Physics 139 Relativity
Relativity Notes 2003

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Notes to be found at:
http://aether.lbl.gov/www/classes/p139/homework/homework.html

1 The Kaluza-Klein Argument

Kaluza (1921) considered the case of a (4 + 1)-dimensional spacetime. We define the 5-dimensional position vector as (x^μ, y). We call the components of the 5-vector as x^A or x^B such that A and B vary according to 0, 1, 2, 3, 4. We assume that the space is Riemannian so that line element is

$$ds^2 = g_{AB}dx^A dx^B = g^{AB}dx_A dx_B \tag{1}$$

The symmetric metric temperature of the 5-space can be written in the general form

$$g^{(5)} = \left(g^{AB}\right) = \phi^{-1/3}\begin{pmatrix} g^{\mu\nu} + \phi A^\mu A^\nu & \phi A^\mu \\ \phi A^\nu & \phi \end{pmatrix}. \tag{2}$$

At this stage, the 4×4 matrix $g^{\mu\nu}$, the vector A^μ and the scalar ϕ are arbitrary.

Or we can consider our 5D metric and consider only transformations that do not depend upon the 5th dimension. Then g^{44} is the scalar and $g^{4\mu} = g^{\mu 4}$ is the 4-vector, and $g^{\mu\nu}$ is the 2nd rank 4-tensor that corresponds to our usual GR (3+1)-dim metric.

Now write down the 5D action of General Relativity

$$S = -\frac{1}{16\pi G^{(5)}}\int R^{(5)}\sqrt{\pm g^{(5)}}d^4x^\mu dy \tag{3}$$

Rewriting this interms of the righthand decomposition of the metric tensor, the action can be rearranged into

$$S = -\frac{1}{16\pi G}\int\left(R + \frac{1}{4}\phi F^{\mu\nu}F_{\mu\nu} + \frac{1}{6\phi^2}\partial^\mu\phi\partial_\mu\phi\right)\sqrt{g}d^4x^\mu \tag{4}$$

Where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. To obtain this expression, it is necessary to assume that there is no dependence on y. This first appeared as an unjustified assumption and had the problem that there was no obvious 5-th dimension. Klein (1927) modified the idea by considering the y (4th spatial) dimension to be compact in the sense of too small to be detected (with existing technology and (perhaps) rolled into a cylinder = edges identified. The cylinder condition requires periodic boundary conditions set by the circumference of the cylinder which with the smallness naturally explains why there is no dependence on y. Given that assumption and a lot of algebra, one arrives at the formula above.

The physical content of the result is astonishing: one gets General Relativity, electromagnetism, plus a scalar field, whose quantum is know as the dilaton. In some ways this is a very radical innovation.

One can think of extending this approach to include the other forces. We can add extra dimensions to create the vector fields that carry the forces. The most common assumption is that, if the extra dimensions are real, they must be compactified - i.e. the universe must be very highly curved so that its typical extent in the extra dimensions may be on the order of the Planck length. Usually the Planck length is picked out as the conventional Einstein style is unstable and it is assumed that quantum effects come in to stablize at the Planck length. There are other approaches that have larger (but still small - less than a TeV) dimensions. Others make the extra dimensions large but confine most of the forces are confined to the (1+3) dimensions we know and love.

1.1 Decomposition of 2nd rank Tensors

We consider here the decomposition by transformation properties of second rank tensors for transformations that do not depend upon one of the coordinates. There is a natural dcomposition into three classes of components: a second rank tensor of one less dimension, two vectors of one less dimension, and a scalar.

Note that this is generically true for any second rank tensor tensor. it is left to the student to show this rigorously but here is an outline of the argument. Remember the transformation law for a second rank tensor is

$$A'_{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} A_{\alpha\beta}$$

Thus if there is no dependence on coordinate x^n where for convenience we take $n = 0$ for metric pertubations and $n = 4$ (the fifth dimension) for the Kaluza-Klein approach. Then we must have $\partial x'^{\mu}/\partial x^n = \delta^{\mu}_{n}$, that is to say that there is no dependence and no change in the n-th coordinate in the transformations considered only the other coordinates are changed in going from x^{μ} to x'^{μ}. Then the components of $A_{\mu\nu}$ transform according to

$$\begin{aligned}
A'_{nn} &= A_{nn} \quad \text{transforms as a scalar} \\
A'_{\mu n} &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A_{\alpha n} \quad \text{transforms as a vector} \\
A'_{\mu n} &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A_{n\alpha} \quad \text{transforms as a vector} \\
A'_{\mu\nu} &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} A_{\alpha\beta}
\end{aligned} \tag{5}$$

where the last are the components of $A_{\mu\nu}$ with with $\mu \neq n$ which transform as a second rank tensor. Thus if there is not change (dependence) on one coordinate in the transformations a second rank tensor of m dimensions can be separated (decomposed) into a second rank tensor of $m-1$ dimensions, two vectors (one if the tensor is symmetric) of $m-1$ dimensions and one scalar A_{nn}. Clearly the same is true for the contravariant tensor.

For simplicity consider a 4-metric tensor of the form

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \text{scalar} & h_{0i} \text{ vector} \\ h_{0i} = h_{i0} & g_{ij} \end{pmatrix} = a(t)^2 \begin{pmatrix} g_{00} = 1 + 2\Phi/c^2 & w_i \\ w_i & -(1-2\Phi/c^2)\delta_{ij} + \end{pmatrix} \tag{6}$$

where the last h_{ij} must be arranged so that it is transverse traceless as the background (show as flat version of Robertson-Walker and using conformal time η for the 0th coordinate) makes up the symmetric trace portion. Usually the spatial scalar potential is designated by Ψ as it can evolve differently in time than the g_{00} Φ.