P139 - Relativity

Problem Set 14: Cosmology 1998 April

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1 Radius in Isotropic, Homogeneous Space

The Robertson-Walker metric in (r, θ, ϕ) form is

$$ds^2 = c^2dt^2 - \frac{dr^2}{1 - k\left(r/R\right)^2} - r^2\left(d\theta^2 + sin^2\theta^2 d\phi^2\right) \quad \text{where} \quad k = \{ \begin{array}{cc} +1 & \text{closed} \\ 0 & \text{flat} \\ -1 & \text{open} \end{array} \tag{1}$$

Thus the distance from the origin to $r = b$ with t, θ, ϕ constant is the radius and

$$\text{radius} = \int_0^b dl = \int_0^b \frac{dr}{\sqrt{1 - k\left(r/R\right)^2}} \tag{2}$$

The solution is

$$\text{radius} = \{ \begin{array}{cc} Rsin^{-1}\frac{b}{R} & \text{positive curvature} = \frac{1}{R^2} \\ b & \text{flat or infinte curvature} \\ Rsinh^{-1}\frac{b}{R} & \text{negative curvature} = -\frac{1}{R^2} \end{array} \tag{3}$$

Plot the radius as a function of b for the three geometries ($0 \leq b \leq \pi R$). Then make separate plots of (1) the circumference of a circle centered on the origin divided by $2\pi radius$, (2) the area of a sphere centered on the origin divided by $4\pi radius^2$ (3) the volume of a sphere centered on the origin divided by $4\pi radius^3/3$ for each of the three geometries.

2 True Distance

Find the true (measured) distance from the Earth to the quasare QSO(OH471) which has a redshift of $z = 3.4$ ($z + 1 = R_0/R_{QSO}$) Assume that our universe is a closed Friedman universe at current epoch $t_0 = 1.2 \times 10^{10}$ years and is matter dominated.

3 Cosmic Strings

These are essentially one dimensional regions of the higher energy vacuum state. Compute the bending of light by a cosmic string of mass per unit length μ. How does this depend upon impact parameter?

The metric line element in cyclindrical coordinates for a straight cosmic string on the z-axis is

$$ds^2 = dt^2 - dr^2 - (1 - 8G\mu/c^2)r^2 d\theta^2 - dz^2$$

where the mass per unit length is $\mu = \pi r_0^2 \rho_0$ under the condition $G\mu << 1$ This shape for the line element can be derived from the cyclindrical symmetry and mirror/translational symmetry. Compute the circumference of a circle of radius r around the string at fixed values of z and t. Does this match the geometry of a conical space? That is, a sheet with an angle $\Phi = 8\pi G\mu/c^2$ cut from it and the edges identified to make a conical surface?

A static string has no active gravitational mass, because the tension along the string just cancels the energy per unit length. However, if a string is not straight as would be expected due to the random walk of its production in causally disconnected regions, the tension will cause the string to want to straighten. Since the tension and mass per unit length match, the string will try to straighten at relativistic speeds.

What would happen if such a string passed through you as an unbiased observer? Hint: We can assume that the cosmic string is moving at a speed comparable to the velocity of light. The passage of the string should causes no immediate rearrangement of material in you, because space is flat with no tidal forces, outside the very narrow coherence width of the string. However, the deficit angle leaves the two sides of the observer approaching at a speed approximately $v\Phi$. For $G\mu/c^2 \sim 10^{-6}$ corresponding to a GUT energy scale of about 10^{16} GeV, this corresponding to a cut closure velocity of about 1 kilometer per second.

This same process would happen to the matter in the universe leaving a wake behind a moving string. It is possible to produce galaxies as a result of string wakes after matter-radiation decoupling at a redshift of about 1000. A Rough estimate is that the wake from a string with $G\mu/c^2 = 10^{-6}$ moving speed about the speed of light will at a redshifit of about 1000 produce a density fluctuation on the scale of $1h^1$ Mpc a density enhancement per unit area of about $10^{11}h^{-1}\ M_\odot$ Mpc^{-2}. This is about the right scale for a galaxy.

4 Expansion Rate and Temperature

In the radiation dominated era we know that $a \propto t^{1/2}$ So that

$$H = \frac{\dot{a}}{a} = \frac{1}{2}\frac{1}{t}$$

One Friedmann equation (energy conservation) gives us

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi}{3}G\rho$$

for the early universe the curvature is negligible (see the notes). Taking the square root gives

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3}G\rho} = \sqrt{\frac{8\pi}{3}G\frac{g_{eff}}{2}\sigma_{SB}}\, T^2$$

Thus we have the relationship

$$2H = \frac{1}{t} = \sqrt{\frac{16\pi G\sigma_{SB}}{3c^2}g_{eff}}T^2 \cong 3.027 \times 10^{-21}[\frac{T}{1K}]^2 sec^{-1}$$

By dimensional analysis this should depend on the inverse square of the kinetic energy kT in terms of the the Planck energy (Planck mass, $M_{Pl} = \sqrt{\hbar c/G}$ times c^2), times the Planck time $t_{Pl} = \sqrt{\hbar G/c^5}$, and clearly from above the coefficient includes $1/\sqrt{g_{eff}}$. g_{eff} is the effective number of degrees of freedom. Find the proportionality factor for the dimensional analysis relation

$$t \propto \frac{1}{\sqrt{g_{eff}}}(\frac{kT}{M_{Pl}c^2})^{-2}t_{Pl}$$