Physics 139 Relativity
Relativity Notes 2002

G. F. SMOOT
Office 398 Le Conte
Department of Physics,
University of California, Berkeley, USA 94720
Notes to be found at
http://aether.lbl.gov/www/classes/p139/homework/homework.html

1 Lorentz Transformations

The Lorentz transformations may be obtained in one of several ways which includes (1) fitting to experimental observations, (2) using the two postulates of special relativity, or (3) assuming Minkowski (4-dimensional) space and finding what are the transformations that leave 4-D vectors lengths invariant.

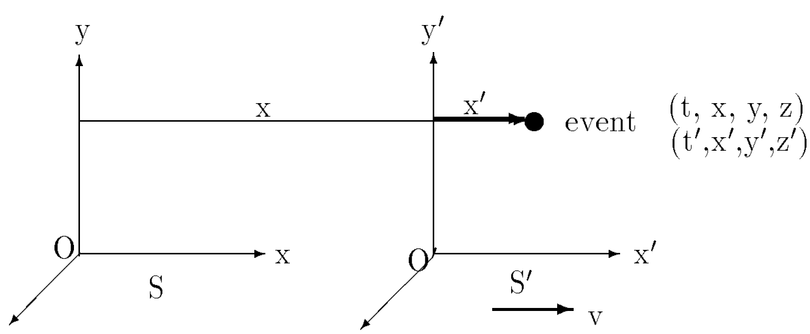
History: Lorentz transformation was derived by Lorentz before Einstein's work. Lorentz obtained them by considering invariance of Maxwell's Equations and the Michelson and Morely experimental results.

Significance: The Lorentz Transformations define the mathematical specification required to discuss a kinematic occurrence – a sequence of space-time events.

1.1 Experimental Lorentz Transformation Derivation

The Lorentz transformations are demanded and supported by experimental observations. The Lorentz transformation equations can readily be derived from length contraction and time dilation after taking a short detour to discuss clock synchronization.

Consider two frames of reference: S, the laboratory frame and S′ a frame of reference moving with velocity $\vec{v}$ in the $\hat{x}$ direction as shown in the following figure.



Arrange things so that at $t = 0$ and $t' = 0$ that the two origins O and O$'$ coincide.

Consider each reference system to be an actual lattice of meter sticks and clocks, e.g. each reference system is filled with these space and time measuring devices at every point.

Get clocks in S to agree. Identical clocks set by sending out a light pulse from origin O and also from the midpoint between any two clocks. Check times, reflect back to midpoint, if pulses arrive together, then clocks agree.

System S$'$ does the same with his clocks.

We have for the space-time event in the figure above

$$x = vt + x' \times \sqrt{1 - v^2/c^2} \tag{1}$$

where the second term takes into account length contraction of a moving frame. We can use our arguments about the transverse directions to show that they are unchanged and then have the spatial Lorentz transformations:

$$\begin{aligned} x' &= \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt) \\ y' &= y \\ z' &= z \\ t' &= t\sqrt{1 - v^2/c^2} + \text{synchronization effect} \end{aligned} \tag{2}$$

1.1.1 Synchronizing Clocks in Moving Frame

Our approach to get our system of reference made of a grid of meter sticks and synchronized clocks requires that we synchronized the clocks. An approach to synchronizing the clocks is: bring the clock together, match their readings, then move into place. Move them slowly and gently so as not to disturb their operation.

Consider the simple case of two clocks brought together at the origin of the moving system S$'$. When they are together, from the laboratory frame S both clocks read same time and are going slow by a factor $\sqrt{1 - (v/c)^2}$ as a result of **time dilation**. Now very slowly and gently move one clock back (in negative x'-direction; toward the laboratory system origin) a distance ℓ in elapsed time $\ell = \delta v\ \tau$. The clock at the origin has its rate slow by $\sqrt{1 - v^2/c^2}$ relative to the laboratory frame. Clock moving back in negative x'-direction has its rate slowed by the factor $\sqrt{1 - (v - \delta v)^2/c^2}$

$$f_A = \sqrt{1 - v^2/c^2} f_o \qquad f_B = \sqrt{1 - (v - \delta v)^2/c^2} f_o \tag{3}$$

The difference in the clocks' rates is

$$f_A - f_B = f_o \left[\sqrt{1 - v^2/c^2} - \sqrt{1 - (v - \delta v)^2/c^2}\right]$$

$$
\begin{aligned}
&= \frac{f_o}{\sqrt{1-v^2/c^2}}\left[1-\frac{v^2}{c^2}-\left((1-\frac{v^2}{c^2})(1-\frac{(v-\delta v)^2}{c^2}\right)^{1/2}\right] \\
&= \frac{f_o}{\sqrt{1-v^2/c^2}}\left[1-\frac{v^2}{c^2}-\left((1-\frac{v^2}{c^2})(1-\frac{v^2}{c^2}+\frac{2\delta v v}{c^2}-\frac{\delta v^2}{c^2})\right)^{1/2}\right] \\
&= -f_a\frac{\delta v\ v/c^2}{1-v^2/c^2} = -f_o\frac{\delta v\ v/c^2}{\sqrt{1-v^2/c^2}} \qquad (4)
\end{aligned}
$$

If it takes a time $\tau = \ell_o/\delta v$ to separate the clocks, the time difference between them is

$$
\begin{aligned}
\Delta t &= \frac{f_A - f_B}{f_A}\tau = \left(\sqrt{1-v^2/c^2} - \sqrt{1-(v-\delta v)^2/c^2}\right)\tau \\
&= -\frac{\delta v\ v/c^2}{\sqrt{1-v^2/c^2}} \times \frac{\ell_o}{\delta v} = -\frac{\ell_0\ v/c^2}{\sqrt{1-v^2/c^2}} = -\ell' v/c^2 \qquad (5)
\end{aligned}
$$

Note that the speed with which the clock moves drops out and the change in reading is proportional only to the distance displaced and the velocity of the moving system.

Clocks get out of synchronization (phase) by an amount proportional to their separation ℓ_o and v. If brought back together, the clocks will go into synchronization. **The clock that is farther behind in space is further ahead in time.**

Note that in the frame S′ the difference in rate of time kept between the clock at the origin and the one being moved back to its place is second order in v/c rather than first order:

$$
f'_B = \frac{f'_A}{\sqrt{1-(\delta v)^2/c^2}} \simeq f'_A \times \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)
$$

So that by moving with a very, very slow velocity the integrated effect in the S′ frame can be made arbitrarily small while the effect as observed in the S frame is always $-\ell' v/c^2$ independent of δv. That is because the effect in frame S is first order in $\delta v/c$ and integrated over time equals the displacement.

The final Lorentz transformations are:

$$
\begin{aligned}
t' &= t\sqrt{1-v^2/c^2} - \frac{x' v}{c^2} = t\sqrt{1-v^2/c^2} - \frac{v^2}{c^2}\frac{1}{\sqrt{1-v^2/c^2}}(x - vt) \\
&= \frac{1}{\sqrt{1-v^2/c^2}}\left[t - \frac{vx}{c^2}\right] = \gamma\left[t - \frac{vx}{c^2}\right] \qquad (6)
\end{aligned}
$$

Notice for $v << c$ get Galilean transforms and there is also a symmetry between the transformation equations.

$$
\begin{array}{cc}
t' = \gamma(t - vx/c^2) & t = \gamma(t' + vx'/c^2) \\
x' = \gamma(x - vt) & x = \gamma(x' + vt') \\
y' = y & y = y' \\
z' = z & z = z'
\end{array} \qquad (7)
$$

1.2 Postulate Lorentz Transformation Derivation

We want to consider the transformation from one inertial coordinate frame S with coordinates (ct, x, y, z) more generally (x_0, x_1, x_2, x_3) to another inertial coordinate frame S' with coordinates (ct', x', y', z') more generally (x'_0, x'_1, x'_2, x'_3)

First we establish that the transformation must be linear. This can be shown many ways - for example, from Newton's first law and the idea of temporal and spatial homogeneity: An ideal standard clock is one that runs at a constant rate, e.g. ticking off a second at a fixed interval independent of the absolute time. We would like for its rate not to depend its position in space or time as an indication of spatial and temporal homogeneity. Consider a standard clock C moving through frame S, its motion being given by $x_i = x_i(t)$, where x_i $(i = 1, 2, 3)$ stand for the three spatial coordinates (x, y, z). Then $dx_i/dt =$ constant. If τ is the time indicated by the clock C itself, homogeneity requires the constancy of $dt/d\tau$. Equal outcomes here and there, now and later, of the experiment that consists of timing the ticks of a standard clock moving at constant speed.

Together these results imply $dx_\mu/d\tau =$ constant and thus $d^2x_\mu/d\tau^2 = 0$, where we have written x_μ $(\mu = 0, 1, 2, 3)$ for $ct, x, y, z)$. In frame S' the same argument yields $d^2x'_\mu/d\tau^2 = 0$. We also have

$$\frac{dx'_\mu}{d\tau} = \sum_\nu \frac{\partial x'_\mu}{\partial x_\nu}\frac{dx_\nu}{d\tau}, \qquad \frac{d^2x_\mu}{d\tau^2} = \sum_\nu \frac{\partial x'_\mu}{\partial x_\nu}\frac{d^2x_\nu}{d\tau^2} + \sum_\nu\sum_\sigma \frac{\partial^2 x'_\mu}{\partial x_\nu \partial x_\sigma}\frac{dx_\nu}{d\tau}\frac{dx_\sigma}{d\tau}. \tag{8}$$

Thus for any free motion of such a clock the last term in the equation must vanish. This can only happen if $\partial^2 x'_\mu/\partial x_\nu \partial x_\sigma = 0$: that is, if the transformation is linear.

An immediate consequence of linearity is that all the defining particles (that is, those at rest in the lattice) of any inertial S' move with identical, constant velocity through any other inertial frame S. Suppose that the coordinates of S and S' are related by

$$x_\mu = \left(\sum_\nu A_{\mu\nu} x'_\nu\right) + B_\mu \tag{9}$$

Then setting $x'_i =$ constant $(i = 1, 2, 3)$ for a particle fixed in S', we get $dt = A_{00}dt'$, $dx_j = A_{j0}dt'$, and thus $dx_j/dt = A_{i0}/A_{00} =$ constant, as asserted. The defining particles of S' thus constitute, as judged in S, a rigid lattice whose motion is fully determined by the velocity of any one of its particles.

Another consequence of linearity (plus symmetry) is that the standard coordinates in two arbitrary inertial frames S and S' can always be chosen so as to be in standard configuration with respect to each other.

It is always possible to chose the line of motion of the spatial origin of S' as the x-axis of S, and to choose the zero points of time in S and S' so that they two origin clocks both read zero when they pass each other. Any two orthogonal planes intersecting along the x-axis can serve as the coordinate planes $y = 0$ and $z = 0$ of S. The two planes, fixed in S, plus the moving plane $x = vt$ (v being the velocity ofS' relative to S) correspond to plane sets of particles fixed in S'. Moreover, the planes

$y = 0$ and $z = 0$ must also be regarded as orthogonal in S', otherwise the isotropy of S (in particular, its axial symmetry about the x-axis) would be violated. So we can takes these planes as the coordinate planes $y' = 0$ and $z' = 0$, respectively of S', otherwise the projection of that axis onto that plane would violate the isotropy of S. Hence we can take $x = vt$ as $x' = 0$. In what follows, we assume S and S' to be in standard configuration.

The Relativity Postulate implies that the transformation between any pair of inertial frames in standard configuration, with the same v, must be the same. Suppose we reverse the x- and z- axes of both S and S', by symmetry and reciprocity, this operation produces an identical pair of inertial frames with the roles of the 'first' and 'second' interchanged. So if we then interchange primed and unprimed coordinates, the transformation equations must be unchanged. In other words, the transformation must be invariant under what we shall call an xz reversal:

$$x \leftrightarrow -x', \quad y \leftrightarrow y', \quad z \leftrightarrow -z', \quad t \leftrightarrow t' \tag{10}$$

The same is true for an xy reversal.

Now by linearity, $y' = Ax + By + Cz + Dt + E$, where the coefficients are constants, with some dependence on v. Since, by our choice of coordinates, $y = 0$, must entail $y' = 0$, we have $y' = By$. Applying an xz reversal yields $y = By'$ and so $B = \pm 1$. However when $v \to 0$ one must continuously go to the identity transformation and to $y' = y$ and thus the only choice is $B = 1$. The argument for z and z' is similar, and we arrive at the 'trivial' members of the transformation:

$$y' = y, \qquad x' = z \tag{11}$$

just as in the Galilean/Newtonian case and for the same reasons.

Next suppose linearity so $x' = \gamma x + Fy + Gz + Ht + J$, where for tradition we have used γ as the coefficient for x. By our choice of coordinates, $x = vt$ must imply $x' = 0$, so that $\gamma v + H,\ F,\ G,\ J$ all vanish and

$$x' = \gamma\,(x - vt) = \gamma\,(x - \beta ct) \tag{12}$$

An xz reversal then yields

$$x = \gamma\,(x' + vt') = \gamma\,(x' + \beta ct') \tag{13}$$

At this stage Newton's axiom $t' = t$ would lead to $\gamma = 1$ and $x' = x - vt$; that is, to the Galilean transform. Instead we make the real use of Einstein's law of light propagation - the second postulate of Special Relativity. According to it, $x = ct$ and $x' = ct'$. Both frames S and S' have the speed of light as c and the two equations are simply descriptions of the same light signal in each reference frame. Substituting these expressions back into the transformation equations for x' and x we find the relations $ct' = \gamma t(c - v)$ and $ct = \gamma t'(c + v)$, whose product divided by tt', yields

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{14}$$

since $v \rightarrow 0$ must lead to $x' = x$ continuously, we must chose the positive root. This is the famous 'Lorentz factor' gamma (γ), which plays such an important role in Special Relativity. Previously it was found to satisfy experiment. Here it appears to satisfy the second postulate of the speed light being the same **finite** value in all inertial frames and in the context of the relativity postulate.

The elimination of the cross reference frame coordinate, e.g. x', finally leads to the most revolutionary of the four equations

$$t' = \gamma \left(t - vx/c^2\right) \tag{15}$$

Since in the above derivation we have used Einstein's light propagation postulate only on the x-axis, must still check whether the transformations respects it generality. First, the linearity of the transformation implies that any uniformly moving point transforms into an uniformly moving point. This, incidently recovers the invariance of Newton's first law, but, of course, it also applies to light signals. Next, one easily derives from the transformations the enormously important fundamental identity

$$c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2 dt - dx^2 - dy^2 - dz^2 \tag{16}$$

The distance dr between neighboring points in a Euclidean frame S is given by the 'Euclidean' metric

$$cr^2 = dx^2 + dy^2 + dz^2 \tag{17}$$

From the identity we have $dr^2 = c^2 dt^2$, which is characteristic of any effect traveling at the speed of light, implies that $dr'^2 = c^2 dt'^2$ and vice versa. So the Euclidicity of the metric and the invariance of the speed of light are jointly respected by the Lorentz transform.

1.3 Minkowski SpaceTime Derivation

The (Minkowski) geometry of space-time is constructed so that the interval: $dx^2 + dy^2 + dz^2 - c^2 dt^2$ is **invariant** under a Lorentz transformation. And the signature is invariant under all real transformations of coordinates.

In more general form the signature is written as a bilinear transformation or a matrix:

$$(ds)^2 = \sum_{\mu\nu} \eta_{\mu\nu} \left(dx_\mu\right) \left(dx_\nu\right) \tag{18}$$

where the Minkowski metric term $\eta_{\mu\nu}$ can be expressly written as

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{19}$$

for Euclidean (pseudo-Euclidean because of sign difference) coordinates.

We consider that inertial coordinate systems are those for which the four-dimensional length defined by the metric is invariant. Immediately, this gives us time dilation:

$$
\begin{aligned}
(cd\tau)^2 = -(ds)^2 &= (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \\
&= (dt)^2 \left[c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 \right] \\
&= (cdt)^2 \left[1 - \frac{v^2}{c^2} \right] \\
d\tau &= dt\,\sqrt{1 - v^2/c^2}
\end{aligned} \tag{20}
$$

Thus we see immediately that the rest frame elapsed (rate) time τ will be dilated by the factor $\sqrt{1 - v^2/c^2}$.

If proper time is invariant, then we can show Lorentz transformation is linear. The conversion from one coordinate system to another

$$dx'_\alpha = \sum_\beta \frac{\partial x'_\alpha}{\partial x_\beta} dx_\beta \equiv \frac{\partial x'_\alpha}{\partial x_\beta} dx_\beta$$

where the second right hand side defines the Einstein summation convention that a repeated index (in this case β) mean summation on that index. The Greek symbol index sums over four (4-D) going 0, 1, 2, 3 and Roman letters sum over three spatial coordinates going 1, 2, 3.

$$
\begin{aligned}
c^2 d\tau^2 &= \sum_\alpha (dx'_\alpha)^2 = \sum_\alpha \sum_\beta \sum_\delta \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial x'_\alpha}{\partial x_\delta} dx_\beta dx_\delta \\
&= \sum_\alpha dx_\alpha^2 = \sum_\alpha \sum_\beta \sum_\delta \delta_{\beta\alpha} \delta_{\delta\alpha} dx_\beta dx_\delta
\end{aligned} \tag{21}
$$

Therefore

$$\sum_\alpha \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial x'_\alpha}{\partial x_\delta} = \sum_\alpha \delta_{\beta\alpha} \delta_{\delta\alpha} \tag{22}$$

implying if one takes the derivative: $\frac{\partial}{\partial x_\epsilon}$ one finds

$$\frac{\partial^2 x'_\alpha}{\partial x_\beta \partial x_\epsilon} \frac{\partial x'_\alpha}{\partial x_\delta} + \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial^2 x'_\alpha}{\partial x_\delta \partial x_\epsilon} = 0$$

Now one can then shift through the indices: $\epsilon \rightarrow \beta \rightarrow \delta \rightarrow \epsilon$ and get generically

$$\frac{\partial^2 x'}{\partial x \partial x} \frac{\partial x'}{\partial x} = 0$$

and the determinant of $\partial x'/\partial x = \pm 1$ which implies

$$\frac{\partial^2 x'}{\partial x \partial x} = 0$$

and

$$x'_\alpha = A_\alpha + \sum_\beta A_{\alpha\beta} x_\beta$$

Which shows that the coordinate (Lorentz transformation) must be linear to preserve invariant the proper distance and time. Thus

$$dx'_\alpha = \sum_\beta A_{\alpha\beta} dx_\beta$$

and

$$\sum_\alpha A_{\alpha\beta} A_{\alpha\delta} = \delta_{\beta\delta}$$

The solution to these equations is

$$A = \begin{bmatrix} cosh\psi & -sinh\psi & 0 & 0 \\ -sinh\psi & cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or equivalently

$$[ct', x', y', z',] = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

The Lorentz transformation can be derived as the transformations that are velocity boosts from one inertial frame to another.

1.3.1 *ict* Derivation of L.T.

In this subsection we derive the Lorentz transformation the way that undergraduates were often introduced to the subject and this shows an alternate coordinate definition. First consider four dimensional Euclidean space with coordinates (x_1, x_2, x_3, x_4) which in Cartesian coordinates are $x, y, z, ict)$. Then the length of a vector with one end on the origin and the other at point $P = (x_1, x_2, x_3, x_4)$ is given by the Pythorean (Euclidean/Cartesian) metric Define $x_4 = T \equiv ict$, then

$$\begin{aligned} d^2 &= \sum_{\alpha=1,4} x_\alpha^2 \\ &= x^2 + y^2 + z^2 + T^2 \end{aligned}$$

$$
\begin{aligned}
&= x^2 + y^2 + z^2 + (ict)^2 \\
&= x^2 + y^2 + z^2 - (ct)^2
\end{aligned} \tag{23}
$$

Thus the distance squared is just the same as before but with the opposite sign. If we want to consider all the transformations that leave the four-D distance invariant:

$$
\begin{aligned}
d^2 &= (d')^2 \\
x^2 + y^2 + z^2 - (ct)^2 &= (x')^2 + (y')^2 + (z')^2 - (ct')^2
\end{aligned} \tag{24}
$$

The most general linear transformation for a Euclidean space (excluding translations) that leave vector lengths invariant is a simple rotation.

$$
\begin{aligned}
x' &= x cos\alpha + T sin\alpha \\
T' &= -x sin\alpha + T cos\alpha
\end{aligned} \tag{25}
$$

Consider the point(s) $x' = 0$. By definition of the relationship between the reference systems $x = vt = vT/(ic)$ for same point. The rotation angle is then

$$
tan\alpha = i\frac{v}{c} \equiv i\beta \tag{26}
$$

Thus α is an imaginary angle ($tan\alpha = iv/c \rightarrow \alpha^* = v/c$).

$$
cos\alpha = 1/sec\alpha = 1/(1 + tan^2\alpha)^{1/2} = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma \tag{27}
$$

$$
sin\alpha = tan\alpha \times cos\alpha = i\frac{v}{c}\gamma = i\beta\gamma \tag{28}
$$

Putting these into the rotation equations above

$$
\begin{aligned}
x' &= x cos\alpha + T sin\alpha = \gamma x + T i\beta\gamma = \gamma(x - vt) \\
T' &= -x sin\alpha + T cos\alpha = -x i\beta\gamma + T\gamma = \gamma(T - i\beta x) \\
ct &= \gamma(ct - \beta x)
\end{aligned} \tag{29}
$$

1.4 General Lorentz transformations: the Lorentz group

One can explicitly verify that the Lorentz transformation

$$
\begin{aligned}
ct' &= \gamma(ct - \beta x) \qquad t' = \gamma(t - vx/c^2) \\
x' &= \gamma(x - \beta ct) \qquad x' = \gamma(x - vt) \\
y' &= y \\
z' &= z
\end{aligned} \tag{30}
$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$, leaves the space-time interval $ds = (cd\tau)$ invariant. By choosing space coordinates so that the relative velocity of two inertial frames is along the x direction, it follows that all Lorentz transformations leave the interval invariant.

We can explicitly write this out as a matrix equation:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \equiv \sum_\mu \sum_\nu \eta_{\mu\nu} dx^\mu dx^\nu \tag{31}$$

where a repeated index means a summation in the Einstein convention.

Specifically, in Cartesian coordinates

$$ds^2 = [cdt \quad dx \quad dy \quad dz] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{32}$$

The Lorentz transformation in matrix notation for Cartesian coordinates is

$$\Lambda_\alpha^\beta = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

Since

$$\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{34}$$

it is clear that the Lorentz transformation leaves the interval $(ds')^2 = ds^2$ invariant.

Now we can define the Lorentz group as the group of matrix transformations that leave the interval ds invariant.

$$(ds')^2 = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma dx^\rho dx^\sigma \rightarrow= ds^2 \tag{35}$$

But ds^2 may be written as: $ds^2 = \eta_{\rho\sigma} dx^\rho dx^\sigma$ thus we may infer that

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \tag{36}$$

Hence, any matrix Λ which leaves the metric invariant under the transformation represents a Lorentz transformation. These matrices form a group of transformations known as the Lorentz group. When combined with translation symmetry, $x^\mu \rightarrow (x')^\mu = x^\mu + \alpha^\mu$, with α^μ as the components of a constant 4-vector, it forms a larger group known as the Poincare group.

1.5 Lorentz Group

In Mathematics the Lorentz Group is the the group of all linear transformations of the vector space R^4 that leave the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2$ invariant. The Lorentz group is isomorphic to $O(1,3,R)$, a real form of the complex orthogonal group $O(4)$.