Physics 139 - Relativity
Problem Set 11 Due Week April 19 ,1999

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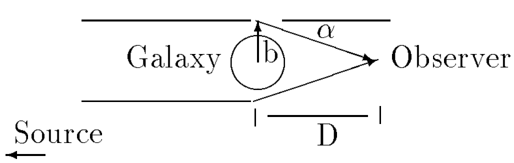
1 Bending of Path by the Sun

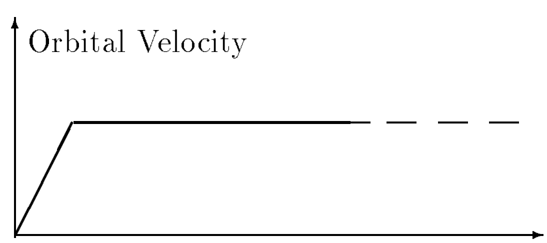
A *relativistic* particle of mass m and velocity v passes by the Sun with impact parameter (closest approach) b. The particle comes from infinitely far away and goes to infinitely far away. Find the angle of deflection caused by the Sun using the impact parameter approach used in the bending of light. Evaluate and show that in the two limits:
(a) Non-relativistic limit $\to 2GM_{\odot}/c^2b$
(b) Highly relativistic limit $\to 4GM_{\odot}/c^2b$

Hint: Use Newtonian physics for **(a)** and relativistic physics for **(b)**. Much less clearly solve the bending in terms of (E, p) and consider the limits $E/pc = 1$ for highly relativistic and $E/pc << 1$ for non-relativistic to find the factor of two.

2 Galaxy as Gravitational Lens

(a) A very distant quasar ($z > 3$) is located nearly directly behind an elliptical galaxy of mass $M_g = 10^{12} M_{\odot} = 2 \times 10^{42}$ kg which is 100 milion light years away. Assume that the galaxy is compact relative to the light path. What would be the angular radius of the Einstein ring of light from the quasar observed around the lensing galaxy?
(b) What would be angular radius of the Einstein ring, if the galaxy-observer distance were twice as great?
(c) How would the answer change, if the outside the nucleus the mass distribution causes a flat rotation curve. That is the orbital velocity is independent of radius.





Show in this case that the mass inside of impact parameter b is $M(r < b) \propto b$, where r is the radius from the center of the galaxy. (Derive a formula for the Einstein ring angular radius as a function of b and D the distance between lens and observer.)

(d) What would be the results in the unphysical case of a galaxy which rotated as a solid object?

3 Lensing Probability

If the density of ellipitical or spiral galaxies is about 0.02 h^3 Mpc^{-3}, what is the probability that a very distant source (10^{10} light years) is lensed? A Mpc = megaparsec = 3.26 million light years.

4 Schwartzschild Metric

The geodesic equation

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{1}$$

for the Schwartzschild metric produces four second-order differential equations for the motion of a test particle in the field of a spherically symmetric mass.

$$\frac{d^2r}{ds^2} + \frac{1}{2}.... \tag{2}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - sin\theta cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \tag{3}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \tag{4}$$

$$c^2\frac{d^2t}{ds^2} + \frac{d\nu}{dr}\frac{dr}{ds}\frac{dt}{ds} = 0 \tag{5}$$

Chose a path (orbit) which moves along $\theta = \pi/2$ with $d\theta/ds = 0$. In that case $d^2\theta/ds^2 = 0$ and $\theta = \pi/2$ permanently. This is motion in an equatorial plane. Then with the 2nd equation solved we move to the third:

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0 \tag{6}$$

which may be integrated once to yield

$$r^2\frac{d\phi}{ds} = constant = \ell \tag{7}$$

The first equation of motion becomes

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - \frac{2GM}{c^2r}\left[1 + r^2\left(\frac{d\phi}{ds}\right)^2\right] = k^2 - 1 \tag{8}$$

with r, ϕ being the cordinates of the Schwartzschild line element and ds is the proper time on a clock moving with the test body.

The similar Newtonian equations are:

$$r^2 \frac{d\phi}{dt} = constant = \ell \tag{9}$$

and

$$\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 - \frac{2GM}{c^2 r} = constant \tag{10}$$

where r, ϕ are ordinary polar coordinates and dt is the Newtonian (pre-relativistic) time.

Combine these two relativistic equations to show that the relativistic equation of motion is

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{c^2\ell^2} + 3GMu^2/c^2 \tag{11}$$

where $u = 1/r$ and this equation is the same as the Newtonian one except for the added $3GMu^2/c^2$

From the result of a previous problem set (relativistic hydrogen atom) take the solution and show the precession of Mercury in seconds of arc per century.

This is also provides a convenient way to calculate the gravitational deflection of light by a spherically symmetric mass. First note that the angular momentum forces $GM/c^2\ell^2 \to 0$. Then successive approximation. Find the solution to the case without the non-Newtonian term and then put the solution u_1 into the non-Newtonian term.