$

Using the original pair of unit vectors,

$$\begin{aligned} S^2 &= (S^1)^2 + (S^2)^2 + 2(S^1)(S^2)cos\theta \\ &= \sum_{i,j=1}^{2} g_{ij} S^i S^j \end{aligned} \tag{51}$$

with the metric tensor

$$g_{ij} = \begin{bmatrix} 1 & cos\theta \\ cos\theta & 1 \end{bmatrix} \tag{52}$$

Defined to be symmetric.

The tensor g^{ij} is defined by

$$\sum_{j=1}^{2} g^{ij} g_{jk} = \delta^i_k. \tag{53}$$

It is easy to find that

$$g^{ij} = \frac{1}{sin^2\theta} \begin{bmatrix} 1 & -cos\theta \\ -cos\theta & 1 \end{bmatrix}. \tag{54}$$

From this relation one finds that

$$S_i = \sum_j g_{i,j} S^j \tag{55}$$

and

$$S^i = \sum_j g^{i,j} S_j. \tag{56}$$

The components S^i are **contravariant** and the components S_i are **covariant**. The square of the length of $\vec{\mathbf{S}}$ is (as given above)

$$|\vec{\mathbf{S}}|^2 = \sum_{i,j} g_{ij} S^i S^j = \sum^{i,j} g^{ij} S_i S_j, \tag{57}$$

but is given more compactly by

$$S^2 = \sum_j S_j S^j \tag{58}$$

Other relations of interest are:

$$g^{ij} = \frac{\text{Signed Minor of } g_{ij}}{\text{Det } g_{ij}} = \frac{\text{Cofactor of } g_{ij}}{g} \tag{59}$$

For this example Det $g_{ij} = g = sin^2\theta$; the cofactor of g_{ij} is $(-1)^{i+j} g_{ji} = (-1)^{i+j} g_{ij}$ because g_{ij} and g^{ij} are symmetric.

Returning to the original sets of basis vectors

$$\vec{a}^1 = \frac{\hat{a}_2 \times \hat{a}_3}{\hat{a}_1 \times \hat{a}_2 \cdot \hat{a}_3} = \frac{\hat{a}_2 \times \hat{a}_3}{[\;]_{123}} \tag{60}$$

and others by cycling indices, by substitution one has:

$$\hat{a}_1 = \frac{\hat{a}^2 \times \hat{a}^3}{\hat{a}^1 \times \hat{a}^2 \cdot \hat{a}^3} = \frac{\hat{a}^2 \times \hat{a}^3}{[\;]^{123}} \tag{61}$$

$$[\;]^{123} = \frac{1}{[\;]_{123}} = \frac{1}{sin\theta} \tag{62}$$

Also one has

$$\mathrm{Det}(g^{ij}) = \frac{1}{\mathrm{Det}(g_{ij})} = \frac{1}{sin^2\theta}. \tag{63}$$

5.12 Electric Charge

We now consider the implications for electric charge. We define electric charge density as the charge per volume, ρ. We have a law of conservation of charge: Charge cannot be created or destroyed. Thus

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0. \tag{64}$$

So the charge-current density Lorentz 4-vector

$$\tilde{j} \equiv \rho_0 \tilde{u} = (\rho c, \rho v_x, \rho v_y, \rho v_z) = (j_0, j_1, j_2, j_3) \tag{65}$$

(where $\rho = \gamma \rho_0$) and

$$\Box \tilde{j} = 0 \qquad j^{\mu}_{,\mu} = 0 \tag{66}$$

is the equation for the conservation of charge. $\tilde{j}$ is the 4-vector charge current.

Now consider the vector and scalar potentials of the electromagnetic fields.

$$\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \qquad \text{where} \qquad \vec{A} = \frac{1}{c} \int\int\int \frac{\vec{j}\, dV}{r} \\ \vec{E} &= -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \text{where} \qquad \Phi = \int\int\int \frac{\rho\, dV}{r} \end{aligned} \tag{67}$$

The Lorentz 4-vector potential is

$$\tilde{A} = (\Phi, A_x, A_y, A_z) = (A_0, A_1, A_2, A_3) \qquad \text{where} \qquad A^{\mu} = \frac{1}{c} \int\int\int \frac{j^{\mu}\, dV}{r} \tag{68}$$

Then the inner product gives

$$\begin{aligned}
\Box \cdot \tilde{A} &= \Box_\mu A^\mu \\
&= \frac{\partial \Phi}{\partial ct} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
&= \frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0
\end{aligned} \tag{69}$$

This is the equation of Lorentz gauge invariance.

5.12.1 Box on $\tilde{A}$ is a four vector

It is clear that $\tilde{j} = \rho_0 \tilde{u}$ is a four vector since $\tilde{u}$ was constructed to be one and we constructed $\tilde{j}$ as a scalar (rest frame charge density) times that four vector. However, I merely asserted that $\tilde{A}$ was a four vector. That is true only if dV/r is invariant under Lorentz transforms. We have this as an exercise for the student to show that is true. The following are hints: Show that $dV' = (1 + \beta cos\theta)\gamma dV$ and that $r' = r\gamma(1 + \beta cos\theta)$ and thus $dV'/r' = dV/r$.

5.13 Lorentz Force Law

The 3-D vector form of the force law is

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{70}$$

We need to write this in 4-D vector form to show that it is Lorentz invariant. The relativistic force law must involve the particle velocity and the simplest form is linear in the 4-D velocity. The 4-D vector form then would be

$$\tilde{F} = \frac{q}{c}\tilde{\tilde{F}} \cdot \tilde{u}, \qquad F_\mu = \frac{q}{c} F_{\mu\nu} u^\nu \tag{71}$$

To obtain the 4-D expression for the electromagnetic fields we need second rank tensors, i.e. $F_{\mu\nu}$.

Since we want the force F_μ to be rest-mass preserving, we have the requirement that $F_\mu u^\mu = 0$ and thus $F_{\mu\nu} u^\mu u^\nu = 0$. Since this must hold for all u^μ, the $F_{\mu\nu}$ must be antisymmetric.

A cartesian flat-space second rank tensor has components C_{ij}. The tensor is the sum of a symmetric tensor S_{ij} and an antisymmetric tensor A_{ij}:

$$\begin{aligned}
C_{ij} &= \frac{1}{2}(C_{ij} + C_{ji}) + \frac{1}{2}(C_{ij} - C_{ji}) \\
&= S_{ij} + A_{ij}
\end{aligned} \tag{72}$$

$$S_{ij} = S_{ji}; \quad A_{ij} = -A_{ji} \tag{73}$$

The property of being symmetric or of being antisymmetric is preserved under orthogonal transformations.

Now construct the antisymmetric tensor in a generalized curl

$$F_{\mu\nu} = \Box_\mu A_\nu - \Box_\nu A_\mu = \partial^\mu A_\nu - \partial^\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu} \tag{74}$$

Note that

$$F_{00} = F_{11} = F_{22} = F_{33} = 0$$

$$F_{23} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = (\vec{\nabla}\times\vec{A})_x = B_x$$

Similarly, $F_{31} = B_y$, $F_{10} = B_z$.

$$F_{10} = \frac{\partial A_0}{\partial x_1} - \frac{\partial A_1}{\partial x_0} = \frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial ct} = -E_x$$

and similarly $F_{20} = -E_y$ and $F_{30} = -E_z$. So the full tensor is

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \tag{75}$$

$F_{\mu\nu}$ is the electromagnetic field tensor.

The contravariant form of the electromagnetic field tensor is

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{76}$$

One can raise and lower indices by use of the metric tensor.

$$F_{\mu\nu} = \sum_\gamma \sum_\delta g_{\mu\gamma} F^{\gamma\delta} g_{\delta\nu} \tag{77}$$

In 3-D Maxwell's equations are:

$$\begin{aligned} \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} &= \rho\frac{\vec{v}}{c} = \frac{\vec{j}}{c} \\ \vec{\nabla}\cdot\vec{E} &= \rho \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla}\cdot\vec{B} &= 0 \end{aligned} \tag{78}$$

Now we take the 4-D divergence of the electromagnetic field tensor

$$\Box \cdot \tilde{\tilde{F}} = \tilde{j}/c \tag{79}$$

which reduces to the first two Maxwell equations. The continuity equation is simply

$$j^{\mu}_{,\mu} = 0. \tag{80}$$

Since there were actually two possible ways to unify the electric and magnetic fields into a single entity, we now define the dual electromagnetic field tensor:

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix} \tag{81}$$

The second set of Maxwell's equations can be simply written as

$$\sum_{\nu} \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{82}$$

Or, if one does not wish to resort to the dual electromagnetic field tensor, then the second set of Maxwell's equations can be simply written as

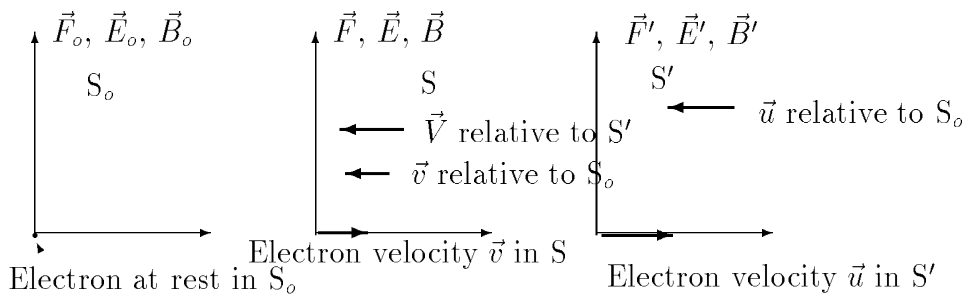
$$\partial^{\alpha} F^{\beta\gamma} + \partial^{\beta} F^{\gamma\alpha} + \partial^{\gamma} F^{\alpha\beta} = 0 \tag{83}$$

a generalized curl.

5.14 Transformation of the EM Fields

One can derive the transformation of the electromagnetic field by using the Lorentz force law $\vec{F} = q(\vec{E} + \vec{V} \times \vec{B})$ as a definition of the $\vec{E}$ and $\vec{B}$ (and by the transformation of second rank tensors as shown below.) To derive the $\vec{E}$ and $\vec{B}$ requires using three reference frames in order to see how both transform.

Do use the Lorentz force law we need a test electron or charge to probe the force and thus how the fields must transform. We consider the field acting on an electron located at the origin of three reference frames in relative motion.



The electron is at rest relative to reference frame S$_o$, moving with velocity $\vec{v}$ with respect to reference frame S, and moving with velocity $\vec{u}$ with respect to reference frame S$'$. We arrange the coordinate systems so that the velocities all lie along the

x axes. Thus the relative velocity $\vec{V}$ of the frames S and S′ is given by the velocity addition formula as

$$V = \frac{u+v}{1+uv/c^2}$$

We can write simple expression for the Lorentz force components in frames S, S′, and S_o, respectively:

$$\begin{array}{ccc} S & S' & S_o \\ F_x = eE_x & F'_x = eE'_x & F_{ox} = eE_{ox} \\ F_y = e(E_y - vB_z) & F'_y = e(E'_y - uB'_z) & F_{oy} = eE_{oy} \\ F_z = e(E_z + vB_y) & F'_z = e(E'_z + uB'_y) & F_{oz} = eE_{oz} \end{array}$$

Note that in S_o the electron is not moving so that the magnetic field does not produce a force.

The equations for the transformation of force (for $u'_x = 0$) give

$$\begin{array}{cc} F_x = F_{ox} & F'_x = F_{ox} \\ F_y = F_{oy}\sqrt{1-v^2/c^2} & F'_y = F_{oy}\sqrt{1-u^2/c^2} \\ F_y = F_{oy}\sqrt{1-v^2/c^2} & F'_z = F_{oz}\sqrt{1-u^2/c^2} \end{array}$$

Then we have

$$\begin{array}{cc} E_x = E_{ox} & E'_x = E_{ox} \\ E_y - vB_z = E_{oy}\sqrt{1-v^2/c^2} & E'_y - uB'_z = E_{oy}\sqrt{1-u^2/c^2} \\ E_z + vB_y = E_{oz}\sqrt{1-v^2/c^2} & E'_z - uB'_y = E_{oz}\sqrt{1-u^2/c^2} \end{array}$$

We can see at once that $E_x = E'_x$. From the velocity addition law we have

$$\frac{v}{c} = \frac{u/c + V/c}{1+(u/c)(V/c)}$$

and thus

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1+\frac{u}{c}\frac{V}{c}}{\sqrt{1-u^2/c^2}\sqrt{1-V^2/c^2}}$$

Thus

$$\frac{E_y - vB_z}{\sqrt{1-v^2/c^2}} = E_{oy} = \frac{E'_y - uB'_z}{\sqrt{1-u^2/c^2}}$$

so that

$$\left[E_y - \frac{u+V}{1+uV/c^2}B_z\right] \times \left[\frac{1+\frac{u}{c}\frac{V}{c}}{\sqrt{1-u^2/c^2}\sqrt{1-V^2/c^2}}\right] = \frac{E'_y - uB'_z}{\sqrt{1-u^2/c^2}}$$

If these equations are to hold true for all values of u, then since the terms which contain u must be equal and those that do not must also be equal:

$$E'_y = \frac{E_y - VB_z}{\sqrt{1 - V^2/c^2}}$$

$$B'_z = \frac{-(V/c)E_y + B_z}{\sqrt{1 - V^2/c^2}}$$

Similarly by equating the expression for E_{oz} one finds

$$E'_z = \frac{E_z + VB_y}{\sqrt{1 - V^2/c^2}}$$

$$B'_y = \frac{(V/c)E_z + B_y}{\sqrt{1 - V^2/c^2}}$$

This gives the transformation law for 5 of the six components of the electromagnetic field. We are missing B_x since we started with a stationary electron in frame S_o. This can be found by considering an electron moving at right angles to B_x and recalling that the force is unchanged in the x direction. Thus $B'_x = B_x$.

Now do the derivation of field transformation from the transformation of a second rank tensor and apply that to $F_{\mu\nu}$.

$$F'_{\mu\nu} = \sum_\alpha \sum_\delta a_{\mu\alpha} a_{\nu\delta} F_{\alpha\delta} \tag{84}$$

applied to either the electromagnetic field tensor $\tilde{\tilde{F}}$ or its dual gives

$$\begin{array}{cc} E'_x = E_x & B'_x = B_x \\ E'_y = \gamma(E_y - \beta B_z) & B'_y = \gamma(B_y + \beta E_z) \\ E'_z = \gamma(E_z + \beta B_y) & B'_z = \gamma(B_z - \beta E_y) \end{array} \tag{85}$$

5.15 The Equations of Motion for a Charge Particle

The 3-D Lorentz force law

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \tag{86}$$

We can turn this into 4-D vector equation by first replacing $dt = \gamma d\tau$ and 3-vector velocity $\vec{v}$ by the 4-vector velocity $\tilde{u}$.

$$F_\mu = \frac{dp_\mu}{d\tau} = qF_{\mu\nu} u^\nu \tag{87}$$

5.16 The Energy-Momentum Tensor

First a brief review to provide motivation for the study and understanding of tensors:
(1) Electromagnetism described by a tensor field (4 by 4)
(2) Gravity represented by a tensor field (4 by 4)
(3) elastic phenomena in continuous media mechanics (classical 3 x 3)
(4) metric tensor for generalized coordinates

First we found a 4-vector equation of motion for a single particle:

$$\frac{d\tilde{p}^{\square}}{d\tau} = \tilde{F}^{\square} \qquad \frac{d\tilde{p}}{d\tau} = \tilde{F} \qquad \frac{dp^{\alpha}}{d\tau} = F^{\alpha} \tag{88}$$

Next we found the equation of motion for a single particle in an electromagnetic field as:

$$\frac{dp^{\alpha}}{d\tau} = m_0 \frac{du^{\alpha}}{d\tau} = \tilde{\tilde{F}}^{\alpha\beta} u_{\beta} \tag{89}$$

Later we will find that the equation of motion for a single particle in a weak gravitation field is

$$\frac{dp_{\mu}}{d\tau} = m_0 \frac{du_{\mu}}{d\tau} = \frac{1}{2}\kappa h_{\alpha\beta,\mu} m_o u^{\alpha} u^{\beta} \tag{90}$$

The last equation the second rank tensor $h_{\alpha\beta}$ is obvious but there is another simple second rank tensor there $m_o u^{\alpha} u^{\beta}$. This is an important tensor. The next paragraph supplies a little more motivation to study this important and one of the simplest that one could think to form.

In classical mechanics one has the concept that the integral of the force times distance is the work done (energy gained) and that the gradient of the potential is the force.

$$W = \Delta E = \int \vec{F} \cdot d\vec{x} \qquad \vec{F} = -\vec{\nabla} V \tag{91}$$

All this points to the need to develop the same concept in 4-D.

$$\Delta E = \int \vec{F} \cdot d\vec{x} = \int \frac{d\vec{p}}{dt} \cdot d\vec{x} = \int \frac{d\vec{x}}{dt} \cdot d\vec{p} = \int \vec{u} \cdot d\vec{p} \tag{92}$$

From the last part of the equality one finds that the integral to get the "4-potential" will involve $p^{\alpha} u^{\beta}$. The tensor $p^{\alpha} u^{\beta}$ is labeled the energy-momentum tensor. We can write out explicitly the tensor for a particle.

$$\begin{aligned} T^{\alpha\beta} &= p^{\alpha} u^{\beta} = m_o u^{\alpha} u^{\beta} \\ &= m_o c^2 \gamma^2 \begin{bmatrix} 1 & \beta_x & \beta_y & \beta_z \\ \beta_x & \beta_x^2 & \beta_x\beta_y & \beta_x\beta_z \\ \beta_y & \beta_x\beta_y & \beta_y^2 & \beta_y\beta_z \\ \beta_z & \beta_x\beta_z & \beta_y\beta_z & \beta_z^2 \end{bmatrix} \end{aligned} \tag{93}$$

since $(u^{\alpha}) = \gamma c(1, \beta_x, \beta_y, \beta_z)$.

The quantity, $\gamma^2 m_o c^2 = \gamma E$, seems a bit strange but not so when we consider a collection of particles or a continuum in density of material, ρ. $\rho = \gamma^2 \rho_o$ since one factor of γ comes from the mass increase and another factor of γ comes from the volume contraction due to length contraction along the direction of motion.

$$T^{\alpha\beta} = \rho c^2 \begin{bmatrix} 1 & \beta_x & \beta_y & \beta_z \\ \beta_x & \beta_x^2 & \beta_x\beta_y & \beta_x\beta_z \\ \beta_y & \beta_x\beta_y & \beta_y^2 & \beta_y\beta_z \\ \beta_z & \beta_x\beta_z & \beta_y\beta_z & \beta_z^2 \end{bmatrix} \tag{94}$$

and now we see that the energy-momentum tensor components are the transport of energy-momentum-component in α-direction into the β-direction.

Consider an interesting case: a large ensemble of non-interacting (elastic scattering only) particles – an ideal gas. For an ideal gas, $< \beta_i >= 0$ and $< \beta_i\beta_j >= 0$, for $i \neq j$, and $< v_x^2 >=< v_y^2 >=< v_z^2 >$, so that the energy-momentum tensor is diagonal

$$T^{\alpha\beta}_{\text{ideal gas}} = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & \rho < v_x^2 > & & \\ 0 & & \rho < v_y^2 > & \\ 0 & & & \rho < v_z^2 > \end{bmatrix} = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & & \\ 0 & & P & \\ 0 & & & P \end{bmatrix} \tag{95}$$

where ϵ is the full energy density due to the mass density, and $P = \rho < v_i >^2$ which is easily derived for an ideal gas ($PV = nkT = nm < v_i^2 >$).

We can write a simple formula for the energy-momentum tensor for a perfect fluid in a general reference frame in which the fluid moves with 4-D velocity u^μ as

$$T^{\mu\nu} = \left(\rho_0 + p/c^2\right) u^\mu u^\nu - p g^{\mu\nu} \tag{96}$$

which reduces to the equation above in its rest frame.

We want the full generalized relation between the energy-momentum tensor, $T^{\alpha\beta}$, and the 4-force to be:

$$\tilde{F} = \Box \cdot \tilde{\tilde{T}} \tag{97}$$

$$f^\mu = \sum_\nu \frac{\partial T^{\mu\nu}}{\partial x_\nu} \equiv \sum_\nu T^{\mu\nu}_{,\nu} \equiv T^{\mu\nu}_{,\nu} \tag{98}$$

where the last term represents the repeated indices summation convention. One uses $_{,\text{index}}$ indicates partial derivative with respect to x_{index} and repeated index to indicate summation on that index to make the equations easier to write and view.

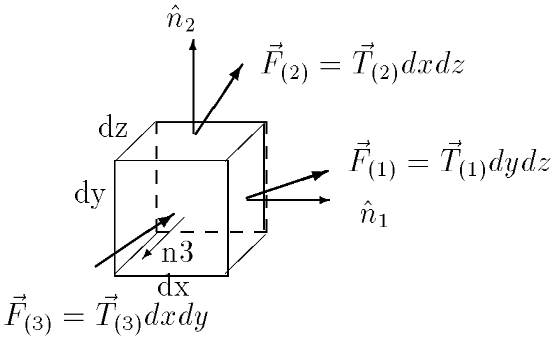
5.17 The Stress Tensor

Now we can consider the case of a medium or field that can have non-zero off-diagonal components. First it is good to review the concept of stress. Stress is defined as force per unit area, (same a pressure which is a particularly simple stress),

Imagine a distorted elastic solid or a viscous fluid such as molasses in motion. Imagine a surface (conceptual/mathematical) in the medium (The surface can and will be curved or distorted.) with a plus and a minus side and unit normal vector for every point on it. A differential area element dA, with normal $\hat{n}$ will exert forces on each of its sides. The forces are equal and opposite by Newton's second law, since the mass of the element is zero. $\vec{F}_{\text{total}} = m\vec{a} = 0$, so $\vec{F}_{+\text{ on }-} + F_{-\text{ on }+} = 0$

The force per unit area on the small element of the surface is the stress. It is a vector, not necessarily known. It underlies the dynamics of continuous media.

Consider a small piece of material at the surface



We define stress which stretches as positive and stress which compresses as negative.

Clearly each of the three axes has a vector force associated with it so that we have a second rank tensor field associated with the stress. We define the stress tensor, $E_{ij} \equiv T_{(i)j}$. Normal Stress is when the vector $T_{(i)}$ is co-directional with the normal $\pm\hat{n}_{(i)}$.

If $E_{ij} = C\delta_{ij}$, C is the hydrostatic pressure, if $C > 0$.

Simple Tension Consider $E_{ij} = C\hat{n}_i\hat{n}_j$, then $T_{(j)} = \vec{\vec{E}}_{ij} \cdot \hat{n}_i = C\hat{n}_i\hat{n}_j \cdot \hat{n}_i = C\hat{n}_j$ thus is co-directional with $\pm\hat{n}_i$. If $\hat{m}_i$ has directional orthogonal to n_i, then $\vec{T}_{(j)} = C\hat{n}_i\hat{n}_j \cdot \hat{m}_i = 0$.

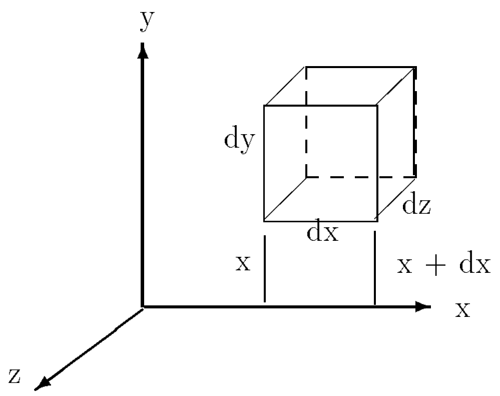
If C is negative ($C < 0$), the stress is simple compression.

Shearing Stress is specified by $\vec{\vec{E}}_{ij} = C(\hat{n}_i\hat{m}_j + \hat{n}_j\hat{m}_i)$

We will see by example the following generalization: **A simple tension in one direction and a single compression along an orthogonal direction is equivalent to a shearing stress along along shearing stress along the direction bisecting the angle between the two directions.**

In anticipation of later integration to 4-D we can call the stress tensor $E_{ij} = T_{ij} \equiv$ Force per area on the surface along the i-axis along the surface with normal in the j-direction by the material on the side with smaller x_j. Since action must equal reaction $-T_{ij}$ = force by material on the side of larger x_j.

Now return to our infinitesimal cube of the medium, with sides lined up along the cartesian coordinate planes:



The force on the back face is

$$F_x(x) = +T_{xx}(x)dydz = T_{xx}(x)dydz \tag{99}$$

The force on the front face: F_x is exerted on it toward inside in the x-direction is

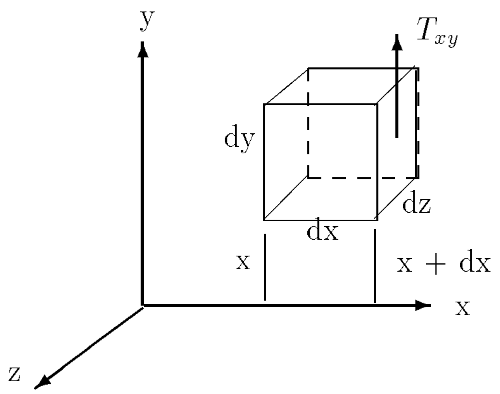
$$F_x(x+dx) = -T_{xx}(x+dx)dydz = -\left(T_{xx}(x) + \frac{\partial T_{xx}}{\partial x}dx\right)dydz \tag{100}$$

The net force on the cube is F_x is exerted on it toward inside in the x-direction is

$$F_x = -\frac{\partial T_{xx}}{\partial x}dxdydz \tag{101}$$

If $T_{xx} > 0$, inside pushes on the outside, pressure: compressive stress. If $T_{xx} < 0$, inside pulls on the outside, tension: tensile stress.

T_{xy} and T_{yx} are shear stresses.



Similarly to the treatment above the net force in the y-direction, F_y, on the front and back face is

$$F_y = -\frac{\partial T_{xy}}{\partial x}dxdydz \tag{102}$$

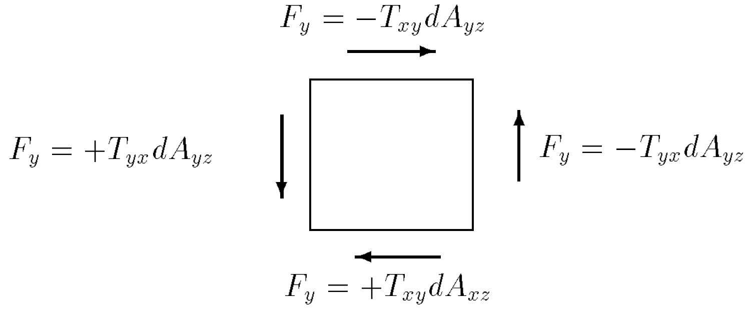
and

$$F_z = -\frac{\partial T_{xz}}{\partial x} dx dy dz \tag{103}$$

Thus the total F_x on the material inside is

$$\begin{aligned} F_x \text{ total} &= -\left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{zx}}{\partial x}\right) dx dy dz \\ F_i &= -\sum_{i=1}^{3} \frac{\partial T_{ij}}{\partial x^i} dV \end{aligned} \tag{104}$$

Now consider F_y on the two faces perpendicular to x and F_x on the two faces perpendicular to y as exerted from the outside.



The sign changes because from the surface the force is toward the inside. Now calculate the net torque. The two x faces have a counter-clock-wise torque:

$$\text{torque from x} - \text{face} = \text{Force } \times \text{ moment arm} = (T_{xy} dy dz) dx/2 \tag{105}$$

$$\text{torque from y} - \text{face} = -(T_{yx} dx dz) dy/2 \tag{106}$$

To the net torque is

$$\tau = (T_{xy} - T_{yx}) dx dy dz/2 = I \frac{d\omega}{dt} \tag{107}$$

where $I \propto mr^2 \sim \rho(dxdydz)r^2$ is the moment of inertia and $d\omega/dt$ is the angular acceleration so that

$$T_{xy} - T_{yx} \propto \rho r^2 \frac{d\omega}{dt} \tag{108}$$

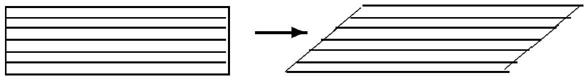
as we consider an infinitesimal cube, $r^2 \rightarrow 0$ so that

$$T_{xy} = T_{yx} \tag{109}$$

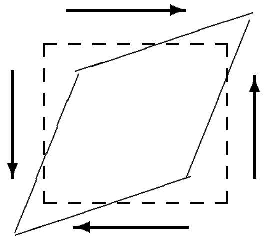
which means the stress tensor must be symmetric. The stress tensor is symmetric, so only six independent components.

5.18 Consideration of Shear

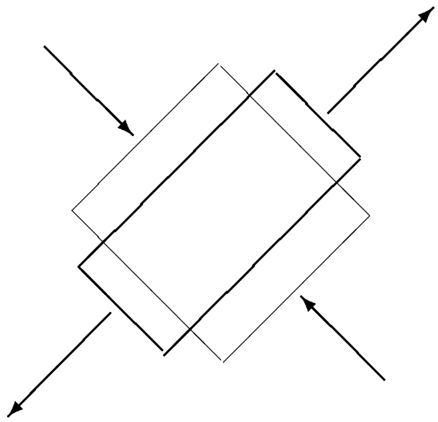
Simple shear displacement is like sliding a deck of cards.



A pure shear displacement keeps the center at the same place and is what our four forces try to do:



If the little cube is cut differently, e.g. cut at 45° to the previous cube, a different effect occurs:



Thus pure shear is a superposition of tensile and compressive stresses of equal size at right angles to each other.

Let us follow our example of shear a little further:

$$T_{ij} = \begin{bmatrix} 0 & T_{xy} & 0 \\ T_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{110}$$

We can look at the transformation properties by considering on the 2×2 portion. Now rotate the axes 45°. How do the tensor components change?

$$S'_{ij} = \sum_k \sum_l a_{ik} a_{jl} S_{kl} \tag{111}$$

where a_{ik} is the matrix for the coordinate transformation, rotation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{112}$$

For 45°, the rotation matrix is:

$$[A_{ij}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{113}$$

so that

$$\begin{aligned} T_{11} &= (a_{11})^2 T_{11} + a_{12}a_{11}T_{21} + a_{11}a_{12}T_{12} + a_{12}a_{21}T_{22} \\ &= \frac{1}{2}(T_{11} + T_{21} + T_{12} + T_{22}) = T_{21} = T_{12} \end{aligned} \tag{114}$$

$$\begin{aligned} T_{12} &= a_{11}a_{21}T_{11} + a_{11}a_{22}T_{12} + a_{12}a_{21}T_{21} + a_{12}a_{22}T_{22} \\ &= \frac{1}{2}(-T_{11} + T_{12} - T_{21} + T_{22}) = 0 \end{aligned} \tag{115}$$

$$\begin{aligned} T_{22} &= a_{21}a_{21}T_{11} + a_{21}a_{22}T_{12} + a_{22}a_{21}T_{21} + a_{22}a_{22}T_{22} \\ &= \frac{1}{2}(T_{11} - T_{12} - T_{21} + T_{22}) = -T_{21} = -T_{12} \end{aligned} \tag{116}$$

So that for the 45° rotation we have

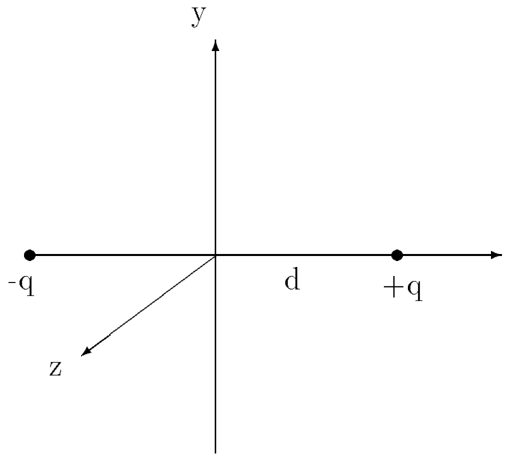
$$T'_{ij} = \begin{bmatrix} T_{21} & 0 \\ 0 & -T_{21} \end{bmatrix} \tag{117}$$

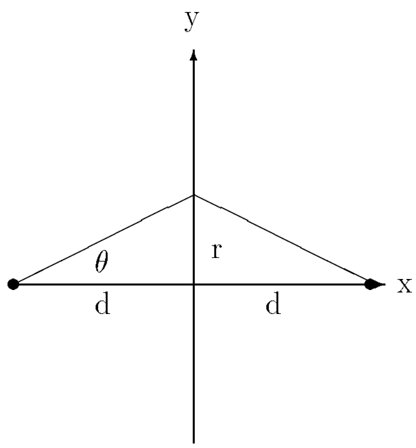
Thus we have shown that a pure shear stress rotated by 45° is equivalent to equal amounts of tension and compression stress at right angles to each other with the pure shear bisecting the angle they make.

5.19 Electric and Magnetic Stress

In this section we see that using the Faraday lines of force concept that both the electric and magnetic field lines can be under tension or compression and thus by the argument just above under shear stress.

First consider two opposite charges, magnitude q, a distance $2d$ apart, located symmetrically opposite the origin on the x-axis. The force between them is $F = q^2/(4d^2)$ according to the Coloumb law. We can imagine putting a metal plate (perfect conductor) in the $y - z$ plane and know that an image charge will form and have the same force on it and thus the plate. This makes sense in terms of the Faraday lines of force. We can calculate the total integrated mean square value of the electric field in the $y - z$ plane.





The only non-zero component is $E_x = 2qcos\theta/r^2 = 2qd/r^3$ where $r^2 = \rho^2 d^2$.

$$\int E_x^2 dA = 4q^2d^2 \int_0^\infty \frac{2\pi\rho d\rho}{(\rho^2+d^2)^3} = 4\pi q^2 d^2 \int_{\rho=0}^\infty \frac{d[\rho^2+d^2]}{(\rho^2+d^2)^3} = 4\pi q^2 d^2 \frac{1}{2(\rho^2+d^2)^2}|_{\rho=\infty}^{\rho=0} = \frac{2\pi q^2}{d^2} \tag{118}$$

The actual force between the charges is $q^2/(4d^2)$, so that the force per unit area in field must be $\frac{E^2}{8\pi}$ which is a tensile stress and is along the lines of electric field.

Now consider the same situation but with both charges having the same sign. In this case the lines bend and become tangent to the $y-z$ plane and are clearly in compression. By symmetry the only non-zero component of the electric field is that that goes radially (in the $\hat{\rho}$ direction).

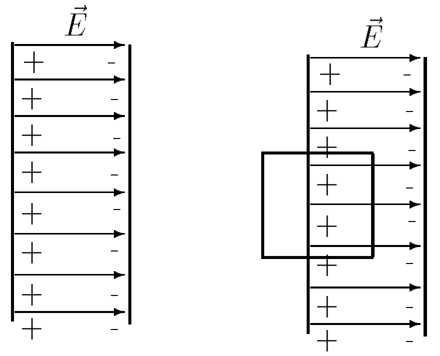
$$E_\rho^2 = \frac{2q}{d^2} sin\theta = \frac{2q}{d^2}\frac{\rho}{r} = \frac{2q\rho}{r^3}$$

where $E_\rho^2 = E_y^2 + E_z^2$. Again we can compute the total integrated mean square electric field strength in the $y-z$ plane:

$$\int E_\rho^2 dA = 4q^2 \int_0^\infty \frac{\rho^2}{r^6} 2\pi\rho d\rho = 4\pi q^2 \int_{d^2}^\infty \frac{(r^2)-d^2}{(r^2)^3} d(r^2) = 4\pi q^2 \left[\frac{1}{r^2} - \frac{d^2}{2(r^2)}\right]|_\infty^{d^2} = \frac{2\pi q^2}{d^2} \tag{119}$$

Thus again we find the compressive stress perpendicular to the electric field lines is $E^2/8\pi$.

Consider another simple case of tension along the lines of electric field, which is the familar simple capacitor.

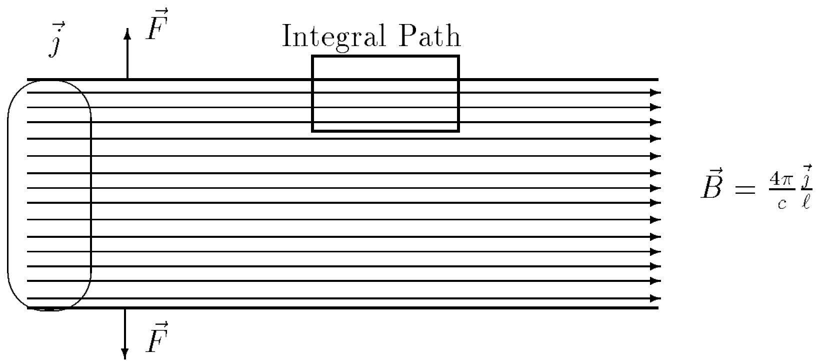


Clearly the lines of force, electric field lines are under tension. We can consider the charge on each of the capacitor faces to have a surface charge density equal to σ. Then by Gauss's law we can construct the usual pill box which has a uniform electric field passing though the face with area A and not on the sides or outside face. Thus in Gaussian units $4\pi\sigma = E$ (in Heavyside-Lorentz units, $\sigma = E$) and the force between the plates per unit area is

$$\frac{F}{A} = \frac{E\sigma}{2} = \frac{E^2}{8\pi} \tag{120}$$

(or in Heavyside-Lorentz units, $E^2/2$).

Now we turn to magnetic stress. First consider a very long solenoid or a current sheet.



The magnetic field is parallel to the solenoid and

$$\int \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c}\vec{j}$$

so that $B = 4\pi j/c\ell$. The Lorentz force on the current is

$$\vec{F} = q(\vec{v}\times\vec{B}) = \vec{j}\times\vec{B}$$

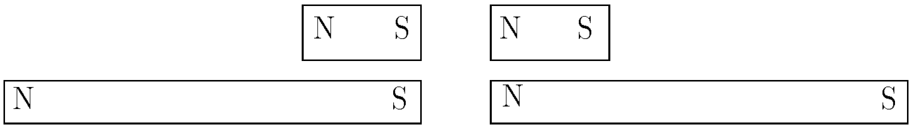
The force per unit area is equal to the average of the magnetic field at each edge of the solenoid or for an ideal solenoid this is half the internal magnetic field. We then have pressure stress

$$P_{\text{magnetic}} = \frac{cB^2}{8\pi} \tag{121}$$

The factor c depends upon the units one uses. Thus we see that like the electric field, the magnetic field can have compression perpendicular to the magnetic field lines.

Now we observe tension along magnetic field lines. Consider two magnets placed with poles near each other. If the poles are opposite, the magnets are attracted – tension in the direction of the lines. If the poles are the same, the magnets are repulsed – compression perpendicular to the lines.

We can see that this reduces to exactly the same case as for the charges calculated above by considering two long magnets.



As the magnets get longer and longer, each pole acts exactly as if it is an isolated charge and the math is the same.

Now we see that we need to have a momentum-energy tensor or more properly stress-energy tensor for electromagnetism.

5.20 Stress-Energy Tensor

We need to generalize this to 4-vectors and Lorentz invariance. This will require the use of second rank tensor - the stress-energy tensor.

In relativistic mechanics for continuous media the energy-momentum or stress-energy tensor, $T^{\alpha\beta}$, is usually defined as:

$$T^{ij} = \rho u^i u^j - E^{ij}; \qquad T^{i0} = T^{0i} = \rho u^i; \qquad T^{00} = \rho \tag{122}$$

where ρ is the density and E^{ij} is the Cartesian stress tensor usually defined as the tensor that describes the surface forces on a differential cube around the point in question. The normal surface force is pressure but there can be terms for tension/compression and shearing stress.

Then the equations of motion of a continuous medium is

$$\sum_\alpha \frac{\partial T^{\alpha\beta}}{\partial x_\alpha} \equiv T^{\alpha\beta}_{,\alpha} = f^\beta \tag{123}$$

where f^β is the 4-force density. That is the net force on material in a volume V is

$$F^\beta = \int\int\int_V f^\beta d^3V = \int\int_{surface} T^{\alpha\beta} dA_\beta \tag{124}$$

where the last equality comes from invoking Stoke's theorem.

In the case of electromagnetism in the 3-dimensional form the parallel equations are

$$\vec{F} = \int\int\int_V \left(\vec{E} + \vec{v} \times \vec{B}\right) \rho d^3V = \int\int\int_V \left(\rho\vec{E} + \vec{j} \times \vec{B}\right) d^3V \qquad (125)$$

Thus the force density $\vec{f}$ is

$$\vec{f} = \rho\vec{E} + \vec{j} \times \vec{B} \qquad (126)$$

Now we want to replace ρ and $\vec{j}$ by the fields via Maxwell's equations.

$$\rho = \vec{\nabla} \cdot \vec{E}, \qquad \vec{j} = \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Thus

$$\vec{f} = (\vec{\nabla} \cdot \vec{E})\vec{E} + (\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

Through suitable use of Maxwell's equations this can be recast to

$$\vec{f} = (\vec{\nabla} \cdot \vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E}) + (\vec{\nabla} \cdot \vec{B})\vec{B} - \vec{B}\times(\vec{\nabla}\times\vec{B}) - \frac{1}{2}\vec{\nabla}\left(E^2 + B^2\right) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

This is not a particularly elegant expression but is symmetrical in $\vec{E}$ and $\vec{B}$. The approach can be simplified by introducing the **Maxwell Stress Tensor**,

$$T_{ij} = \left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \qquad (127)$$

For example the indices i and j can refer to the coordinates x, y, and z, so that the Maxwell Stress Tensor has a total of nine components (3×3). E.g. with ϵ_0 and μ_0 explicitly stated instead of the units we usually use with c

$$T_{ij} = \begin{bmatrix} \frac{1}{2}\epsilon_0\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right) & \epsilon_0\left(E_xE_y\right) + \frac{1}{\mu_0}\left(B_xB_y\right) & \\ \epsilon_0\left(E_xE_y\right) + \frac{1}{\mu_0}\left(B_xB_y\right) & \frac{1}{2}\epsilon_0\left(E_y^2 - E_z^2 - E_x^2\right) + \frac{1}{2\mu_0}\left(B_y^2 - B_z^2 - B_x^2\right) & \\ \epsilon_0\left(E_xE_z\right) + \frac{1}{\mu_0}\left(B_xB_z\right) & \epsilon_0\left(E_yE_z\right) + \frac{1}{\mu_0}\left(B_yB_z\right) & \frac{1}{2}\epsilon_0\left(\right. \end{bmatrix}$$

And thus the force per unit volume is then

$$\vec{f} = \vec{\nabla} \cdot \vec{\vec{T}} - \frac{1}{c^2}\frac{\partial\vec{S}}{\partial t} \qquad (128)$$

And by Stoke's Law

$$\vec{F} = \int\int_{surface} \vec{\vec{T}} \cdot d\vec{A} - \frac{1}{c^2}\frac{d}{dt}\int\int\int_V \vec{S}d^3V \qquad (129)$$

This turns out to be a much more compact equation in 4-D vector notation.

For 4-dimensions the force law is $f^\mu = F^{\mu\nu} j_\nu$.

We want the full generalized relation between the energy-momentum tensor, $T^{\alpha\beta}$, and the 4-force to be:

$$\tilde{F} = \Box \cdot \tilde{\tilde{T}} \tag{130}$$

$$f^\mu = \sum_\nu \frac{\partial T^{\mu\nu}}{\partial x_\nu} \equiv \sum_\nu T^{\mu\nu}_{,\nu} \equiv T^{\mu\nu}_{,\nu} \tag{131}$$

where the last term represents the repeated indices summation convention. One uses $_{\text{,index}}$ indicates partial derivative with respect to x_{index} and repeated index to indicate summation on that index to make the equations easier to write and view.

For example,

$$\begin{aligned} f_x &= T_{xx,x} + T_{xy,y} + T_{xz,z} \\ \text{force}_{\text{x}} &= \Delta\text{pressure} + \Delta\text{shear stress} \end{aligned} \tag{132}$$

For electromagnetism the force equation is

$$f_\mu = F_{\mu\nu} j_\nu = F_{\mu\nu} F_{\nu\sigma,\sigma} \tag{133}$$

since $F_{\nu\sigma,\sigma} = j_\nu$. Thus we have

$$T_{\mu\nu,\nu} = F_{\mu\nu} F_{\nu\sigma,\sigma} \tag{134}$$

A tensor satisfying this equation is

$$T_{\mu\nu} = -\frac{1}{4\pi}\left[F_{\mu\alpha} F^{\alpha}_{\nu} - \frac{1}{4}\delta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right] \tag{135}$$

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\alpha} F^{\nu}_{\alpha} - \frac{1}{4}\delta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}\right] \tag{136}$$

$$T^{\mu\nu} = -\frac{1}{4\pi}\left[F^{\mu\alpha} F^{\nu}_{\alpha} - \frac{1}{4}g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}\right] \tag{137}$$

First consider the Maxwell stress tensor,

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right) \tag{138}$$

$$T_{xx} = \frac{\epsilon_0}{2}\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right) \tag{139}$$

$$T_{xy} = \epsilon_0 \left(E_x E_y\right) + \frac{1}{\mu_0}\left(B_x B_y\right) \tag{140}$$

and so on. Bear in mind that the stress tensor is symmetric. It is also possible to add some additional terms.

$$T^{00} = \frac{1}{8\pi}\left(E^2 + B^2\right) + \frac{1}{4\pi}\vec{\nabla} \cdot (\Phi \vec{E}) \tag{141}$$

$$T^{0i} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)_i + \frac{1}{4\pi}\vec{\nabla}\cdot(A_1\vec{E}) \tag{142}$$

$$T^{i0} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)_i + \frac{1}{4\pi}\vec{\nabla}\times(\Phi\vec{B}) - \frac{\partial}{\partial x_0}(\Phi E_i) \tag{143}$$

The added terms uses the free field $\vec{j} = 0$ Maxwell equations and included for completeness. If the fields are reasonably localized, then T^{00} is the field energy density, and the $T^{0i} = cP^i_{field}$ is the components of the field momentum density or the Poynting vector $\vec{S}$. Thus a simplified form is

$$T_{\mu\nu} = \begin{bmatrix} \frac{1}{8\pi}\left(E^2+B^2\right) & \vec{S} \\ \vec{S} & \text{Maxwell Stress Tensor} \end{bmatrix} \tag{144}$$

$$T_{\mu\nu} = \frac{1}{4\pi}\begin{bmatrix} \frac{E^2+B^2}{2} & E_yB_z - E_zB_y & E_zB_x - E_xB_z & E_xB_y - E_yB_x \\ E_yB_z - E_zB_y & \frac{(E_x^2-E_y^2-E_z^2)+(B_x^2-B_y^2-B_z^2)}{2} & E_xE_y + B_xB_y & E_xE_z + B_xB_z \\ E_zB_x - E_xB_z & E_xE_y + B_xB_y & \frac{(E_y^2-E_x^2-E_z^2)+(B_y^2-B_x^2-B_z^2)}{2} & E_yE_z + B_yB_z \\ E_xB_y - E_yB_z & E_xE_z + B_xB_z & E_yE_z + B_yB_z & \frac{(E_z^2-E_x^2-E_y^2)+(B_z^2-B_x^2-B_y^2)}{2} \end{bmatrix} \tag{145}$$

5.21 Bopp Theory

In classical electromagnetic theory there are two additional factors that must be taken into account: (1) the finite speed of light which means that the charge distribution can change and the change only propagates at the speed of light and (2) the 1/r form of the potential means that any point charge has infinite energy. To take into account the motion of charges one must end up using retarded potentials. In 3-D one has:

$$A_i(t, \vec{x}_1) = \frac{1}{c} \int \frac{j_i(t - r_{12}/c, \vec{x}_2)}{r_{12}} dV_2 \tag{146}$$

Bopp suggested a simpler form of the 4-vector potential which he thought might handle both problems:

$$A_\mu(\vec{x}_1) = \int\int\int\int j_\mu(t_2, \vec{x}_2) f(s_{12}^2) dV_2 dt_2 \tag{147}$$

Where $f(s_{12}^2)$ is a function which is zero every where but peaks when the square of the 4-vector distance s_{12}^2 between the source (2) and the point of interest (1) is very small. The integral over $f(s_{12}^2)$ is also normalized to unity. The Dirac delta function is the limiting case for $f(s_{12}^2)$. Thus $f(s_{12}^2)$ is finite only for

$$s_{12}^2 = c^2(t_1 - t_2)^2 - r_{12}^2 \approx \pm\epsilon^2 \tag{148}$$

Rearranging and taking the square root

$$c(t_1 - t_2) \approx \sqrt{r_{12}^2 \pm \epsilon^2} \approx r_{12}\sqrt{1 \pm \frac{\epsilon^2}{r_{12}^2}} \approx r_{12}(1 \pm \frac{\epsilon^2}{2r_{12}^2}) \tag{149}$$

So

$$(t_1 - t_2) \approx \frac{r_{12}}{c} \pm \frac{\epsilon^2}{2cr_{12}} \tag{150}$$

which says that the only times t_2 that are important in the integral of A_μ are those which differ from the time t_1, for which one is calculating the 4-potential, by the delay r_{12}/c ! – with negligible correction as long as $r_{12} \gg \epsilon$. Thus the Bopp theory approaches the Maxwell theory as long as one is far away from any particular charge.

By performing the integral over time one can find the approximate 3-D volume integral by noting that $f(s_{12}^2)$ has a finite value only for $\Delta t_2 = 2 \times \epsilon^2/2r_{12}c$, centered at $t_1 - r_{12}/c$. Assume that $f(s_{12}^2 = 0) = K$, then

$$A_\mu(\vec{x}_1) = \int j_\nu(t_2, \vec{x}_2) f(s_{12}^2) dV_2 dt_2 \approx \frac{K\epsilon^2}{c} \int \frac{j_\nu(t - r_{12}/c, \vec{x}_2)}{r_{12}} dV_2 \tag{151}$$

which is exactly the 3-D version shown above if we pick K so that $K\epsilon^2 = 1$.

This manner of thinking eventually leads one to the interaction Lagrangian as a the product of the two currents (electrical, matter, strong, weak, gravitational).

5.22 The Principle of Covariance

The laws of physics are independent of the choice of space-time coordinates.

Special Relativity applies this only to the choices of Euclidean (pseudo-Euclidean), non-rotating coordinate moving with constant velocities with respect to each other. That is by definition inertial frames.

General Relativity applies this to all conceivable space-time coordinates: rotating, accelerating, distorting, non-Euclidean, non-orthogonal, etc.

Einstein said that this principle is an inescapable axiom, since coordinates are introduced only by thought and cannot affect the workings of Nature.

Therefore the Principle of Covariance cannot have Physical Content to determine the laws of any part or field of physics.

Tensors are essential because all tensor equations of proper form are manifestly covariant; their functional form does not change when coordinates are changed. (Proper form means that both sides of the equation result in tensors of the same rank and, if the equation matches the classical limit formula, then it is the **only** correct form. *get the stuff in these parentheses, precisely right.*)

The form of a tensor equation provides no guide for selecting a particular "fixed" or "at rest" coordinate system. However, its content may provide this.

Covariance Language has heuristic invariance:

(1) It guides in proceeding, without telling where to go.

(2) It helps to prevent errors from staying with particular coordinates (through oversight or error).

(3) One should take as a first approximation to physical laws those which are *simple* in tensor language, but not necessarily simple in a particular coordinate system.