Physics 139 Relativity
Problem Set 7 Due Week March 9, 2003

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1 Alpha-Particle Decay

A polonium nucleus at rest in the laboratory disintegrates (radioactive decay) into a lead nucleus and an alpha particle: ${}_{84}Po^{210} \rightarrow {}_{82}Pb^{206} +_2 He^4$. The kinetic energy of the alpha particle is measured to be 5.30 MeV. Calculate to three significant figures the "Q" of the reaction, defined as the loss of rest energy: $\mathrm{Q}/c^2 = m_{Po} - m_{Pb} - m_{He}$. The final state is a very nonrelativisitic (both decay products have small velocity - $\beta \ll 1$) but the observed loss of proper mass is a purely relativisitic effect.

2 Four Force Example

A nuclear power plant is stationary and experiences no net force ($\vec{F}$) in system S'. The reactor exhausts heat through its cooling tower at the rate of 900 MW. If System S' has a velocity $\beta = 4/5 = 0.8$ relative to frame S.
(a) Evalute the force 4-vector in the S' and S frames.
(b) What is the vector force (in Newtons) acting on the plant in S?
(c) What is the power output (in megawatts) in S?

3 Relativistic Torque and Angular Momentum

Explain why torque in special relativity is a second rank antisymmetric tensor of the form

$$N_{\mu\nu} = x_\mu f_\nu - f_\mu x_\nu$$

If relativistic angular momentum is given by

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$$

show that

$$\frac{dL_{\mu\nu}}{d\tau} = N_{\mu\nu}$$

Hint: Compare this to the 3-vector formulae.

4 Linear vs. Circular Accelerator

The 2-mile-long Stanford Linear Accelerator accelerates electrons to an energy of 40 GeV as measured in the rest frame of the accelerator. Idealize the accelerator as a constant electric field over 2 km length. Assume that the equation of motion is

$$\frac{d\vec{p}}{dt} = e\vec{E}$$

where $\vec{p}$ is the relativistic three momentum.

What is the electric field necesary?

What is the acceleration very near the end of the accelerator?

What is the power loss in radiation for this linear acceleration?

How does this compare to the synchrotron radiation from a circular accelerator with a circumference equal to SLAC's length?

5 Relativistic Beaming of Radiation

On the next page is a set of two parallel plots: one for perpendicular power divided by γ^8 and for parallel power divided by γ^{10}. The normalization is such that the curves would fit on the plot and be visible. Label the plots for perpendicular and parallel acceleration and label the curves for $\beta = 0$, 0.5, 0.9, and 0.999. Does the angle of maximum radiation appear to be at $\theta = 1/\gamma$?

6 Frequency Spectrum of Relativisitic Beaming

An electron moves in a circle of radius R at relativisitic speed v due to a uniform magnetostatic field. The energy spectrum of radiated photons peaks at a laboratory frequency about γ^3 times the circulation frequency of the electron.

(a) Give a simple qualitatively accurate explanation of this fact.

(b) Asssuming that all radiated photons have about this frequency, calculate from the energy-loss formula the number of quanta emitted in one complete circle. Evaluate for a 300 MeV electron and a 3 GeV electron, each in a 1 Tesla (10 kGauss) magnetic field.

(c) Still under this assumption, calculate the number of photons emitted in an arc of the circle of angle γ^{-1} associated with the recepton by an observer looking tangent to the electron's path. Evaluate it for each of the two cases in (b).

(d) Calculate in each case the fractional rate of shrinkage ($\frac{1}{R}\frac{dR}{dt}$) of the orbit in one second due to the energy loss by radiation.

(e) For the same energies but in different fields giving different orbit radii, find for each energy the radius at which the peak of the radiated spectrum is roughly in the center of the visible optical spectrum.

7 Energy Loss for High-Energy Electrons

An ultrarelativisitc electron emits synchrotron radiation.
(a) Show that its energy decreases in time according to:

$$\gamma = \frac{\gamma_o}{1 + A\gamma_o t} \qquad A = \frac{2e^4 B_\perp^2}{3m_e^3 c^5} \tag{1}$$

Here γ_o is the intitial value of γ and $B_\perp = B sin\alpha$.
(b) Show that the time for the electron to lose half of its energy is

$$t_{\frac{1}{2}} = (A\gamma_o)^{-1} = \frac{5.1 \times 10^8}{\gamma_o B_\perp^2} \text{ sec} \qquad B_\perp \text{ in Gauss} \tag{2}$$

(c) How does one reconcile the decrease of γ here with the result of constant γ implied in the derivation of the synchrotron energy loss formula?
(d) If the initial cosmic ray electron spectrum was a power law

$$\frac{dN(E)}{dE} = KE^{-p},$$

how does the power spectrum go after a time Δt of propagation in a magnetic field? Evaluate for a magnetic field of a 3 μGauss and a time period $\Delta t = 10^7$ years. Hint make a plot (sketch) of log(dN/dE) vs. log(E), first with the original power spectrum and then sketch the curve of an electron moving perpendicular to the field lines and one at a typical inclination angle.

8 Polarization of Synchrotron Radiation

beginning of extra credit problems Radio astronomers can tell if the radiation they are seeing is synchrotron radiation because it is polarized.
(a) Why is it polarized? (Hint: remember the factors in front of $a_\perp^2$ and $a_\|^2$.) Show in a sketch the direction of polarization as a function of observation angle to the magnetic field direction.
(b) The linear polarization for the frequency-integrated synchrotron emission of particles of the same γ is 75%. For particles with a power law distribution of energies (see problem above $dN/dE \propto E^{-p}$), the degree of polarization is

$$\Pi = \frac{p+1}{p+7/3}$$

Can you provide an argument why this is true?

9 High Energy Muon Collider

Imagine you are a high-energy physicist who would like to probe the unification of the electromagnetic and weak forces (electroweak unification) and to discover the Higgs boson and thus determine the origin of mass. Current theory indicates that colliding beams with energies above 1 TeV should show unification effects and produce Higgs bosons.

(a) It is clear that a muon-on-muon ($m_\mu c^2 = 105.66$ MeV) collider is better than an electron on electron collider, due to synchrotron losses. Find the ratio of synchroton losses for an electron ($m_e c^2 = 0.511$ MeV) to a muon in the same accelerator. Assume a magnetic field of 8 Tesla and find the following for an electron, a muon, and a proton ($m_p c^2 = 938.272$ MeV): (i) radius of the machine, (ii) orbit frequency or period, (iii) radiated synchrontron power, (iv) energy loss per orbit (per turn around the accelerator) in electron volts. Make a table with the headings, quantity, formula, electron, muon, proton.

(b) It is also true that muons are better than protons since at high energies a proton first behaves as if it is made of three quarks rather than behaving as a point-like particle. By an energy of 1 TeV it typically behaves as if it were made of 10 quarks (the original three plus virtual quark-antiquark pairs made real). How much more energy would one have to provide to each proton to get a typical quark on quark collision to have the same c.m. energy as a 1 TeV on 1 TeV muon collision?

(c) The Higgs boson coupling strength to a particle is proportional to its mass. The Higgs is the source of the mass. Thus the probability of producing a Higgs is proportional to the product of the colliding masses. How many more particles does one need in a beam to get the same number of Higgs produced per second as a muon-on-muon beam in an electron-on-electron beam and in an proton-on-proton beam (read quark-on-quark beam with a typical effective quark mass of 5 MeV)? Thus how much more initial accelerator energy? Then when including the synchrotron radiation losses.

(d) If mass and point-like is good, why not use a τ-lepton with a mass of 1.777 GeV/c^2 and a lifetime of $(291 \pm 1.5) \times 10^{-15}$ second? The muon also decays with a mean lifetime of 2.2×10^{-6} seconds, why can it be accelerated and stored? How long does the average muon last in the laboratory frame? How many times does the average muon go around the accelerator?

(e) A muon in the beam decays into two neutrinos (a muon-neutrino and an antielectron-neutrino) and an electron. What is the approximate energy of each decay product?

The decay electron immediately begins to emit synchroton radiation and curve inward toward the inner beam pipe wall. What is the rough split between the synchroton radiation loss by the electron to the beam pipe's outer wall and its kinetic energy that it deposits in the beam pipe's inner wall? Assume the muon beam is concentrated in the center of a 2-cm wide beam pipe.