Physics 139 Relativity
Problem Set 6   Due Week March 1, 2003

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1 Force between moving charges

In frame S, two identical point charges q move abreast along lines parallel to the x-axis, a distance r apart and with velocity v. Determine the force in frame S that each exerts on the other, and do this in two ways: (i) by use of the Lorentz force in conjunction with the field; and (ii) by transforming the Coulomb force from the rest-frame to the lab frame S. Note that this force is smaller than in the rest-frame, while the mass is greater. Here we see the dynamical resons for the 'relativistic focusing' effect whose existence we recognize as inevitable by purely kinematic considerations. Show that the dynamics leads to exactly the expected time dilation of and 'electron clock'.

Hint Try a transform of $\vec{E}' = q\vec{r}/r^3$ and $\vec{B}' = 0$.

2 Relativistic Ohm's Law

The relativistic Lorentz force law derived from the 3-D force law is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \rightarrow \quad \frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \tag{1}$$

What is the relativistic generalization of Ohm's law?

$$\vec{j} = \sigma\vec{E} \quad \rightarrow \quad ? \tag{2}$$

It must be expressible invariantly in terms of j^μ, $F^{\mu\nu}$, σ (electrical conductivity), and u^μ (the 4-D velocity of the conducting element).

Hint: The expression must reduce to 3-D Ohm's law in the rest frame of the conducting element. If the expression is written in manifestly covariant (tensor) form, then if it is correct in one frame, it will be correct in all frames.

Additional Hint: In the simple electric field case one has $\vec{j} = \sigma\vec{E}$ for Ohm's law. When magnetic fields are present, e.g. the Hall Effect, then there is an additional term $\vec{j} = \sigma\left(\vec{E} + \vec{\beta} \times \vec{B}\right)$ Also the simple case is for a conductor at rest, if the conductor is moving, then there is a term for the motion of the charge carriers: $\vec{j} = \vec{j}' + \rho_0\vec{v}$. The rest charge in a typical conductor is zero as the carriers (electrons) and matrix (nuclei) balance each other out. In the Special Relativistic case relative motion of the carriers changes this. This difference is why some say that using currents to produce magnetic fields is a 100% relativistic effect even for relatively slow moving charges.

3 Scalar 4-Potential Flaw

If a scalar field Φ is the source of a 4-force $F_\mu = \Phi_{,\mu}$, prove that the rest mass m_0 of particles subject to this force must vary as $m_0 = \Phi/c^2 + \text{constant}$.

Hint: Show that

$$F_\mu u^\mu = c^2 \frac{dm_0}{d\tau} = \gamma^2 \left(\frac{dE}{dt} - F_i v_i \right)$$

starting with the defintion of 4-force $F_\mu = dp_\mu/d\tau$ and allowing rest mass to vary with time.

4 Relativisitic Motion in an Electric Field

In an inertial frame S there is a uniform electric field E in the direction of the positive x-axis. (Electromagnetic fields are rest mass preserving.) A particle or rest mass m_0 and rest charge q_0 is projected into this field in the y-direction with an initial velociy u_i and corresponding $\gamma_i = 1/\sqrt{1 - u_i^2/c^2}$. Prove that its trajectory is a catenary whose equation relative to a suitably chosen origin is

$$x = \frac{c^2 m_0 \gamma_i}{qE} cosh \left(\frac{qEy}{cm_0 u_i \gamma_i} \right)$$

Hint:

$$\frac{d^2 y}{d\tau^2} = 0, \qquad m_0 \frac{d^2 x}{d\tau^2} = \gamma q E, \qquad \gamma = \frac{1}{c} \sqrt{\left(\frac{dx}{d\tau} \right)^2 + c^2 \gamma_i^2}$$

Show that in the particle instanteous rest frame the acceleration of the particle is $qE/m_0\gamma(u_2)$ and is thus not constant.

5 Special Relativistic Hydrogen Atom

The Bohr theory of hydrogenic atoms (atoms having a nucleus with charge +Ze and a single electron) was extended by Sommerfeld using the classical dynamics of Special Relativity. He derived the same energy levels that were later shown to follow from Dirac's relativistic quantum wave equation. THIS IS A VERY REMARKABLE COINCIDENCE.

Sommerfeld's calculation differs from Bohr's only in use of the relativistic expressions for energy and momentum in terms of velocity. Reproduce the derivation. Key equations along the way are [with (r, ϕ being the plane polar coordinates]:

$$m_o c^2 \left[1 - (v/c)^2 \right]^{-1/2} + V = \text{constant} = \text{W} = \text{energy}$$

$$m_o \left[1-(v/c)^2\right]^{-1/2} r^2 \frac{d\phi}{dt} = \text{constant} = \text{L} = \text{angular momentum}$$

$$dt = m_o \gamma r^2 d\phi / L, \qquad \gamma = \left[1-(v/c)^2\right]^{-1/2}$$

$$\gamma = \frac{[W + Ze^2/r]}{m_o c^2}$$

Define $u = 1/r$. After some calculation,

$$1 + \left(\frac{L}{m_o c}\right)^2 \left[(\frac{du}{d\phi})^2 + u^2\right] = \left[\frac{W + Ze^2 u}{m_o c^2}\right]^2$$

taking $d/d\phi$, and dividing by a common factor, one finds the orbit equation:

$$\frac{d^2 u}{d\phi^2} + \left[1 - \left(\frac{Ze^2}{cL}\right)^2\right] u = \frac{Ze^2 W}{(cL)^2}$$

The coefficient of u is

$$\lambda^2 = 1 - Z^2 \left(\frac{e^2}{\hbar c}\right)^2$$

is very near to unity, where $L = n\hbar$, $\hbar = h/2\pi$, $n = 1$ (the lowest value), and $e^2/\hbar c$ is the famous Sommerfield fine structure constant, closely equal to to 1/137.

Show that the orbit equation yields a slowly precessing ellipse, whose point of closest approach to the nucleus (assumed to be at rest) advances in one period by an angle

$$\Delta\phi = 2\pi(1-\lambda)/\lambda$$

Use this method to calculate the advance of the perihelion of the planet Mercury orbiting about the Sun in units of arcseconds per century. For this purpose you will need to know the semi-major axis, the period, and the eccentricity of Mercury's orbit, ($a = 0.387094$ A.U., A.U. $= 1.49597892 \times 10^{13}$ cm, period $= 0.241$ years, $\epsilon = 0.20561421$), and you will need to use Kepler's laws. The answer you should get is quite wrong when compared with the known advance; this is because the Special Theory of Relativity is inadequate to deal with gravitational fields. Einstein's General Theory of Relativity predicts precisely the observed advance of Mercury's perihelion (when taken together with the known perturbations of the other planets).

6 Electrostatic Particle Velocity Separator

This and next problem are extra credit. Some experiments want to separate beams of particles by mass and momentum. In an electrostatic velocity separator ("Wein filter"), a homogeneous vertical electrostatic field E_y exerts a vertical force on a horizontally moving particle with velocity, v_x, which is opposed by the oppositely directed force exerted by a homogeneous horizontal transverse magnetostatic field B_z. For particles having the selected velocity the forces are equal and opposite; these particles are not deflected and thus pass through the slits at each end of the separator.

For this problem consider a separator with length 10 m and electric field $E_y = 60$ kV/cm, designed to select protons with a momentum of 8.0 GeV/c.

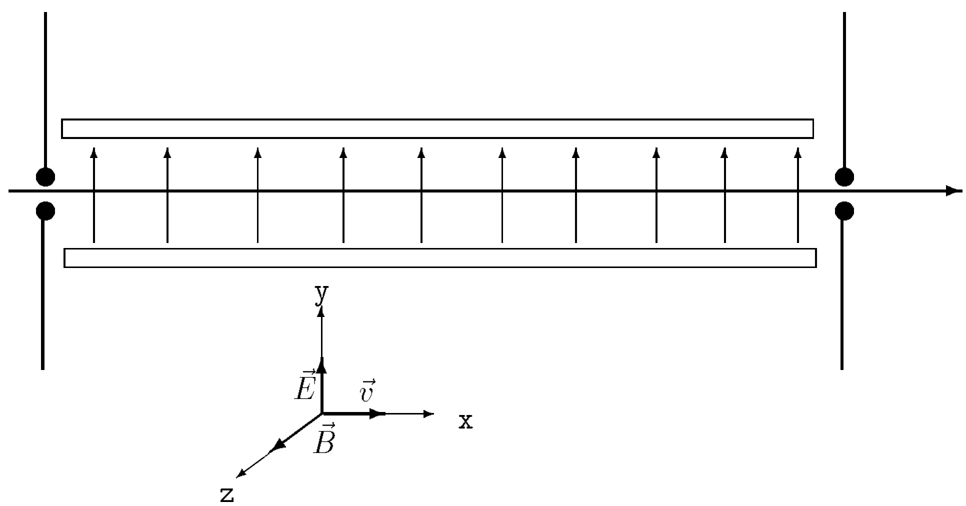
a. Find the velocity of these protons ($m_p c^2 = 938.272$ MeV) relative to that of light. Evaluate $1 - (v/c)$ to three significant figures.

b. Find their kinetic energy in GeV.

c. Find the required magnetic field B_z.

d. Find the time to pass through the separator.

e. Consider a "wrong" particle – a pion ($\pi^{\pm}$ $m_\pi c^2 = 139.57$ MeV) having the same momentum. What is its velocity relative to that of light? Find the difference to three significant figures. What is the time from the first to the second slit? What is its transverse displacement at the second slit? Find its transverse momentum there. What angle does the pion's path make with the proton beam there?



7 Rest Frame Separator

In the preceding problem transform to the rest frame of the proton. In that frame find the electric field, the magnetic field, the force on the proton, and the time of passage between the slits.

Find the velocity of the pion, the force on it, and its transverse displacement at the slit in the proton rest frame.