

Physics 139 Relativity
 Problem Set 9 Due Week April 2, 1998

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1 Christoffel Symbols

Make a guided calculation of the Christoffel symbols, Γ^m_{ij} , for a spherical surface:

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (1)$$

The metric tensor is then

$$g_{ij} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{bmatrix} \quad g^{ij} = \begin{bmatrix} r^{-2} & 0 \\ 0 & (r \sin \theta)^{-2} \end{bmatrix} \quad (2)$$

and

$$\Gamma^m_{ij} = \frac{1}{2} g^{mk} [g_{ik,j} + g_{jk,i} - g_{ij,k}] \quad (3)$$

Show that there are only three non-zero Christoffel symbols:

$$\Gamma^1_{22} = -\sin \theta \cos \theta ; \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot \theta \quad (4)$$

2 Riemann-Christoffel Curvature Tensor

Calculate the Riemann-Christoffel curvature tensor, R^k_{ars} , for the spherical surface of problem 1.

$$R^k_{ars} = \Gamma^k_{ar,s} - \Gamma^k_{as,r} + \Gamma^b_{ar} \Gamma^k_{sb} - \Gamma^b_{as} \Gamma^k_{rb} \quad (5)$$

Show that

$$R^1_{212} = -R^1_{221} = \cos^2 \theta - \sin^2 \theta + (\cot \theta)(-\sin \theta \cos \theta) = -\sin^2 \theta \quad (6)$$

$$R^1_{112} = -R^2_{121} = \csc^2 \theta - \cot^2 \theta = 1 \quad (7)$$

$$R^1_{112} = R^1_{121} = R^2_{212} = R^2_{221} = R^i_{j11} = R^i_{j22} = 0 \quad (8)$$

3 Ricci Curvature Tensor & Contraction

Contraction of the Riemann-Christoffel curvature tensor, R^k_{ars} , of the contravariant index with the last covariant index gives the Ricci tensor.

For the spherical case

$$R_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \quad (9)$$

Find the mixed Ricci tensor and show it is

$$R^i_j = \begin{bmatrix} 1/r^2 & 0 \\ 0 & 1/r^2 \end{bmatrix} \quad (10)$$

Contract the mixed Ricci tensor to find the scalar curvature.

4 Ellipsoidal Surface

What do you think the mixed Ricci tensor would be for an ellipsoidal surface?