Physics 139 Relativity
Relativity Notes 2003

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Notes to be found at:
http://aether.lbl.gov/www/classes/p139/homework/homework.html

Prologue

This course provides an introduction to Relativity (Special and General). This course covers the historical, experimental basis for relativity and an exposition of the major concepts and features of relativity. As an instructor I think it important to include material that involves practical and important applications as well as the material that brings out the content of the concepts of relativity. The natural applications include high energy physics, astrophysics, and cosmology. The last two are particularly relevant for applications of General Relativity.

At Berkeley this course originated in 1973 as a result of Chairman Eugene Commins discussions with undergraduate physics majors who felt that they had an inadequate view of Special Relativity in that it was treated piecewise in mechanics, E & M, quantum mechanics, atomic physics, and the nuclear and high energy. However, there was no overall all view of Special Relativity. Eugene Commins then asked David Judd to prepare and give an experimental course for graduating seniors in their last semester (Spring 1973). It was successful and became a regular course – Physics 139. The course has been given in the spring ever since.

Spring 1998 is the first time that I have taught the course and added significant astrophysics material at the request of the students taking the course.

Special Relativity can be taught (or learned) from many perspectives. The most basic of these is a rigorous investigation of the experimental basis for the physics of Relativity. A second approach is to start with the postulates of Einstein and derive the consequences and an understanding of Relativity. A third approach is top down. It begins with assumptions about space-time being $3 + 1$ pseudo-Euclidean space and formulates physics in terms of a 4-dimensional space-time. This leads to the powerful and useful concept of 4-D vectors. In this course you will exposed to all three of these approaches and occasionally some others. These notes are meant to provide much of the experimental background and some explanation of the approaches. Lectures focus primarily on the second and especially the third approach as a natural lead into the geometrical version of General Relativity.

We emphasize the experimental basis because a scientific theory is a living entity; it grows and changes with time. Physics is a description of Nature. The final arbiteur of its validity is Nature, that is observations of Nature and not aesthetic principles or pronouncements from the prominent. Thus no matter how beautiful, economic, consistent, or other wise pleasing a model or theory construct might be, it must agree with experimental observations. The second and third approaches assume principles and postulates and derive a consistent picture. That picture has to agree with observation and the logical consequences of those observations. Thus the early lectures and notes emphasize the experimental basis to the later logical deductions and tools developed and as a balance to the postulates of Special Relativity and the more extended approach following Minkowski geometry.

1 Introduction

<u>The Special Theory of the Relativity of Motion</u> is confined to relativity of uniform motion translatory motions of coordinates in free ("No Gravity") space.

1.1 General Ideas of Space and Time

We usually use concepts arising from spatial and temporal measurements without considering their philosophical implications, if any.

1. Concept of Time
2. Concept of Space
3. The Space and Time of Newton and Galileo
4. The Space and Time of the Ether Theory

1.1.1 Properties of Time

1. Time is a **continuum**. One can find a time between any two times.
2. Time is **one dimensional**. A single number defines time uniquely.
3. Time is **homogeneous**. It has the same properties in the past, present and future.
4. Time is **anisotropic**. Forward and backward in time are different. This is actually controversial since the the laws of physics seem to be invariant (to high order) to the direction of time.
5. Time is **single-valued**. This is the assumption, not necessarily founded, that a completely cyclic universe is ruled out. We do not revisit a previous state.

How do we get knowledge?

$$\begin{array}{ccc} & \rightarrow & time \\ \underline{Past} & \underline{Present} & \underline{Future} \\ Memory & \rightarrow Experience \rightarrow & prediction \\ Inference & & premonition \end{array}$$

Irreversibility: Evidenced by second law of thermodynamics. Entropy increases with time.
Psychology: Memory of past times distinguish them from others to be encountered later.

1.1.2 Properties of Space

1. Space is a **continuum**. One can find a point between any two points. [1]
2. Space is **three dimensional**. Three numbers specifies a point.
3. Space is **homogeneous**. It has the same properties in all regions.
4. Space is **isotropic**. There is no spatial "arrow". All directions are equivalent.
5. Space is **single-valued**. Point labels are unique.
6. Space is **Euclidean**. The differential distance is given by Pythagoras by

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1}$$

Most of these are called into question by things that we know.

1. Uncertainty Principle from Quantum Mechanics
2.
3. Gravity: Strong in some places, weak in others.
4. Electric, Magnetic, and Gravitational fields.
5.
6. "Curved Space" due to energy density distribution in General Relativity.

1.2 The Space & Time of Galileo & Newton

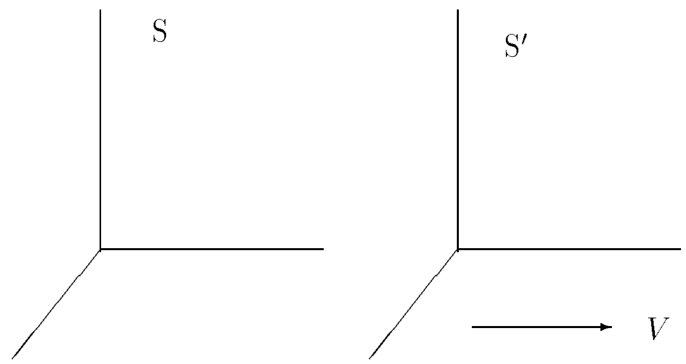
1.2.1 The First Law of Motion

If no force, bodies remain at rest or have uniform straight-line motion.
Aristotle: The **natural** state of a body is a **state of rest**.

But a body in a natural state in reference frame S is also in a natural state in

[1] Strictly continuum needs a more precise definition. To physicists actually space is a continuous manifold. The mathematical property is (local) completeness. It is not enough that between any two points there is another. Mathematically we require that if we have a sequence of points that gets closer and closer together (a Cauchy sequence), then there is some point to which the sequence converges; i.e. limits exists.

The property of what it means to be a continuum or not is best borne out by the Intermediate Value Theorem, which may be stated (in physical terms, in a 1-dimensional system): if an object is moving along a straight line (possibly changing directions) and is recorded to have been at point a and subsequently point b, then the object passed through every point in between. Space being a continuum defines what we mean by "every". Usually, what this means is that the points between a and b are labelled by the real numbers between 0 and 1, and the object passed through a point with each such real number label. The distinction that is made in, say, quantum mechanics, is that there may be *no* points between a and b, and furthermore, there may have been *no* times between when the object was measured at point a and point b. Of course quantum mechanics takes care of this discreteness by being probabilistic, but the distinction from being a continuum is there, nonetheless.



the frame of S′

1.2.2 The Second Law of Motion

$$\vec{F} = m\vec{a}; \quad F_x = m\frac{d^2x}{dt^2}, \quad F_y = m\frac{d^2y}{dt^2}, \quad F_z = m\frac{d^2z}{dt^2} \tag{2}$$

This is actually a definition of force. This definition of force provides the same force in reference frames S and S′, because the acceleration is the same in either.

It also provides a definition of inertial mass m. Masses can be compared with a standard mass – the unit of mass.

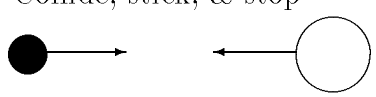
There are many methods:

Static:

Pan balance is used and one assumes $F_{gravity} \propto m$.

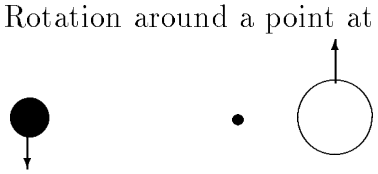
Spring balance which assumes $F_{gravity} \propto m$ and Hooke's law.

Collide, stick, & stop



$m_1v_1 = m_2v_2$

Rotation around a point at rest



$m_1v_1 = m_2v_2$

Dynamic:

1.2.3 The Third Law of Motion

This law states conservation of momentum in an isolated system. It is equivalent to

$$\vec{F}_{\text{on 1 due to 2}} = -\vec{F}_{\text{on 2 due to 1}} \tag{3}$$

That is for every force there is an equal and opposite reaction. This follows by use of the second law on an infinitesimal mass at the point of contact of 1 and 2. It yields consistency in reference frames S and S$'$.

1.2.4 The Final Picture

1. Nothing exists in space with respect to which one can measure an "Absolute Velocity".
2. Velocity of light could only depend on the velocity of its source
3. Space and time are independent continua.

1.2.5 Space and Time of the Ether Theory

Electromagnetic disturbances propagate with velocity c in accordance to the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{4}$$

A particular solution is a plane wave

$$\phi = \phi_0 sin\left[2\pi\nu\left(t - x/c\right)\right] \tag{5}$$

x, y, and z are to be measured with respect to the medium (ether, or a solid or liquid) in which the waves are propagated.

It is inconceivable to have waves without a medium. Consider sound waves, elastic waves (strings, rods), shock waves, E-M waves. Thus it was necessary for the theory of electromagnetism (EM) to have the ether for light to propagate through and provide a consistent set of theory. Maxwell's Equations do predict light that propagates with a speed c. But the question is what is that speed with respect to?

<u>The Formal Ether Picture</u>

A. Space is filled with an ether with respect to which an "Absolute Velocity" should or could be measured.
B. The velocity of light is independent of the velocity of its sources; always c with respect to the ether or vacuum.
C. Space and Time are independent continua.

Implicitly, in the Ether theory turbulence and relative motion of parts of the Ether are ruled out.

Why was the Ether taken as stationary? That is unaffected by motion of matter and without relative motions of its parts.

We try to create a picture of how inevitable the ether theory seemed for a very long time, and to describe some of the crucial experiments that supported it for so long. Every student should know about the lengthy debate over the nature of light - particles or waves?

Newton thought "particles". His prestige as the greatest physicist of all time was enormous. As we know now, he was not wrong! (Light comes in quanta.)

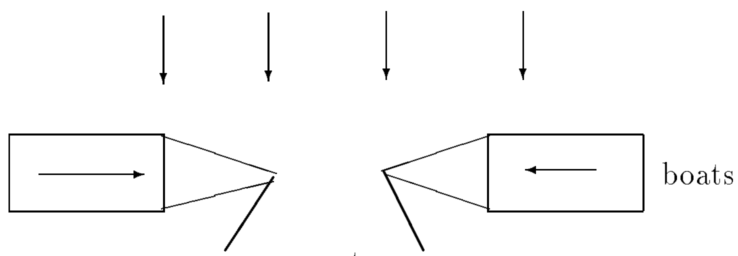
Table 1: Kinds of experiments about the Ether:

A. In the Neighborhood of Moving Matter	
Bradley	1725
Lodge	1892
B. Inside of Moving Media	
Fresnel	1818
Fizeau	1851
Airy	1871
Michelson-Morely	1896
Trouton-Noble	1903

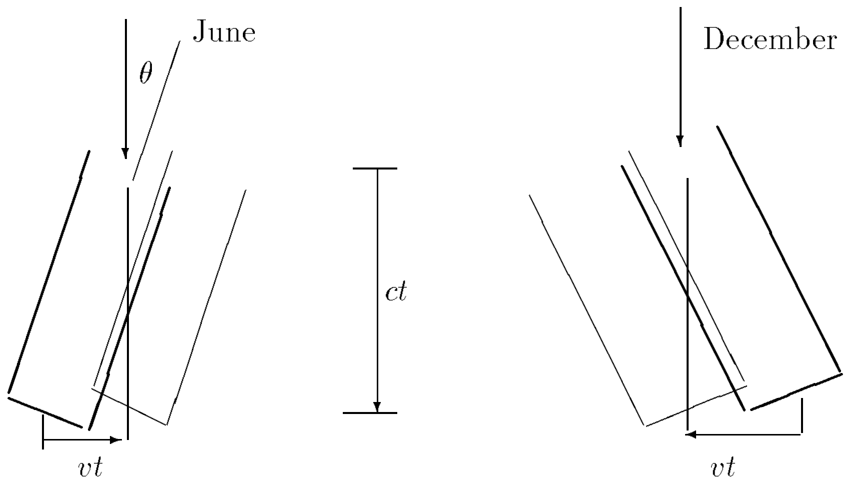
The wave nature of light was finally proved beyond a doubt by Young and Fresnel by display of interference, diffraction, and polarization.

Bradley's Discovery of Aberration

Reasoning by analogy of the behavior of a pennant on a sail boat in the wind:



led Bradley to consider a star's position variation between June and December.



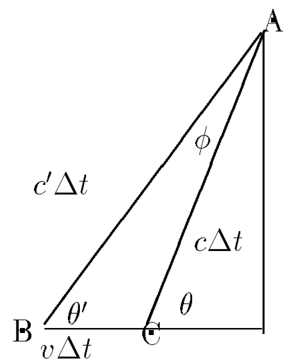
θ is defined as the aberration angle and

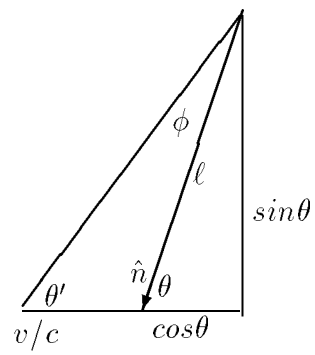
$$tan\theta = \frac{v}{c} = \frac{30 \text{ km/sec}}{3 \times 10^5 \text{ km/sec}} \simeq 10^{-4} \sim 20 \text{ arcsec} \tag{6}$$

Bradley observed it! A motion of a star's position of about 41″ over the course of a year.

Bradley's observation could be explained either by a fixed Ether theory or a corpuscular theory. (But not by a moving ether theory.)

We can derive this carefully in the following manner: Light from the star goes from the top (A) to the bottom (B) of the telescope in Ether system in a time Δt.





It goes from (A) to (C) in the moving telescope system with a speed we can calculate to be

$$c'^2 = c^2 + 2vc\ cos\theta + v^2$$

by the law of cosines. By the law of sines

$$\frac{sin\phi}{sin\theta'} = \frac{v}{c}$$

Thus

$$tan\theta' = \frac{sin\theta}{cos\theta + v/c}$$

Galilean transformation of an ether wave:

$$\begin{aligned} \Delta t' &= \Delta t \\ x_i' &= x_i - v_i t \\ y_i' &= y_i \\ z_i' &= z_i \end{aligned} \tag{7}$$

$$\hat{n} = (-cos\theta,\ -sin\theta,\ 0) \tag{8}$$

Amplitude is proportional $\propto cos\Psi = cos\omega\,(t - \hat{n}\cdot\vec{x}/c)$ in Ether system. Ψ being constant is a fixed phase and thus a wave front.

In the moving system Ether system

$\Psi = \omega'\,(t - \hat{n}\cdot\vec{x}'/c'')$ $\Psi = \omega\,(t - \hat{n}\cdot\vec{x})$

We assert that

$$\omega' = \omega\left(1 + \frac{r}{c}\ cos\theta\right)$$

which is the Doppler effect and

$$c'' = c + v\ cos\theta$$

To show that this is true, plug into the equations.

$$\Psi = \omega\left(1 + \frac{v}{c}cos\theta\right)\left[t - \frac{n_x(x - vt) + n_y y}{c + vcos\theta}\right]$$

$$
\begin{aligned}
&= \omega\left(1+\frac{v}{c}cos\theta\right)\left[t-\frac{\hat{n}\cdot\vec{x}}{c(1+\frac{v}{c}cos\theta)}+\frac{n_x vt}{c(1+\frac{v}{c}cos\theta)}\right] \\
&= \omega\left[t+\frac{vtcos\theta}{c}-\frac{\hat{n}\cdot\vec{x}}{c}+\frac{n_x vt}{c}\right] \\
&= \omega\left[t+\frac{vtcos\theta}{c}-\frac{\hat{n}\cdot\vec{x}}{c}-\frac{cos\theta vt}{c}\right] \\
&= \omega\left(t-\frac{\hat{n}\cdot\vec{x}}{c}\right) \qquad (9)
\end{aligned}
$$

Which checks the first claim. (Writing these equations in reverse order verifies both claims.)

$c'' = c + vcos\theta$= component of the ray velocity perpendicular to the wave front in the moving (telescope) system: The angle between the ray and $\hat{n}$ is $\phi = \theta - \theta'$.

$$
\begin{aligned}
c'' &= c'cos\phi = c'cos(\theta-\theta') \\
&= c'cos\theta' cos\theta + c'sin\theta' sin\theta \qquad (10)
\end{aligned}
$$

From the geometry:

$$
\begin{aligned}
c'cos\theta' &= ccos\theta + v \quad \text{base of right triangle} \\
c'sin\theta' &= csin\theta \quad \text{height of right triangle} \qquad (11)
\end{aligned}
$$

Doppler Shift:

$$\nu' = \nu\left(1+\frac{v}{c}cos\theta\right) \qquad (12)$$

This is the same as for sound with a fixed source and moving observer. For sound with a fixed observer and a moving the source, the difference is second order in v/c.
Oliver Lodge (1892) tried to observe the Ether drag by a nearby heaving moving mass. He used a huge iron sphere of mass 1400 pounds (about 600 kg) in which there were a deep circumferential slot positioned horizontally. He rotated the sphere about a vertical axis and split a beam of light and sent them around in opposite directions through the slot in the sphere via a system of mirrors. He found no difference in the two beams behavior depending upon the rotation of the heavy mass.

Oliver Lodge was a fellow of the Royal Society and a professor of physics at the University College of the University of Liverpool. He published the result of many years of effort as articles in the Philosophical Transaction of the Royal Society of London, Series A. Volume 184 pp. 727-804 (1893) and Volume 189 pp. 149-166 (1897) "Experiments on the Absence of Mechanical Connection Between Ether and Matter". In his experiment Lodge observed the interference between portions of a split light beam traveling in opposite directions around a closed path in the space between two rapidly rotating steel disks. The disks were circular saw disks of diameter 3 feet, rotating in a horizontal plane at up to 3000 r.p.m. The separation was about 1 inch and the beams made four complete circuits around the rotating mass axis. The result of years of experiments was a null effect. The speed of light was unaffected by motion of adjacent matter to the extent of one part in 200 of the speed of the matter.

Lodge then replaced the disc with a heavy (1400 lbs) Swedish-iron oblate spheroid with a half inch width groove cut one foot deep into the sphere. His long experimental program had many problems to over come including: overheated bearings, heated air, miscellaneous vibrations, safety concerns, and the fact that it took one half hour to slow down.

He obtained speeds up to 100 r.p.m. and also considered that drag might take hold slowly so he tried for three hours. Lodge also added magnetic and electric fields perpendicular to the velocity and always found a null effect.

Fresnel (1788-1827)

Fresnel worked upon the theory of the Ether. He indicated that the density of Ether in a transparent material is proportional to the square of the index of refraction n.

$$v_{\text{light in body}} = \frac{c}{n}; \qquad \frac{\rho_{\text{Ether in body}}}{\rho_{\text{Ether in space}}} = n^2 \tag{13}$$

When a body moves through the Ether, part of the Ether is carried along – the part in excess of the vacuum value. The rest of the Ether remains stationary. The density carried along is equal to $\rho_{body} - \rho_{vacuum} = (n^2-1)\,\rho_{vacuum}$. The part that does not move is ρ_{vacuum}.

Thus the center of gravity of the Ether moves with velocity

$$v_{c.m.\ Ether} = \frac{(n^2-1)\,v_b + 1\cdot 0}{(n^2-1)+1} = \frac{n^2-1}{n^2}v_b = \left(1-\frac{1}{n^2}\right)v_b \tag{14}$$

where v_b is the velocity of the body or medium. This velocity is to be added to the wave velocity c/n in the body, so that the light speed in the moving body is

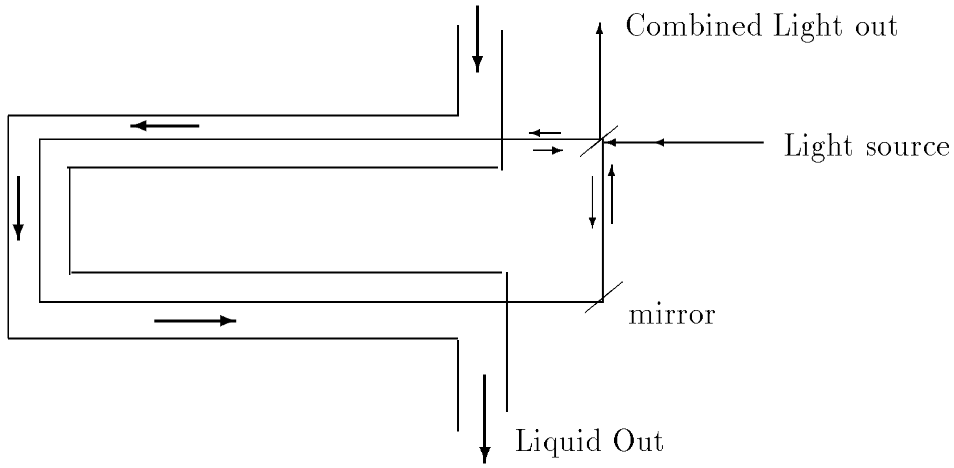
$$v_{\text{light in moving medium}} = \frac{c}{n} + \left(1-\frac{1}{n^2}\right)v_{\text{medium}} \tag{15}$$

The quantity $\kappa \equiv \left(1-\frac{1}{n^2}\right)$ is named the Fresnel Drag Coefficient.

Fizeau (1851)

Fizeau measured the speed of light in a moving transparent medium.

If there is a velocity drag proportional to the medium velocity ($c' = c/n + \kappa v$) the prediction for the experiment as shown in the figure is:



For the counterclockwise traverse, $c' = c/n + \kappa v$. The total number of wavelengths in the horizontal path is $2L/\lambda' = 2Lf/c' = 2Lnf/(c + n\kappa v)$

In the clockwise traverse, $c' = c/n - \kappa v$. The total number of wavelengths in horizontal path is $2L/\lambda' = 2Lf/c' = 2Lnf/(c - n\kappa v)$. The difference in wavelengths of the two paths shows up as the number of interference fringes:

$$\text{Number of fringes} = 2Lnf\left(\frac{1}{c - n\kappa v} - \frac{1}{c + n\kappa v}\right) \simeq \frac{4\kappa n^2 Lv}{c}\frac{f}{c} = \frac{4\kappa n^2 Lv}{\lambda c} \qquad (16)$$

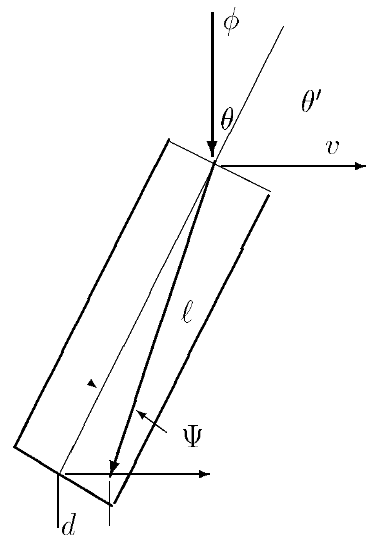
Fizeau (1851) verified Fresnel's drag coefficient using water. Michelson and Morely (1886) repeated the experiment much more accurately using: water, carbon disulfide, and other transparent liquids most with high n.

Stokes (18xx)

Stokes obtained Fresnel's drag coefficient by assuming that the Ether was a compressible but conserved fluid. If the Ether has an apparent velocity v, then for a transparent material $v' = (1 - \kappa)v$. If the Ether density would be $\rho = \rho_0$ in vacuum, then $\rho' = n^2\rho_0$ in a transparent material with index of refraction n. If the Ether is conserved, then $\rho_0 v = \rho' v' = n^2(1 - \kappa)\rho_0 v$ so that $n^2(1 - \kappa) = 1$, and $\kappa = 1 - 1/n^2 = (n^2 - 1)/n^2$ which is Fresnel's value.

Sir George Airy

Sir George Airy, a famous British astronomer, had in 1871 the very clever idea to repeat Bradley's aberration measurements using a water-filled telescope.



Snell's Law (ca. 1600) says that

$$n = \frac{sin\phi}{sin\Psi} = \mu$$

Light travels through the water-filled telescope tube with velocity $c' = c/\mu$ relative to the Ether in the water. The velocity of the Ether with respect to the water is κv where $\kappa = (\mu^2 - 1)/\mu^2$ is Fresnel's drag coefficient. The velocity of the water with respect to the outside Ether is v the nominal speed of the telescope and the velocity of the water relative to the inside Ether is $(\kappa - 1)v$. Distances d and ℓ are in ratio

$$\frac{d}{\ell} = \frac{(1-\kappa)v}{c/\mu}$$

since they take the same Δt.

By the law of sines:

$$\frac{d}{\ell} = \frac{sin\Psi}{sin\theta'} = \frac{(1-\kappa)v}{c/\mu}$$

Note that it is alright to apply Snell's law in the telescope frame. Arago showed in 1810 that, in refraction, light acts as if its source is where it seems to be due to aberration. Thus

$$\frac{sin\Psi}{sin\theta'} = (1-\kappa)\mu v/c$$

so that

$$sin\Psi = sin\phi/\mu \qquad \text{(Snell)}$$

giving

$$\frac{sin\phi}{sin\theta'} = (1-\kappa)\mu^2 v/c$$

If there is no water, $\kappa = 0$ and $\mu = 1$, so

$$\frac{sin\phi}{sin\theta'} = v/c$$

which is Bradley's aberration observation result.

Experimentally, κ is known to be $(\mu^2 - 1)/\mu^2$ so that $1 - \kappa = 1/\mu^2$, which leads to the prediction

$$\frac{sin\phi}{sin\theta'} = v/c$$

Just as before!! Airy's telescope observed the same aberration with water as without.

This seemed to tie down the Ether Theory very well!

Is it plausible that the Ether Density should be proportional to μ^2? $V_{sound} = \sqrt{E/\rho} = \sqrt{Elastic\ Modulus/Density}$ so $\rho \propto 1/v^2 \propto (\mu/c)^2$.

The Ether Theory was brought to its highest point by Lorentz (of the "Lorentz Contraction"). He explained the Fresnel Drag by "Electron Theory". In a moving transparent medium, light interacts with electrons which move along with the medium with velocity v.

Allowing for this but leaving the Ether fixed, you can get $\kappa = (n^2 - 1)/n^2$ but otherwise not.

If the Ether were dragged along, you would get $c' = c/\mu + v$. But you actually get only part of this $c' = c/\mu + \kappa v$, because of the interaction.

Hammer's experiment (1932) was also consistent with the Ether Theory, as was Sagnac's experiment (1915).

1.3 Summary

Postulates a and b together imply that the velocity of light is independent of the relative velocity of source and observer! There are further postulates from mechanics, electrodynamics, and thermodynamics needed to give a complete theory of Special Relativity.

1.4 The Nature of a Deductive System

It is "Universe of Discourse" containing objects, relations between the objects, and rules for getting more relations from previous ones. The relations are statements that take the form of definitions, postulates, and theorems; while the rules are the logic one is allowed to apply for manipulation of these statements. One begins with objects that are undefinable but have certain given relations between them (axioms). In practice, the axioms will depend on which scientific theory we are exploring, whereas the logic we use is independent of which system we are considering.
Desirable properties of a scientific deductive system: [2]

[2] Consistency and Completeness are technical terms in formal logic. I say "desirable properties"

Table 2: SUMMARY

Newton & Galileo	Ether Theory
No Reference System for Absolute Velocity	There is a reference system (ETHER) for Absolute Velocity
The velocity of light depends on the velocity of its source	The velocity of light is independent of that of its source
Space and time are independent	Space and time are independent
Einstein's Special Relativity Theory	
Postulates:	a. No reference system for absolute velocity b. velocity of light independent of source velocity
Result:	c. Space and time are inter-related

(a) Internal Coherence: No contradictions can be reached from the axioms using the given logic.

(b) Completeness: If a true statement can be made, then it can be proved.

(c) Meaning: The true statements have their intended real-world interpretations.

(d) Aesthetic Structure: No superfluous definitions and postulates. i.e. the smallest possible numbers. Fewest number of indefinables. They should be simple, clear, and perhaps chosen to connect to past systems.

(e) A sufficient number of indefinables and a sufficient number of definitions and postulates to produce a structure of theorems.

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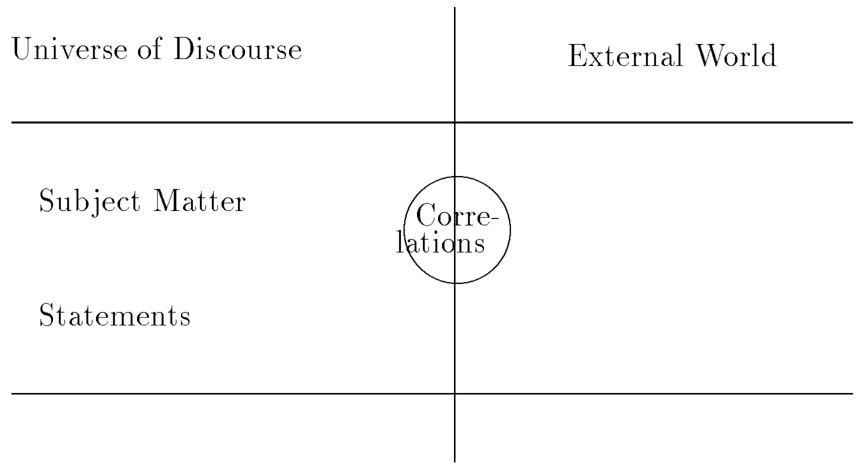
instead of "test of a good" scientific deductive system because it is a theorem of Kurt Godel (1930) that it is impossible to have a meaningful deductive system in which all true statements are provable; in other words, it is impossible to have a (sufficiently complex formal) system which is both consistent and complete. (Needless to say, we usually opt for consistency over completeness.) Nevertheless, it would still be nice if we could prove all true propositions. In any case, it is possible that, in any given system, all of the true statements which we actually care about are provable.

Another disturbing theorem is that in any sufficiently complex consistent system there are statements which are neither true nor false, in the sense that either the statement or its converse could be added as an axiom without making the system inconsistent. There are explicit examples of such statements in very well-known and common-sense theories which we tend to think model the real world. Whenever physicists come up with an undecidable statement, there is usually some concurrence on which (the statement or its converse) "reflects reality", and a new axiom is added. Or, there can be lengthy debate as to what "reflects reality". For instance, the particle/wave postulate for light was for a long time unresolved, and even now, which axiom is chosen depends on the model of physics being used (particles are "good enough for some purposes", as are waves).

This is the end of the line for pure mathematics.

(f) Usefulness in Explaining Phenomena: Providing a map of the external world



We would like to compare and check postulates with the external world, but they are usually too general. But deductions from them can be checked!

1.5 Postulates of Special Relativity

I. It is impossible to measure or detect the absolute velocity of a body in free space. All we can measure is relative velocity of one body with respect to another.

These ideas/principles come from Galileo and Newton.

II. The velocity of light is independent of its source.

This idea comes from the Ether Theory.

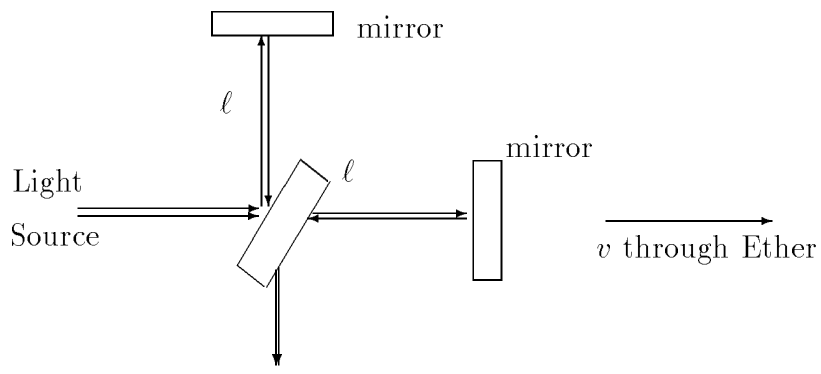
Consequences: Light velocity is independent of relative velocity of source and observer.

1 Tests of the First Postulate

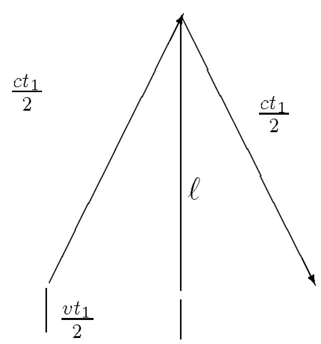
1.1 The Michelson-Morely Experiment

Michelson Am J. Sci 22, 20 (1881) Michelson & Morely Am J. Sci 34, 333 (1887)

The Michelson-Morely Experiment was designed to measure the Earth's velocity, $v_\oplus$, through the fixed Ether due to its orbit around the Sun. To do this Michelson conceived and developed the Michelson interferometer.



The two path lengths (labeled ℓ) are made equal for simplicity and ease of getting a white light fringe.



The travel time t_1 for travel perpendicular to $\vec{v}$ is

$$\left(\frac{ct_1}{2}\right)^2 = \left(\frac{vt_1}{2}\right)^2 + \ell^2 \tag{17}$$

$$t_1 = \frac{2\ell}{\sqrt{c^2 - v^2}} \tag{18}$$

The time t_2 for travel parallel to $\vec{v}$ is

$$t_2 = \frac{\ell}{c-v} + \frac{\ell}{c+v} = \frac{2c\ell}{c^2 - v^2} \tag{19}$$

So the difference in travel times is

$$\begin{aligned} t_2 - t_1 &= 2\ell\left[\frac{c}{c^2-v^2} - \frac{\sqrt{c^2-v^2}}{c^2-v^2}\right] \\ &= \frac{2\ell}{c^2-v^2}\left[c - c + \frac{v^2}{2c} - \cdots\right] \\ &= \frac{\ell}{c}\frac{v^2}{c^2} + \cdots \end{aligned} \tag{20}$$

Now rotate the apparatus through 90° and repeat the measurement. The total time difference is

$$\Delta t = 2(t_2 - t_1) = \frac{2\ell}{c}\frac{v^2}{c^2} \tag{21}$$

and the fringe shift expected is

$$F = \frac{\Delta t}{\tau} = \frac{c\Delta t}{\lambda} = \frac{2\ell}{\lambda}\frac{v^2}{c^2} \tag{22}$$

For the Earth in its orbit around the Sun $(v_\oplus/c)^2 \simeq 10^{-8}$ and for visible light $\lambda \sim 5\times10^{-5}$ cm so that the expected fringe shift is

$$F \sim 2\ell \times \frac{10^{-8}}{5\times10^{-5}\mathrm{cm}} = 4\times10^{-4}\ell/1 \text{ cm}$$

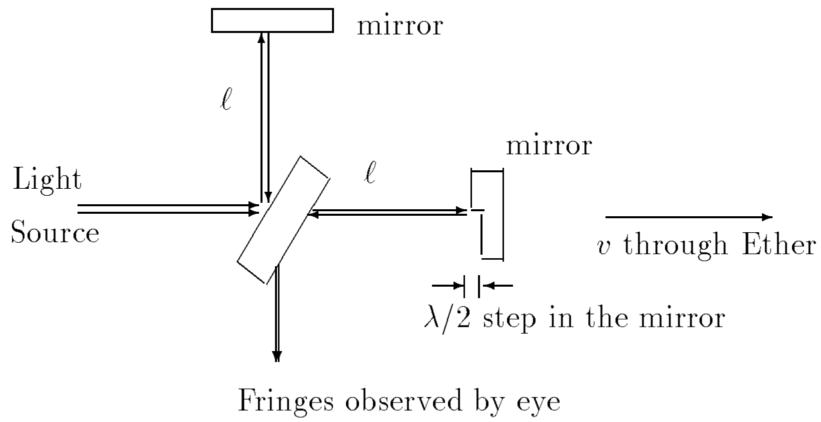
Sophisticated methods allow detection of 1/300th to 1/1500th of a fringe, but detection of 1/100th of a fringe is straightforward. To detect this one needs $\ell > 25$ cm. Michelson and Morely's interferometer had $\ell = 11$ m and used light at 589 nm (589×10^{-9} m) so that they should have seen about one sixth of a fringe shift.

No shift was ever found !!

This work was repeated many times by different workers. Miller obtained $\ell = 65$ m by multiple reflections. The most accurate (in the 1920's) experiments were by Kennedy (Proc. Nat. Acad <u>12</u>, 621 (1926)) and Illingsworth (Physics Rev. <u>30</u>, 692 (1927)).

Consider some of the precautions and sophistications of the best (1920's) experiments (Kennedy and Illingsworth):

For example they introduced a $\lambda/2$ step in the middle of one of the interferometer mirrors.



insert picture here showing two offset sine waves and the fringe patterns varying from top half dark and bottom bright, both medium and equal, and top bright and bottom dark.

The path length of the experiment was four meters lading to a fringe shift of $\lambda/20$ and the could detect between 1/300th and one 1/1500th of a fringe.

The instrument was calibrated by adding small weights to one arm to find that 7500 grams gave one fringe so 5 to 25 grams (1/1500th to 1/300th fringe) was detectable.

The instrument was kept to very accurate constant temperature ($\sim 0.001°$ C?

They tried using polarized light which cuts down on stray light and makes it easier to adjust the intensity.

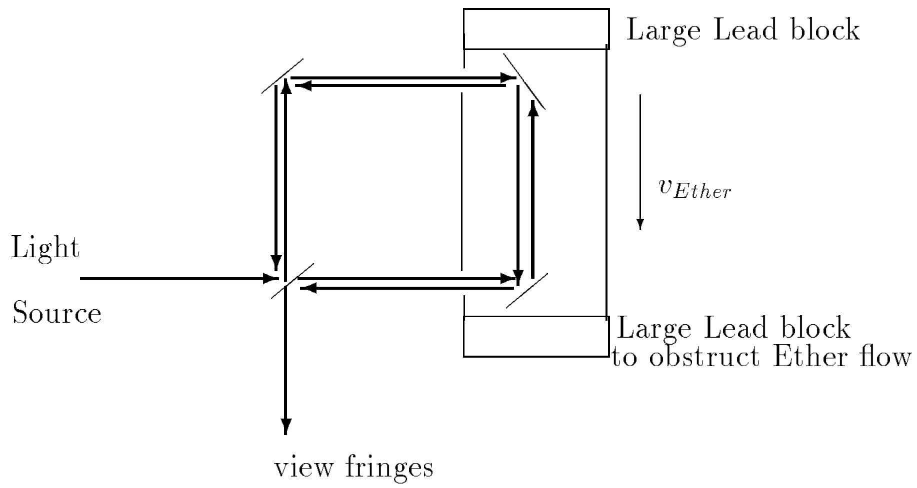
They kept the apparatus in a helium-filled enclosure so that there would be a smaller effect from the gas

$$\frac{n_{He}-1}{n_{Air}-1} \sim \frac{1}{10}$$

The results? Illingsworth found $v < 10$ km/s. Kennedy found $v < 2.5$ km/s. More modern results have for the best optical $v < 1.5$ km/s Charles Townes (Physical Review Letters 1, 342 1958) using maser oscillators found $v < 1/30$ km/s which is equivalent to $v_{Ether}/v_{Earth} < 10^{-3}$ which corresponded to less than 1/50th Hz variation relative to 23,870 MHz.

1.1.1 Auxiliary Experiment of Hamar

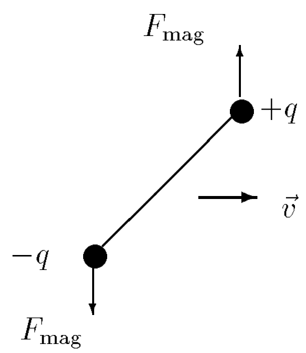
Hamar (Physics Review 48, 462; 1935) did a check to first order in v/c. This tests the ability of matter to obstruct the flow of Ether.



Hamar could detect less than 1/10th fringe and saw no effect which corresponds to less than 1 km/s.

1.2 The Trouton-Noble Experiment

The Trouton-Noble experiment was performed in Great Britain soon (1903) after the Michelson and Morely experiment.

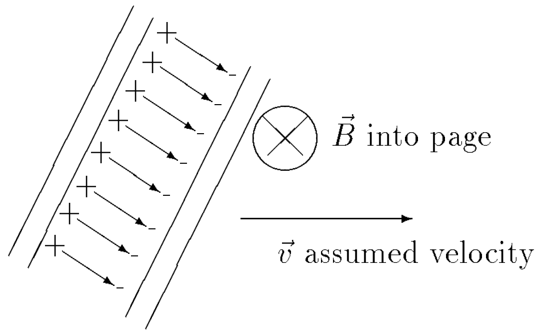


To understand the concept of the experiment, consider two opposite charges held apart by a rod moving at an angle through space. In moving through the ether charge generates a magnetic field (by the Biot-Savart law) and thus each charge experiences a magnetic force $F_{\text{magnetic}} = \pm q\vec{v} \times \vec{B}$. The forces point in different directions and produce a torque on the rod

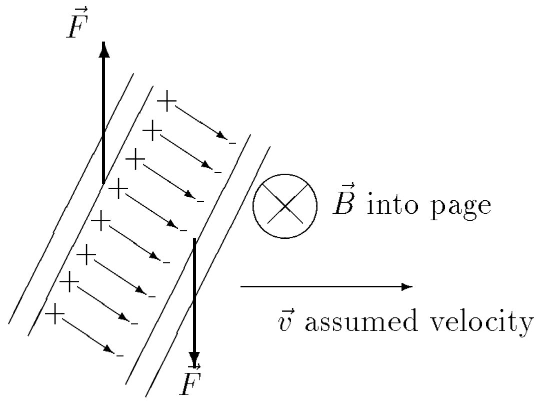
$$\tau = \frac{\mu_o}{4\pi}\frac{q^2v^2}{2}sin\theta cos\theta = \frac{1}{4\pi\epsilon_o}\frac{q}{2}\frac{v^2}{c^2}sin\theta cos\theta$$

If the rod is tilted and moving relative to the Ether frame, then there will be a torque on it. Since the Earth is rotating and orbiting, the rod must sometimes be moving relative to the Ether and so it must have a time varying torque if the Ether exists.

To do this experiment Trouton & Noble used a charged capacitor rather than a rod. The essence is that Trouton & Noble suspended a charged capacitor that would be free to rotate



Assuming motion in the direction shown, the magnetic forces make a counter-clockwise torque on the capacitor.



This is a direct test of the first postulate. **No effect was found.**

1.3 The Kennedy-Thorndike Experiment

The Kennedy-Thorndike experiment results are reported in Phys Rev. 42, 400, (1932).

Thesis: There is a real Ether. There is real motion through it due to the Earth's motion around the Sun. The Michelson-Morely experiment is correct – there is a null effect because there is a real Lorentz-Fitzgerald contraction, just exactly sufficient.

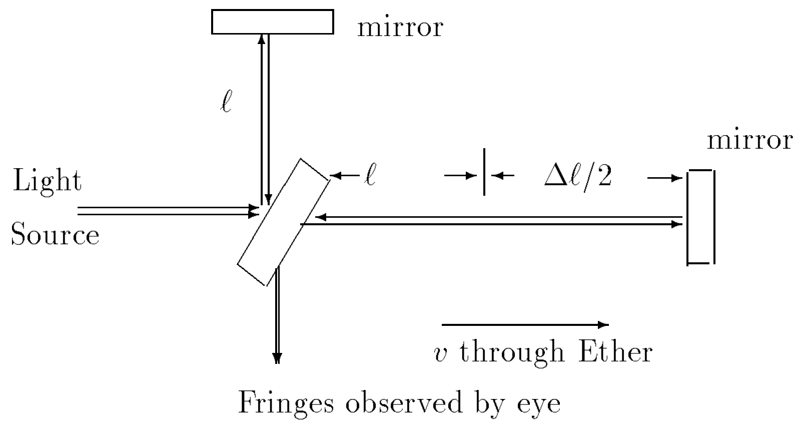
Consequence of Hypothesis: The light travel times for both double traverses of the Michelson-Morely interferometer light paths.

Homework Exercise: Show that the two paths (perpendicular and parallel to direction of motion) are the same with Lorentz-Fitzgerald contraction. ...

The identity of the form for the two paths shows that the postulated Lorentz contraction will give a null result in the Michelson-Morely experiment.

1.3.1 The Kennedy-Thorndike Apparatus:

(Kennedy was the professor and Thorndike was a graduate student at CalTech.)



They did not use a 90° angle but that is not important for this discussion.

$$\Delta t = \frac{\Delta\ell}{c\sqrt{1 - v^2/c^2}} = \frac{\Delta\ell}{c}\left[1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right]$$

Now change $\vec{v}$ to $\vec{v}'$.

$$\Delta t' = \frac{\Delta\ell}{c}\left[1 + \frac{1}{2}\frac{v'^2}{c^2} + \cdots\right]$$

The fringe shift

$$f = \frac{\Delta t' - \Delta t}{\tau} = \frac{1}{2}\frac{\Delta\ell}{c\tau}\left(\frac{v'^2 - v^2}{c^2}\right)$$

where τ is the period of the light (inverse of frequency).

The apparatus on the Earth has a 12 hour reversal of the Earth's rotational velocity which is added in vector form to the Earth's velocity $v_\oplus$around the Sun and thus Sun's velocity through the Ether. Thus the 12 hour modulation is

$$v'^2 - v^2 = [v_E + v_S + v_\oplus]^2 - [v_E + v_S - v_\oplus]^2 \simeq 4(v_E + v_S)v_\oplus \equiv 4v_1 v_\oplus$$

$$f_{12\ hr} = \frac{2\Delta\ell}{\lambda}\frac{v_1 v_\oplus}{c^2}$$

$$f_{6\ mo} = \frac{2\Delta\ell}{\lambda}\frac{v_S v_\oplus}{c^2}$$

The experimental results are summarized as:
Daily (1930-1931): $v_S = 24 \pm 19$ km/s based upon 2500 exposures.
Annual (1931): $v_S = -15 \pm 4$ km/s in opposite direction! Based upon 300 exposures.
Weighted Result: $v_S = 10 \pm 10$ km/s – A NULL EFFECT.

Meaning of this Result: If one keeps the intial hypothesis, one must assume a time contraction.

Experimental Techniques: Quartz base plate, quartz posts to hold mirrors, since quartz is thermally stable and mechanically stable. The temperature control was 10^{-3} °C, since 1°C gives 1/100th of a fringe. An arc light source was not sufficiently stable. They used $\lambda = 5461\mathring{A}$ mercury spectral line from an electrodeless discharge. They took automatic photographs of fringe pattern every 30 minutes. They used $\Delta\ell$ of 31.8 cm which was limited by the coherence of light.

What is the energy variation ΔE of photons whose coherence length is $\Delta\ell$?

$$\Delta p \Delta\ell \sim \hbar \sim \Delta E \Delta\ell / c$$

$$\Delta E \sim \frac{\hbar c}{\Delta\ell}$$

$$E = h\nu = hc/\lambda$$

$$\frac{\Delta E}{E} \sim \frac{\hbar}{h}\frac{\lambda}{\Delta\ell} \sim \frac{\lambda/2\pi}{\Delta\ell}$$

$$\lambda \sim 5460 \times 10^{-8} \text{ cm}, \qquad \Delta\ell = 31.8 \text{ cm}$$

$$\frac{\Delta E}{E} \sim \frac{5460}{63.6\pi} \times 10^{-8} \sim 2.7 \times 10^{-7}$$

Probable fringe comparator error about 1/100th fringe.

The same apparatus can be used to measure frequency shifts in light sources when they are placed in an $\vec{E}$ field. Thorndike did this experiment and found:

$$\frac{\Delta\nu}{\nu} \sim (1.1 \pm 0.8) \times 10^{-14} \text{ per volt/cm}$$

2 Tests of the Second Postulate

2.1 Emission Theory as an Explanation The Michelson-Morely Experiment

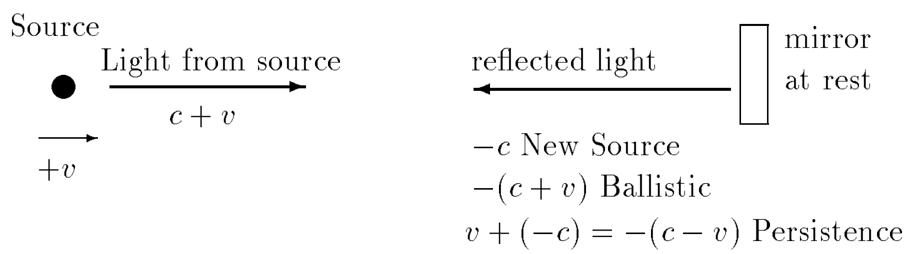
Emission theory: The velocity is c with respect to its source. In the Michelson-Morely experiment all, including the source, are in the same coordinate frame together, so of cause a null fringe shift is expected. There are some difficulties with an emission theory, in general interference and diffraction.

2.2 Different Forms of Emission Theory

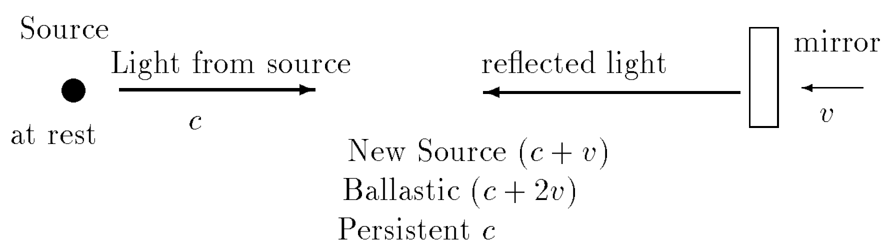
Different forms of emission theory vary regarding velocity of light after reflection from a mirror.
1. New Source Theory (Tolman 1910) Light has velocity c with respect to mirror after reflection.
2. Ballistic Theory (J.J. Thomson 1910) Elastic collision of photon with mirror.
3. Persistence Theory (Ritz 1908) $\vec{c}' = \vec{v}_{source} + \vec{c} =$ velocity of light in one frame. In Ritz's theory $\vec{c}' = \vec{v}_{source} + \vec{c}$ is always the same with respect to the original source of the light.

Summary of Emission Theories Predictions:

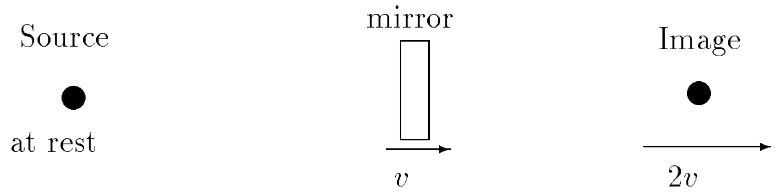
Velocity with respect to mirror:



Velocity with respect to source:



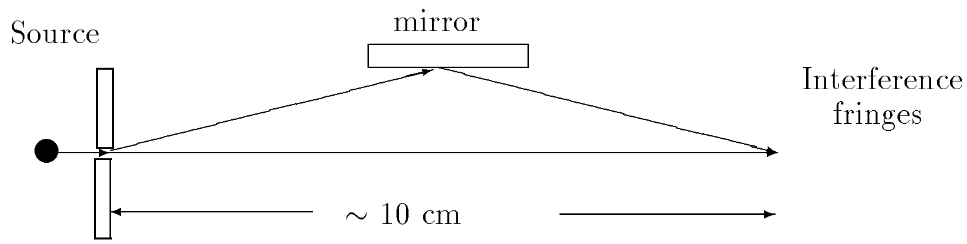
Velocity with respect to mirror image of source:



In ballistic theory, velocity of light is c with respect to source before reflection and c with respect to source after reflection.

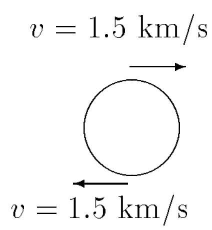
2.2.1 Optical Experiments Testing Emission Theories

The Ritz (persistence) theory is much harder to disprove than the others. It takes an experiment to second order in v/c to distinguish it from the Ether Theory.
1. Interference of Light (Tolman 1910)



2. In the New Source Theory, if the velocity of the source is changed, we expect to observe a fringe shift.

Use light from the two limbs of the Sun:



The fringe shift expected is: $\left\{ \begin{array}{ll} New\ Source & 2\ fringes \\ Ballistic & 0 \\ Ritz & 0 \end{array} \right.$

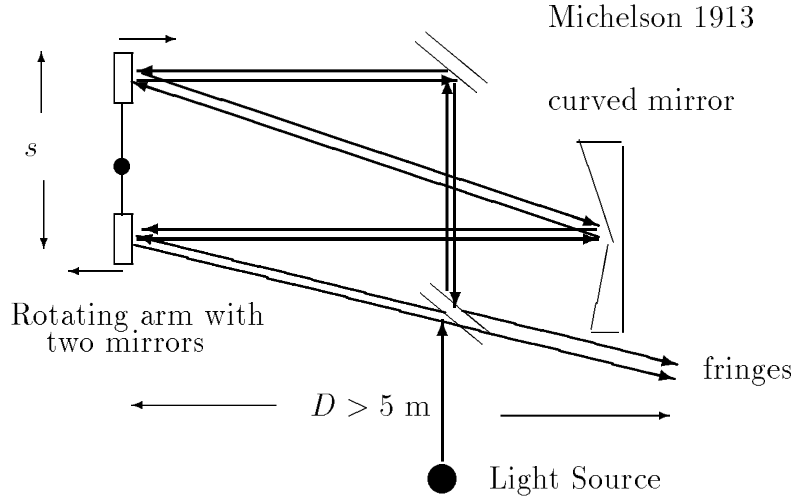
So as a result the New Source Theory is ruled out.
3. Doppler Effect Measurement (Tolman 1910)

Reflected light perpendicular to the axis of a reflection grating.

The emission theory gives a change in frequency but not wavelength, when the source velocity changes.

Expected Result: $\left\{ \begin{array}{ll} \textit{New Source} & \textit{no shift} \\ \textit{Ballistic} & \textit{wrong direction} \\ \textit{Ritz} & \textit{right direction} \end{array} \right.$

4. Velocity of light from a moving mirror (Michelson 1913):



One beam travels the circuit in one direction, while the other travels in the opposite direction. for stationary mirrors the travel times are equal: $t_1 = t_2$.

For moving mirrors the calculations are a bit more complicated. d is the distance that the mirror moves while the light goes a distance $2D$

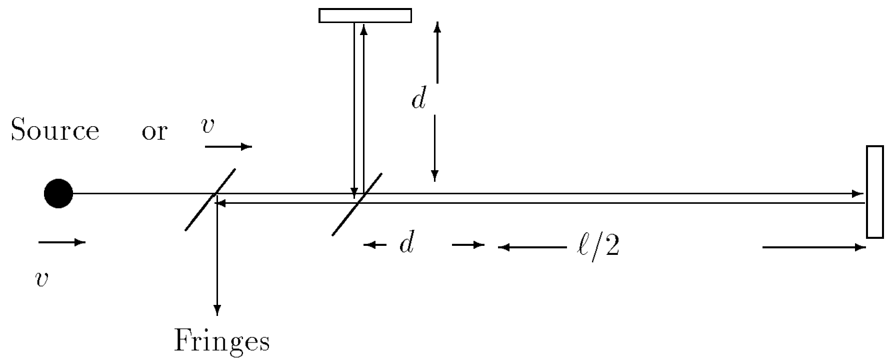
$$d = 2D\frac{v}{c}; \quad d <<< D$$

	New Source Theory	*Ballistic Theory (Ether Theory)*	*Ritz Theory*
t_1	$\frac{D}{c+v} + \frac{D}{c} + \frac{2d}{c}$	$\frac{2D}{c+2v} + \frac{2d}{c}$	$\frac{2(D+d)}{c}$
t_2	$\frac{D}{c-v} + \frac{D}{c} - \frac{2d}{c}$	$\frac{2D}{c+2v} - \frac{2d}{c}$	$\frac{2(D-d)}{c}$
$t_1 - t_2$	$D\left(\frac{1}{c+v} - \frac{1}{c-v}\right) + \frac{4d}{c}$ $= -2\frac{dv}{c^2-v^2} + \frac{8Dv}{c^2} + \frac{8Dv}{c^2}$ $= \frac{6D}{c}\frac{v}{c} + \cdots$	$2D\left(\frac{1}{c+2v} - \frac{1}{c-2v}\right) + \frac{4d}{c}$ $= -\frac{8D}{c}\frac{v}{c} + \frac{4d}{c}$ $0 + \cdots$	$\frac{4d}{c} = \frac{8D}{c}\frac{v}{c}$
Fringes	$\frac{6D}{\lambda}$	0	$\frac{8D}{\lambda}\frac{v}{c}$

The experimental results are:
Observed = 3.81 fringes
Calculated from Ritz = 3.76 fringes.

This result throws out all emission theories except Ritz Persistence theory.

5. Experiments with Light from Moving Mirrors
6. Experiments with Light from Moving Sources



The source or the mirror is stationary.

The time lag for interfering rays is $\Delta t = \ell/2$ for all theories, if both source and mirror are stationary.

Velocity of light before reflection from the most distant mirror

	Moving Mirror	*Moving Source*
New Source	$c+v$	$c+v$
Ballistic	$c+v$ $(for\ 45°)$	$c+v$
Ritz	c	$c+v$

New Source Theory:

$$\Delta t' = \frac{\ell/2}{c+v} + \frac{\ell/2}{c} \rightarrow \frac{v\Delta t'}{c+v}$$

because it does not have to leave so soon.

$$\Delta t' \left[1 - \frac{v}{c+v}\right] = \frac{\ell}{2}\frac{c+c+v}{c(c+v)} = \frac{\ell}{2}\frac{c+v/2}{c+v}$$

$$\Delta t' \frac{c}{c+v} = \frac{\ell(1+v/(2c))}{c+v}$$

$$\Delta t' = \frac{\ell}{c}\left(1 + \frac{v}{2c}\right)$$

The fringe shift is thus

$$F.S. = \frac{\Delta t' - \Delta t}{\tau} = \frac{\ell}{c\tau}\left[1 + \frac{v}{2c} - 1\right] = \frac{\ell}{c\tau}\frac{v}{2c} = \frac{\ell}{2\lambda}\frac{v}{c}$$

Ballistic Theory:

$$\Delta t' = \frac{\ell/2}{c+v} + \frac{\ell/2}{c+v} + \frac{v\Delta t'}{c+v}$$

$$\Delta t' \left(1 - \frac{v}{c+v}\right) = \frac{\ell}{c+v}; \quad \Delta t' \frac{c}{c+v} = \frac{\ell}{c+v}.$$

Thus $\Delta t' = \ell/c = \Delta t$; so that the frame shift is zero.

Ritz Theory:

In this case we must distinguish between moving source and moving mirror.

Moving Mirror	*Moving Source*
$\Delta t' = \frac{\ell}{c} + \frac{v\Delta t'}{c}$	$\Delta t' = \frac{\ell/2}{c+v} + \frac{\ell/2}{c-v} + \frac{v\Delta t'}{c+v}$
$\Delta t'\left(1 - \frac{v}{c}\right) = \frac{\ell}{c}$	$\Delta t'\left(1 - \frac{v}{c+v}\right) = \frac{\ell}{c}\left(\frac{2c}{c^2-v^2}\right)$
	$= \Delta t'\frac{c}{c+v} = \frac{\ell c}{(c+v)(c-v)}$
$\Delta t' = \frac{\ell}{c}\left(1 + \frac{v}{c}\right) + \cdots$	$\Delta t' = \frac{\ell}{c-v} = \frac{\ell}{c}\left(1 + \frac{v}{c}\right) + \cdots$
$F.S. = \frac{\Delta t' - \Delta t}{\tau} = \frac{\ell}{c}\frac{v}{\tau c} = \frac{\ell}{\lambda}\frac{v}{c}$	$F.S. = \frac{\Delta t' - \Delta t}{\tau} = \frac{\ell}{c\tau}\frac{v}{c} = \frac{\ell}{\lambda}\frac{v}{c}$

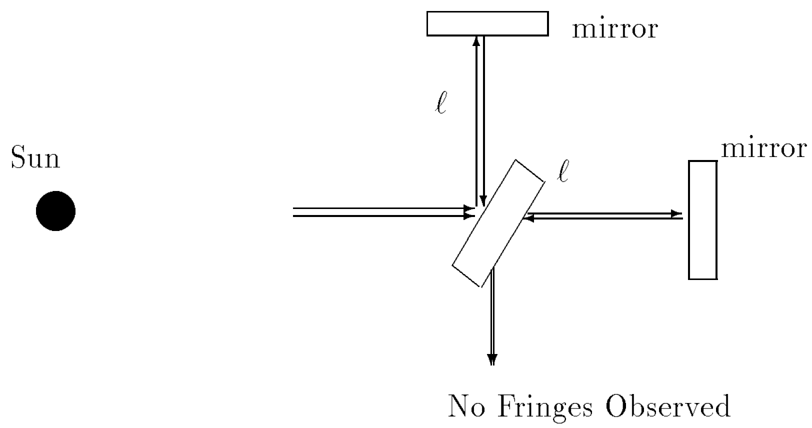
The results are the same!

For a moving mirror: $v = 80$ m/s, $\ell = 23.2$ cm, $\lambda = 5640$ $\mathring{A}$, $F.S._{calc.} = 0.113$ fringes, $F.S._{obs.} = 0.199$ fringe. Similar results were found for moving source.

2.2.2 Michelson-Morely Experiment Using Light from the Sun

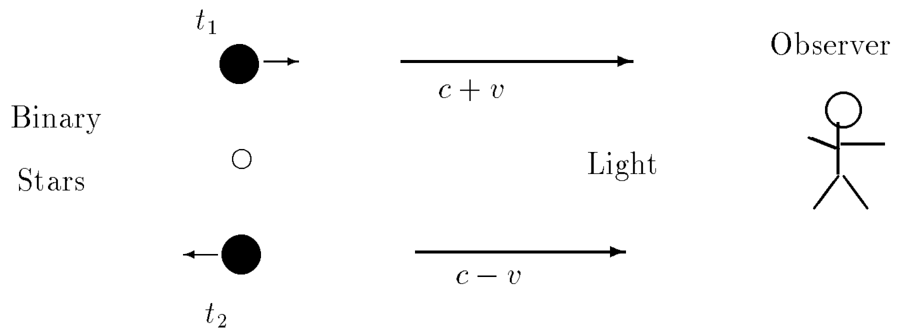
Tolman – Phys. Rev. 35, 136 (1912) – pointed out that a Michelson-Morely experiment using light from the Sun would be a decisive test. This was also pointed out by LaRosa – Phys. Zeitschrift, 18, 1129 (1912).

In the Ritz Theory, light from the Sun behaves as if the Ether were fixed in the Sun. The Earth's velocity through the Ether would be 30 km/second as the Earth orbits the Sun.



2.2.3 Astronomical Evidence

Comstock (1910) and DeSitter(1913) pointed out that the light observed from binary (double) stars provided a test. Consider two stars in orbit about each other.



The upper portion of the orbit seems to be traversed more quickly than the lower half in the Ritz Theory. The actual half period is $t_1 - t_2 = \Delta t$. The observed half period is

$$\begin{aligned}
\Delta t' &= \left(t_1 + \frac{\ell}{c-v}\right) - \left(t_2 + \frac{\ell}{c+v}\right) \\
&= t_1 - t_2 + \ell\left(\frac{1}{c-v} - \frac{1}{c+v}\right) \\
&= \Delta t + \frac{2\ell v}{c^2 - v^2} \\
&\simeq \Delta t \frac{2\ell v}{c^2}
\end{aligned} \tag{23}$$

It turns out that $2\ell v/c^2$ is often greater than Δt for binary stars. So such a term $(2\ell v/c^2)$ would lead to very odd effects; e.g. seeing the start two or three times at once, or not at all other times. Circular orbits would appear elliptical, etc.

Binary stars are not easy to observe. Many stars are "spectroscopic binaries".

DeSitter (1913) studied the data on all know binaries and selected some of low apparent eccentricity (probably nearly circular orbits). His conclusion: c' - on the emission theory $= c + kv$ with $k < 0.002$. While $k = 1$ is predicted by the emission theory.

2.2.4 Final "Box Score"

This summary due to Tolman (1946).
Experiments:
1. Michelson-Morely
2. Trouton-Noble
3. Kennedy-Thorndike

Postulate:

4. Interference (lines of the Sun)
5. Doppler Effect
6. Velocity of light from Moving Mirror
7. Velocity of light from Moving Source

8. Michelson-Morely experiment with light from the Sun
9. Double stars

Experimental Test "Box Score"

Theories to Test:	Experiments	
	Agree	Disagree
Stationary Ether	4, 5, 6, 7, 9	1, 2, 3, 8
Emission Theory - New Source	1, 2, 3	4, 5, 6, 7, 8, 9
Emission Theory - Ballistic	1, 2, 3, 4, 8	5, 6, 7, 9
Emission Theory - Persistence (Ritz)	1, 2, 3, 4, 5, 6, 7	8, 9
Special Relativity - Einstein	all	none

2.3 Transformation of γ or Dilation Factor

Simple transformation of $\sqrt{1-u^2/c^2}$ or γ. Using the transformation laws for velocity we can derive the transformation law for γ or $\sqrt{1-u^2/c^2}$ by simple algebra. First calculate the transformation for $\sqrt{1-u_x^2/c^2}$

$$\begin{aligned} 1-\frac{u_x^2}{c^2} &= 1-\frac{(u'_x+v)^2}{(1+u'_x v/c^2)^2} \\ &= \frac{1+2u'_x v/c^2+\frac{u_x'^2}{c^2}\frac{v^2}{c^2}-\frac{u_x'^2}{c^2}-2u'_x v/c^2-\frac{v^2}{c^2}}{(1+u'_x v/c^2)^2} \\ &= \frac{1-\frac{v^2}{c^2}-\left(1-\frac{v^2}{c^2}\right)\frac{u_x'^2}{c^2}}{(1+u'_x v/c^2)^2} \\ &= \left(1-\frac{u_x'^2}{c^2}\right)\frac{\left(1-\frac{v^2}{c^2}\right)}{(1+u'_x v/c^2)^2} \end{aligned} \tag{24}$$

Now we are ready to do the complete expression

$$1-\frac{u^2}{c^2}=\left(1-\frac{u'^2}{c^2}\right)\frac{\left(1-\frac{v^2}{c^2}\right)}{(1+u'_x v/c^2)^2} \tag{25}$$

The square root of this equation gives the transformation law

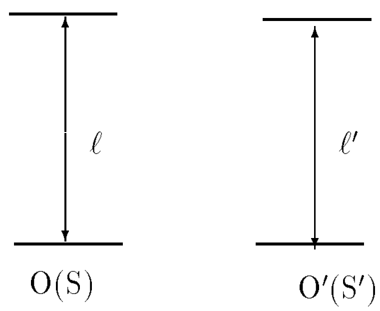
$$\sqrt{1-\frac{u^2}{c^2}}=\sqrt{1-\frac{u'^2}{c^2}}\frac{\sqrt{1-\frac{v^2}{c^2}}}{(1+u'_x v/c^2)} \tag{26}$$

3 Properties of Spatial and Temporal Measurements

In this chapter we explore the properties of spatial and temporal measurements that result from the two postulates of special relativity.

3.1 Comparison of Meter Sticks and Clocks in Relative Motion

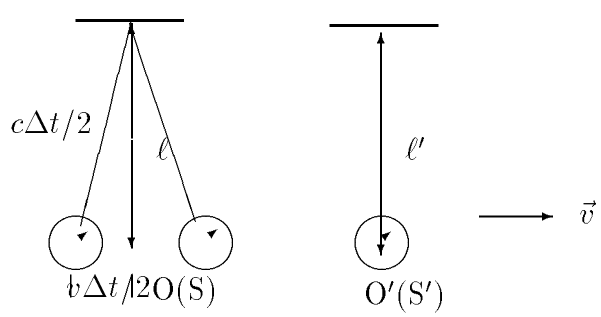
3.1.1 Meter Sticks $\perp$ Motion



By the first postulate the lengths $\ell = \ell'$, since, if one were shorter , it might be absolutely at rest and the other moving with respect to it, or at least they would be distinguishable.

3.1.2 Clock Rates

Consider a clock made by counting reflections between two parallel mirrors moving perpendicularly.



For frame O:

$$
\begin{aligned}
\left(\frac{c\Delta t}{2}\right)^2 &= \ell^2 + \left(\frac{v\Delta t}{2}\right)^2 \\
(\Delta t)^2\left(c^2 - v^2\right) &= 4\ell^2 \\
\Delta t &= \frac{2\ell}{\sqrt{c^2 - v^2}} \\
&= \frac{2\ell}{c}\frac{1}{\sqrt{1-(v/c)^2}} \qquad (27)
\end{aligned}
$$

For frame O′:

$$
\Delta t' = \frac{2\ell'}{c} = \frac{2\ell}{c} \qquad (28)
$$

So one has

$$\Delta t = \frac{\Delta t'}{\sqrt{1-(v/c)^2}} \tag{29}$$

This is called **Time Dilation**. The time for frame O is greater than the time for O′, so that an observer in frame O claims that frame O′'s clocks are running more slowly.

This is termed <u>Time dilation of a moving clock</u>.

Comments:

(1) In Ether theory

$$\Delta t = \Delta t' = \frac{2\ell}{c\sqrt{1-(v/c)^2}}$$

Since both observers would agree on the actual path length through the Ether.

Same for Ether Theory with Lorentz contraction since there is no contraction perpendicular to $\vec{v}$.

(2) Emission Theory gives

$$\Delta t = \Delta t' = \frac{2\ell}{c}$$

(3) What does O′ say about the clocks in frame S?

O′ says clocks in S run slow. This is necessary but the first postulate; Do the experiment the other way and remember that the systems cannot be distinguishable.

(4) Is all this consistent?

Observer O uses two clocks, O′ uses one! O′ blames O's "wrong result" on O's clocks not being properly synchronized. (The second clock is set later.) Thus the systems are not symmetrical and identical. O′ agrees with O as to all the clock readings but explains this differently.

(5) Can the rate of a moving clock be tested experimentally?

<u>Yes</u>. The earliest good work was by Ives and Stilwell using "canal rays". (1928)

Doppler effect makes the result unless the observation is made perpendicular to $\vec{v}$ or one uses the average of parallel and anti-parallel light. The latter approach is better. They used Dempster's velocity selector.

$$\lambda = \frac{\lambda_0\left(1 \pm \frac{v}{c}\right)}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{30}$$

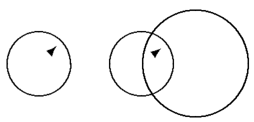
The upper term represents the Doppler effect and the lower the time dilation.

Other early measurements include: Nereson and Rossi published in Physical Review <u>64</u>, 199 (1943) and the direct test with mesons by Neher and Stever published in Physical Review <u>58</u>, 756 (1940).

The height was chosen so that matter traversed was the same difference in rate so decay of cosmic ray mesons in 12,000 feet.

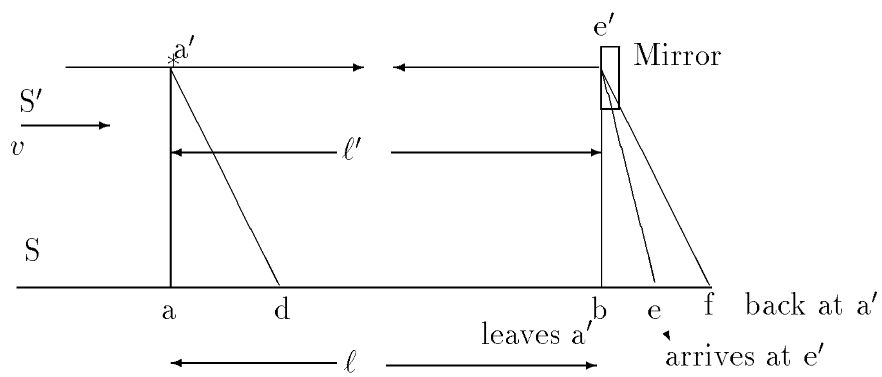
(6) The nature of clocks: Clocks may be mechanical, electrical, chemical, radioactive, biological, atomic, nuclear, etc.: **All clocks obey the same law of time dilation.**

(7) How can one compare clocks in two different systems?



Example, put one clock on a rotating wheel with velocity v and compare after each revolution. The moving clock runs slow by the $\gamma = 1/\sqrt{1 - v^2/c^2}$ factor.

3.1.3 Meter Sticks || Motion



S′ sends light signal to mirror and back, with time $\Delta t'$ in distance $2\ell'$. $\Delta t' = 2\ell'/c$ by second postulate.

By time dilation which is just established:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} = \frac{2\ell'}{c\sqrt{1 - v^2/c^2}} \tag{31}$$

But from the second postulate directly

$$\Delta t = \frac{\overline{ae} + \overline{ed}}{c}$$

$$\overline{ae} = \ell + \overline{be} = \ell + \overline{ae}\frac{v}{c}$$

$$\overline{ae}(1 - \frac{v}{c}) = \ell$$

$$\overline{ae} = \frac{\ell}{1 - \frac{v}{c}}$$

$$\overline{de} = \ell + \overline{be} - \overline{de} = \ell + \overline{ae}\frac{v}{c} - [\overline{ae} + \overline{de}]\frac{v}{c} = \ell - \overline{de}\frac{v}{c};$$

$$\overline{de}(1 + \frac{v}{c}) = \ell$$

$$\overline{de} = \frac{\ell}{1+\frac{v}{c}}.$$

$$\Delta t = \frac{\ell}{c}\left[\frac{1}{1-\frac{v}{c}} + \frac{1}{1+\frac{v}{c}}\right]$$

$$\Delta t = \frac{2\ell}{c\,(1-v^2/c^2)}$$

But

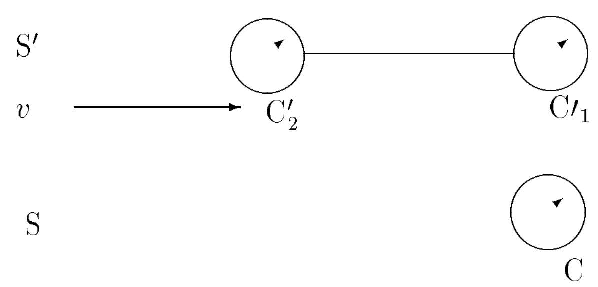
$$\Delta t = \frac{2\ell'}{c\sqrt{1-v^2/c^2}},$$

so the equation

$$\ell = \ell'\sqrt{1-v^2/c^2} \tag{32}$$

give the Lorentz-Fitzgerald contraction.

3.1.4 Setting of Clocks



S notes the clock reading on C and on C'_1 when C'_1 passes C **and** the readings on C and C'_2 when C'_2 passes C. By the first postulate, O and O′ agree on $|v|$.

$$t'_2 - t'_1 = \frac{\ell'}{v}, \qquad t_2 - t_1 = \frac{\ell}{v}$$

with the same v.

But $\ell = \ell'\sqrt{1-v^2/c^2}$ (just established) and

$$t_2 - t_1 = \frac{t'_2 + \Delta t'_2 - t'_1}{\sqrt{1-v^2/c^2}}$$

where the time $\Delta t'_2$ was just established previously in which $\Delta t'_2$ is the correction made by O to the setting by O′ of clock C'_2 in order to get clock synchronization in frame S′. ($\Delta t'_2$ turns out to be negative; O finds t'_2 to be ahead, and must subtract $|\Delta t'_2|$ from its reading.)

$$\frac{t_2' + \Delta t_2' - t_1'}{\sqrt{1 - v^2/c^2}} = \frac{\ell}{v} = \frac{\ell'}{v}\sqrt{1 - v^2/c^2}$$

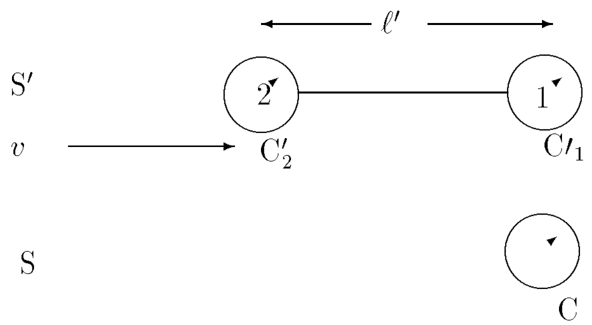
$$\frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}} + \frac{\Delta t_2'}{\sqrt{1 - v^2/c^2}} = \frac{\ell'}{v}\sqrt{1 - v^2/c^2}$$

$$= \frac{\ell'/v}{\sqrt{1 - v^2/c^2}} + \frac{\Delta t_2'}{\sqrt{1 - v^2/c^2}} = \frac{\ell'}{v}\sqrt{1 - v^2/c^2}$$

Solving for $\Delta t_2'$ yields

$$\Delta t_2' = \frac{\ell'}{v}\left[\left(\sqrt{1 - v^2/c^2}\right)^2 - 1\right] = -\frac{\ell' v}{c^2}.$$

O says clock C_2' is set **ahead** by $\ell' v/c^2$ in the time units used by O′.
The clock behind in space is ahead in time.
Now by the Second Postulate A new experiment:



S′ sends a beam of light from 1 to 2 and times how long it takes.

$$t_2' - t_1' = \ell'/c$$

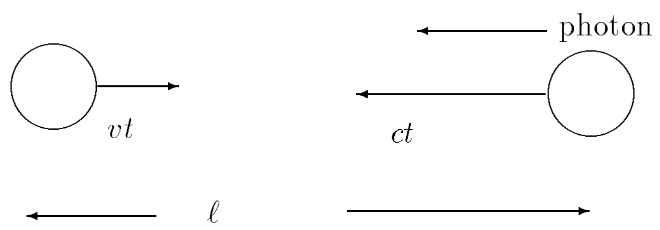
What does O calculate?

$$t_2 - t_1 = \frac{t_2' + \delta t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

remembering that time dilation is previously established. There is a new $\delta t_2'$ for this experiment, the correction calculated by O for C_2'.

$$t_2 - t_1 = \frac{\ell}{c + v} = \frac{\ell'\sqrt{1 - v^2/c^2}}{c + v}$$

making use of the established Lorentz contraction.



$$ct + vt = \ell$$

$$t = \tfrac{\ell}{c+v}$$

$$\frac{t_2' + \Delta t_2' - t_1'}{\sqrt{1 - v^2/c^2}} == \frac{\ell' \sqrt{1 - v^2/c^2}}{c+v}$$

$$\delta t_2' = \frac{\ell' \left(\sqrt{1 - v^2/c^2}\right)^2}{c+v} - (t_2' - t_1')$$

$$= \frac{\ell' \left(\sqrt{1 - v^2/c^2}\right)^2}{c+v} - \frac{\ell'}{c}$$

$$= \ell' \left[\frac{1 - v^2/c^2}{c+v} - \frac{1}{c}\right] = \ell' \frac{c - v^2/c^2 - c - v}{c(c+v)}$$

$$= -\frac{\ell' v(1 + v/c)}{c^2(1 + v/c)} = -\frac{\ell' v}{c^2}$$

Which is exactly the same as deduced from the First Postulate. O says "Clock behind in space is ahead in time."

3.1.5 Operational Explanation of Perceived Synchronization Defect

1) O′ synchronizes two identical clocks at the same place.

2) He carries one slowly to the rear; or both slowly away from each other and they stay synchronized as he sees it.

3) O says that the rear clock moves ahead in time because it runs faster while being moved back to its final position.

Rate of front clock $\equiv r_1'$ and rate of rear clock $\equiv r_2'$ as seen by O.

$$r_1' = r_o \sqrt{1 - v^2/c^2}$$

$$r_2' = r_o \sqrt{1 - (v - \Delta v)^2/c^2}$$

$$\Delta r' \simeq r_o \{ \left[1 - (v - \delta v)^2/(2c)\right] - \left[1 - v^2/(2c^2)\right] \}$$

$$= r_o \left[1 - v^2/c^2 + v\Delta v/c^2 - 1 + v^2/c^2\right]$$

$$\Delta r' = r_o \, \frac{v \Delta v}{c^2}$$

Distance moved is $\ell' = \Delta V \Delta t'$ where $\Delta t'$ is the time to move the clock.

$$\Delta r' = r_o \, \frac{v \ell'}{c^2 \Delta t'}$$

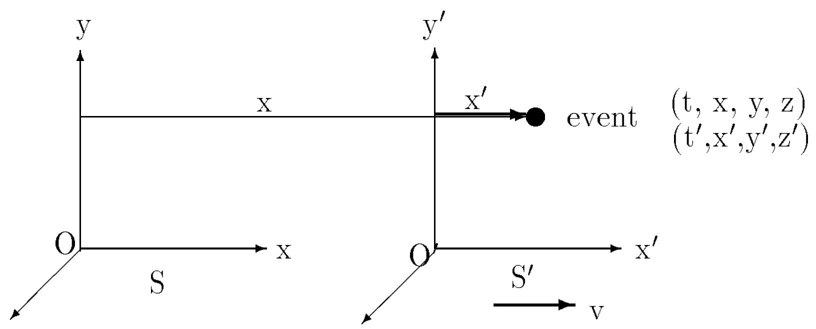
$\Delta r' \Delta t'$ is he synchronization "error" which is $\delta t' = \ell' f / c^2$. The rear clock is ahead in time.

CONCLUSION One must abandon the notion of simultaneity of time for observers in relative motion.

4 The Lorentz Transformation

The Lorentz transformations are demanded and supported by experimental observations. The Lorentz transformation equations can readily be derived from length contraction and time dilation after taking a short detour to discuss clock synchronization.

Consider two frames of reference: S, the laboratory frame and S′ a frame of reference moving with velocity $\vec{v}$ in the $\hat{x}$ direction as shown in the following figure.



Arrange things so that at $t = 0$ and $t' = 0$ that the two origins O and O′ coincide.

Consider each reference system to a an actual lattice of meter sticks and clocks, e.g. each reference system is filled with these space and time measuring devices at every point.

Get clocks in S to agree. Identical clocks set by sending out a light pulse from origin O and also from the midpoint between any two clocks. Check times, reflect back to midpoint, if pulses arrive together, then clocks agree.

System S′ dos the same with his clocks.

We have for the space-time event in the figure above

$$x = vt + x' \times \sqrt{1 - v^2/c^2} \tag{33}$$

where the second term takes into account length contraction of a moving frame. We can use our arguments about the transverse directions to show that they are unchanged and then have the spatial Lorentz transformations:

$$\begin{aligned}
x' &= \frac{1}{\sqrt{1-v^2/c^2}}(x-vt) \\
y' &= y \\
z' &= z \\
t' &= t\sqrt{1-v^2/c^2} + \text{synchronization effect}
\end{aligned} \tag{34}$$

4.1 Synchronizing Clocks in Moving Frame

Our approach to get our system of reference made of a grid of meter sticks and synchronized clocks requires that we synchronized the clocks. An approach to synchronizing the clocks is: bring the clock together, match their readings, then move into place. Move them slowly and gently so as not to disturb their operation.

Consider the simple case of two clocks brought together at the origin of the moving system S′. When they are together, from the laboratory frame S both clocks read same time and are going slow by a factor $\sqrt{1-(v/c)^2}$ as a result of **time dilation**. Now very slowly and gently move one clock back (in negative x'-direction; toward the laboratory system origin) a distance ℓ in elapsed time $\ell = \delta v\ \tau$. The clock at the origin has its rate slow by $\sqrt{1-v^2/c^2}$ relative to the laboratory frame. Clock moving back in negative x'-direction has its rate slowed by the factor $\sqrt{1-(v-\delta v)^2/c^2}$

$$f_A = \sqrt{1-v^2/c^2}\,f_o \qquad f_B = \sqrt{1-(v-\delta v)^2/c^2}\,f_o \tag{35}$$

The difference in the clocks' rates is

$$\begin{aligned}
f_A - f_B &= f_o\left[\sqrt{1-v^2/c^2} - \sqrt{1-(v-\delta v)^2/c^2}\right] \\
&= \frac{f_o}{\sqrt{1-v^2/c^2}}\left[1-\frac{v^2}{c^2} - \left((1-\frac{v^2}{c^2})(1-\frac{(v-\delta v)^2}{c^2}\right)^{1/2}\right] \\
&= \frac{f_o}{\sqrt{1-v^2/c^2}}\left[1-\frac{v^2}{c^2} - \left((1-\frac{v^2}{c^2})(1-\frac{v^2}{c^2}+\frac{2\delta v v}{c^2}-\frac{\delta v^2}{c^2})\right)^{1/2}\right] \\
&= -f_a\frac{\delta v\ v/c^2}{1-v^2/c^2} = -f_o\frac{\delta v\ v/c^2}{\sqrt{1-v^2/c^2}}
\end{aligned} \tag{36}$$

If it takes a time $\tau = \ell_o/\delta v$ to separate the clocks, the time difference between them is

$$\Delta t = \frac{f_A - f_B}{f_A}\tau = \left(\sqrt{1-v^2/c^2} - \sqrt{1-(v-\delta v)^2/c^2}\right)\tau$$

$$= \quad -\frac{\delta v \; v/c^2}{\sqrt{1-v^2/c^2}} \times \frac{\ell_o}{\delta v} = -\frac{\ell_0 \; v/c^2}{\sqrt{1-v^2/c^2}} = -\ell' v/c^2 \tag{37}$$

Note that the speed with which the clock moves drops out and the change in reading is proportional only to the distance displaced and the velocity of the moving system.

Clocks get out of synchronization (phase) by an amount proportional to their separation ℓ_o and v. If brought back together, the clocks will go into synchronization. **The clock that is farther behind in space is further ahead in time.**

Note that in the frame S$'$ the difference in rate of time kept between the clock at the origin and the one being moved back to its place is second order in v/c rather than first order:

$$f'_B = \frac{f'_A}{\sqrt{1-(\delta v)^2/c^2}} \simeq f'_A \times \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

So that by moving with a very, very slow velocity the integrated effect in the S$'$ frame can be made arbitrarily small while the effect as observed in the S frame is always $-\ell' v/c^2$ independent of δv. That is because the effect in frame S is first order in $\delta v/c$ and integrated over time equals the displacement.

The final Lorentz transformations are:

$$\begin{aligned} t' &= t\sqrt{1-v^2/c^2} - \frac{x'v}{c^2} = t\sqrt{1-v^2/c^2} - \frac{v^2}{c^2}\frac{1}{\sqrt{1-v^2/c^2}}(x-vt) \\ &= \frac{1}{\sqrt{1-v^2/c^2}}\left[t - \frac{vx}{c^2}\right] = \gamma\left[t - \frac{vx}{c^2}\right] \end{aligned} \tag{38}$$

Notice for $v << c$ get Galilean transforms and there is also a symmetry between the transformation equations.

$$\begin{array}{cc} t' = \gamma(t - vx/c^2) & t = \gamma(t' + vx'/c^2) \\ x' = \gamma(x - vt) & x = \gamma(x' + vt') \\ y' = y & y = y' \\ z' = z & z = z' \end{array} \tag{39}$$

4.1.1 Remarks on Lorentz Transformation

History: Lorentz transformation was derived by Lorentz before Einstein's work. Lorentz obtained them by considering invariance of Maxwell's Equations.

Significance: They define the mathematical specification required to discus a kinematic occurrence – a sequence of space-time events.

Agreement with First Postulate: If one does the inversion, one obtains the same equations. Try replacing v by $-v$.

.....

Agreement with Second Postulate:

In S, light is described by

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2 \tag{40}$$

This is equivalent to

$$dx^2 + dy^2 + dz^2 - c^2 dt^2 = 0 \tag{41}$$

Substitute in the Lorentz transformations

$$\frac{(dx' + v dt')^2}{\left(\sqrt{1 - v^2/c^2}\right)^2} + dy'^2 + dz'^2 - c^2 \frac{\left(dt' + \frac{v}{c} dx'\right)^2}{\left(\sqrt{1 - v^2/c^2}\right)^2}$$

$$= \frac{1}{\left(\sqrt{1 - v^2/c^2}\right)^2}\left[dx'^2 + 2v dx' dt' + v^2 dt'^2 - \frac{v^2}{c^2} dx'^2 - 2v dx' dt' - c^2 dt'^2\right] + dy'^2 + dz'^2$$

$$= dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = 0$$

Group Property of Lorentz Transformations in a Line
Identity exists: $v = 0$
Inverse exists: $v \to -v$
Transitive: $\mathrm{S} \to \mathrm{S}' \to \mathrm{S}'' \equiv \mathrm{S} \to \mathrm{S}''$
Exhibit as an Exercise?
It is true that all Lorentz Transformations also form a group.

4.2 Composition of Velocities

In Galilean relativity one simply adds velocities when changing frames of reference. Velocity composition is slightly more complicated in Special Relativity. We can readily derive the velocity composition formulae from the Lorentz transformation.

$$dt = \frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1 - v^2/c^2}}; \qquad \frac{dt}{dt'} = \frac{1 + \frac{v}{c^2}\frac{dx'}{dt'}}{\sqrt{1 - v^2/c^2}} = \frac{1 + v u_x/c^2}{\sqrt{1 - v^2/c^2}}$$

$$\frac{dt}{dt'} = \frac{1 + u'_x \frac{v}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$u_x = \frac{dx}{dt} = \frac{dx' + v dt'}{dt' \sqrt{1 - v^2/c^2}} \frac{dt'}{dt} = \frac{\frac{dx'}{dt'} + v}{\sqrt{1 - v^2/c^2}} \frac{dt'}{dt}$$

$$u_x = \frac{u'_x + v}{\sqrt{1 - v^2/c^2}} \frac{\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}$$

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \tag{42}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{dt} = \frac{dy'}{dt'}\frac{dt'}{dt}$$

$$u_y = u'_y \frac{\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}} \tag{43}$$

$$u_z = u'_z \frac{\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}} \tag{44}$$

These three equations are the **Einstein Velocity Addition Law**.

<u>Velocities do not add like vectors!</u>

There are other important quantities for which transformation equations are needed. That is to say that they do not transform like vectors. Can work them out as exercises, e.g. transformation of acceleration and force. We will discuss these more later.

$$a_x = \frac{du_x}{dt}; \qquad a'_x = \frac{du'_x}{dt'}; \qquad \text{etc.}$$

Answers:

$$a_x = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x \tag{45}$$

$$a_y = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^2} a'_y - \frac{\frac{u'_y v}{c^2}\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x \tag{46}$$

$$a_z = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^2} a'_z - \frac{\frac{u'_z v}{c^2}\left(1 - \frac{v^2}{c^2}\right)}{\left[1 + \frac{u'_x v}{c^2}\right]^3} a'_x \tag{47}$$

Note as hint that

$$\sqrt{1 - u^2/c^2} = \sqrt{1 - (u')^2/c^2}\frac{\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}$$

and

$$u^2 = u_x^2 + u_y^2 + u_z^2$$

4.2.1 Transformation of the Lorentz Factor γ

Now that we have the composition of velocities or Lorentz transformation of velocities, we can find the transformation of the Lorentz Factor γ and/or $\sqrt{1 - v^2/c^2}$.

First consider $\sqrt{1 - u^2/c^2}$ where u is the speed of the particle in frame S and u' is the speed of the particle in frame S$'$ and the frames have relative velocity V.

$$u^2 = u_x^2 + u_y^2 + u_z^2 = \left(\frac{u'_x + V}{1 + u'_x V/c^2}\right)^2 + \left(\frac{u'_y\sqrt{1 - V^2/c^2}}{1 + u'_x V/c^2}\right)^2 + \left(\frac{u'_z\sqrt{1 - V^2/c^2}}{1 + u'_x V/c^2}\right)^2$$

so that

$$1 - \frac{u^2}{c^2} = 1 - \frac{(u'_x + V)^2 + \left(u'^2_y + u'^2_z\right)(1 - V^2/c^2)}{c^2\,(1 + u'_x V/c^2)^2}$$

$$= \frac{c^2 + 2u'_x V + u'^2_x V^2/c^2 - u'^2_x - 2u'_x V - V^2 - \left(u'^2_y + u'^2_z\right)(1 - V^2/c^2)}{c^2\,(1 + u'_x V/c^2)^2}$$

$$= \frac{c^2 - V^2 - \left(u'^2_x + u'^2_y + u'^2_z\right)(1 - V^2/c^2)}{c^2\,(1 + u'_x V/c^2)^2}$$

$$= \frac{(1 - V^2/c^2)\,(1 - u'^2)}{(1 + u'_x V/c^2)^2}$$

Taking the square root yields

$$\sqrt{1 - u^2/c^2} = \frac{\sqrt{1 - V^2/c^2}\sqrt{1 - u'^2/c^2}}{1 + u'_x V/c^2} \tag{48}$$

And since $\gamma = 1/\sqrt{1 - u^2/c^2}$ we have

$$\gamma_p = \left(1 + u'_x V/c^2\right)\gamma_f \gamma'_p \tag{49}$$

where γ_p and γ'_p are the Lorentz γ of the particle in the S and S$'$ frames, respectively, and $\gamma_f = 1/\sqrt{1 - V^2/c^2}$ is the Lorentz γ of one frame relative to the other.

We will use these transformations again later.

4.2.2 Velocity of Light as Maximum

The velocity addition law indicates that the velocity of light is the maximum velocity attainable by a material object. (Hence the origin of the t-shirt with Einstein in policeman's cap saying Speed Limit: 186,000 miles/sec! It's not just a good idea; it's the law.)

The velocity addition law for motion in the x-direction is

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$$

If $v = u'_x = c/2$,

$$u_x = \frac{c/2 + c/2}{1 + 1/4} = \frac{4}{5}c.$$

If $v = u'_x = c$,

$$u_x = \frac{c + c}{1 + 1} = c!!$$

That is adding together two velocities that are very near the speed of light only gets one closer to the speed of light; one cannot keep adding velocities and exceed the speed of light.
Exercise: Show that if one has a particle moving at ϵc slower than c ($u' = (1-\epsilon)c$ in the frame S′ moving at speed $v = (1-\delta)c$ just less than the speed of light in the same direction, the velocity observed in frame S is just a little less than c.
Solution: Once can use the composition of velocities formula

$$u = \frac{v + u'_x}{1 + u_x v/c^2} = \frac{(1 - \epsilon + 1 - \delta)c}{1 + (1-\epsilon)(1-\delta)} = \frac{2 - \epsilon - \delta}{2 - \epsilon - \delta + \epsilon\delta}c = \frac{c}{1 + \frac{\epsilon\delta}{2-\epsilon-\delta}} \simeq (1 - \epsilon\delta/2)c$$

4.2.3 Velocity of a Causal Impulse

In Frame S: (From the point of view of observer O in frame S)

$$\Delta t = t_2 - t_1 = \frac{x_2 - x_1}{u}; \qquad u = \frac{x_2 - x_1}{t_2 - t_1}$$

In Frame S′: (From the point of view of observer O′ in frame S′)

$$\Delta t' = t'_2 - t'_1 = \frac{1}{\sqrt{1 - v^2/c^2}}\left[t_2 - \frac{x_2 v}{c^2} - t_1 + \frac{x_1 v}{c^2}\right]$$

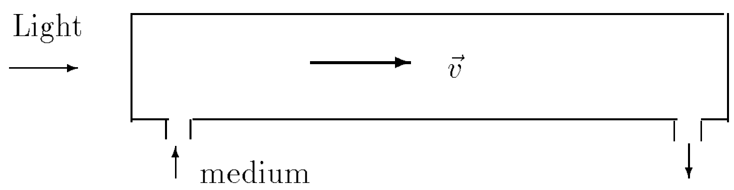
$$= \frac{t_2 - t_1}{\sqrt{1 - v^2/c^2}}\left[1 - \frac{v}{c^2}\left(\frac{x_2 - x_1}{t_2 - t_1}\right)\right] = \frac{1 - uv/c^2}{\sqrt{1 - v^2/c^2}}\Delta t.$$

Now, if the causal impulse velocity u is greater than c (the speed of light), one can choose v to make $\Delta t'$ negative! Effect precedes cause! This is impossible, if we are to keep causality, so the maximum velocity of a causal impulse is c.

This limit is for the group velocity in a medium wth normal disperson - group velocity is the speed with which signals can be sent in that medium. Phase velocities may have any value! In a medium with anomoulus dispersion the situation is more complicated. Needless to say after some investigation, it will be found that the information in the wave will travel at c or less and that the electromagnetic field travels at speed c but the effects of the interaction with the medium and phases of subsequent scattering/re-radiation can give nearly any group velocity if the phasing is arranged properly.

4.2.4 Velocity of Light in a Moving Medium

S S′ moves with medium



$$u' = \frac{c}{n}; \qquad n = \text{index of refraction} \tag{50}$$

$$u = \frac{u'+v}{1+u'v/c^2} = \frac{c/n+v}{1+cv/(nc^2)} \simeq \left(\frac{c}{n}+v\right)\left(1-\frac{v}{nc}\right).$$

$$u \simeq \frac{c}{n} + v - \frac{c}{n^2}\frac{v}{c} - \frac{v^2}{nc}$$

$$\frac{u}{c} \simeq \frac{1}{n} + \frac{v}{c} - \frac{v}{n^2 c} - \frac{1}{n}\frac{v^2}{c^2} \simeq \frac{1}{n} + \left(1-\frac{1}{n^2}\right)\frac{v}{c}$$

$$u = \frac{c}{n} + \left(1-\frac{1}{n^2}\right)v \tag{51}$$

This is exactly Fresnel's drag coefficient from 1818.

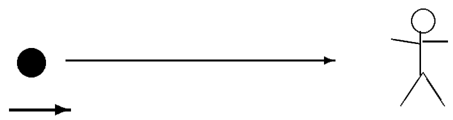
Note that the effect is a little more complicated when dispersion (index of refraction n depends on wavelength/frequency) is taken into account because of the Doppler shift (see next section). The speed c_m of light in a moving medium is equal to

$$c_m = \frac{c}{n} + k v_m \tag{52}$$

where v_m is the speed of the medium and

$$k = 1 - \frac{1}{n(\lambda)^2} - \frac{\lambda}{n(\lambda)}\frac{dn(\lambda)}{d\lambda}$$

4.2.5 Doppler Effect



A stationary observer sees light from a distant source, e.g. a star, The observer sees the light with period P

$$P = P_o \frac{(1 \mp u/c)}{\sqrt{1 - u^2/c^2}} \tag{53}$$

And wavelength λ:

$$\lambda = \lambda_o \frac{(1 \pm u/c)}{\sqrt{1 - u^2/c^2}}. \tag{54}$$

One approach to this result is

$$\lambda = \lambda_o \frac{(1 \pm u/c)}{\sqrt{(1 - u/c)(1 + u/c)}} = \sqrt{\frac{1 + u/c}{1 - u/c}}$$

Remember that $\lambda f = c$ or $\lambda/P = c$.

For the Ether Theory:

$$\lambda = \lambda_o \{ \begin{array}{cc} (1 - u/c) & \text{for moving source} \\ 1/\,(1 + u/c) & \text{for moving observer} \end{array}$$

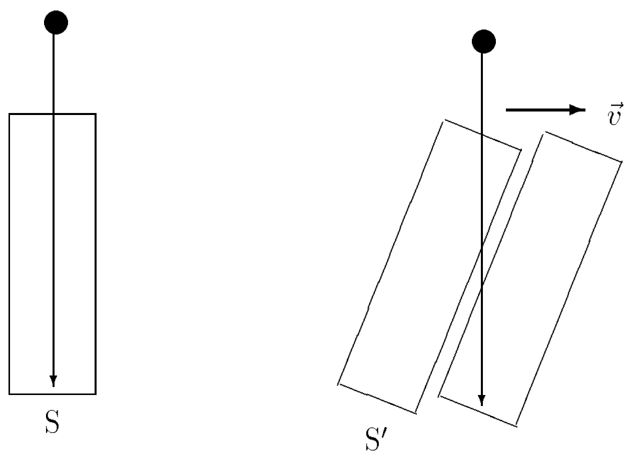
Where u is positive for approach.

The Special Relativity result is the geometric mean of these:

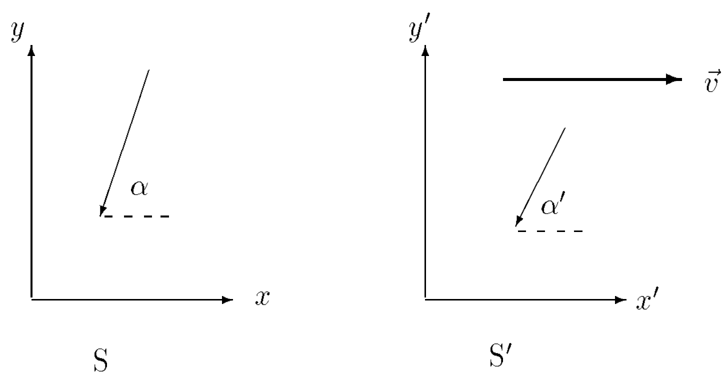
$$\lambda = \lambda_0 \sqrt{\frac{1 - u/c}{1 + u/c}} \tag{55}$$

4.2.6 Aberration of Starlight

First consider light coming from a star perpendicular to the direction of motion of the telescope.



Then consider more general directions.



$$\begin{array}{ll} u_x = -c cos\alpha & u'_x = -c cos\alpha' \\ u_y = -c sin\alpha & u'_y = -c sin\alpha' \end{array}$$

Now apply Einstein velocity addition:

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}$$

$$cos\alpha' = \frac{cos\alpha + v/c}{1 + \frac{v}{c} cos\alpha}$$

$$v'_y = u_y \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$$

$$sin\alpha' = sin\alpha \frac{\sqrt{1 - v^2/c^2}}{1 + \frac{v}{c} cos\alpha}$$

It is easy to check that $sin^2\alpha' + cos^2\alpha' = 1$.

In the simple case $\alpha = \pi/2$ so $cos\alpha = 0$,

$$cos\alpha' = \frac{v}{c}$$

which is the Bradley result.

In the general case, use the trigometric identity

$$tan\frac{\theta}{2} = \frac{sin\theta}{1 + cos\theta}$$

$$tan\frac{\alpha'}{2} = \frac{sin\alpha'}{1 + cos\alpha'} = sin\alpha\frac{\sqrt{1 - v^2/c^2}}{\left(1 + \frac{v}{c}cos\alpha\right)\left[1 + \frac{cos\alpha + v/c}{1 + (v/c)cos\alpha}\right]}$$

$$tan\frac{\alpha'}{2} = \sqrt{\frac{1 - v/c}{1 + v/c}}tan\frac{\alpha}{2} \tag{56}$$

For outgoing rays, $c \rightarrow -c$.

5 Einstein's Special Relativity

It is straight-forward to show that from Einstein's postulates one also obtains the Lorentz transformations.

Two Postulates

1. No physical experiment (without reference to outside) can determine the absolute speed of the frame of reference.
2. The speed of light is independent of the speed of the source (or observer).

Consider an expanding sphere of light

$$c^2t^2 - x^2 + y^2 + z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

viewed by two inertial frames of reference (S and S′) by observers O and O′ respectively with origins coinciding – at $t = t'^2 = 0$, $x = x'^2 = 0$, $y = y'^2 = 0$, $z = z'^2 = 0$.

By simple argument one can see that lengths transverse to the direction of motion must be unchanged only x and t will be modified. One argument is the one made before about considering two identical cylinders aligned with each other and their axes parallel to the direction of motion $\vec{v}$. If the dimension perpendicular to the direction of motion changes, then one cylinder will grow or shrink relative to the other and could pass through the other. If one then switches to the other frame, the opposite should happen or one can determine the absolute velocity. One would be able to tell which one went inside and which outside.

Or consider the earlier discussion of two meter sticks aligned perpendicular to the direction of motion. When the two meter sticks pass by each other one can use them to measure each other and tell which is longer and thus establish the absolute velocity.

Thus by symmetry and logic using postulate (1) we have

$$y = y', \qquad z = z'$$

so that the equation of expanding light sphere reduces to

$$c^2t^2 = c^2t'^2 - x'^2$$

If we accept the second postulate and assume coordinate transformations are linear and homogeneous we have

$$x' = Ax + Bt$$

$$t' = Cx + Dt$$

Now consider special cases:
(1) O′ origin has $x' = 0$, which implies $x = -\frac{B}{A}t$. Since velocity of O′ relative to O is v, so that $v = -\frac{B}{A}$ which yields $B = -Av$.
(2) The origin of O has $x = 0$, which gives $x' = Bt$, $t' = Dt$ implying $x' = \frac{B}{D}t$ or $B = -Dv$.

Combining (1) and (2) yields $D = A$. The linear transformation simplifies to

$$x' = A(x - vt)$$

$$t' = Cx + At$$

(3) Putting this back into the expanding light sphere formula

$$c^2t^2 = c^2t'^2 - x'^2 c^2 [Cx + At]^2 - [A(x - vt)]^2$$

$$= c^2C^2x^2 + 2c^2CAxt + c^2A^2t^2 - A^2x^2 + 2A^2vx - A^2c^2t^2$$

$$= A^2\left(1 - v^2/c^2\right)c^2t^2 + 2c^2A\left(C + \frac{v}{c^2}A\right)xt - \left(A^2 - c^2C^2\right)x^2$$

We can conclude that $A^2(1 - v^2/c^2) = 1$ or $A = 1/\sqrt{1 - v^2/c^2}$ and $A^2 - c^2C^2 = 0$ so that

$$C = -vA/c^2 = -\frac{v}{c^2}\frac{1}{\sqrt{1 - v^2/c^2}}$$

and thus

$$A^2 - c^2C^2 = \frac{1}{1 - v^2/c^2} - \frac{v^2}{c^2}\frac{1}{1 - v^2/c^2} = 1$$

This gives us the Lorentz transformation

$$\begin{array}{ll} t' = \gamma(t - vx/c^2) & t = \gamma(t' + vx'/c^2) \\ x' = \gamma(x - vt) & x = \gamma(x' + vt') \\ y' = y & y = y' \\ z' = z & z = z' \end{array} \tag{57}$$

Thus we have the identical Lorentz transformations from simple logical deduction. We can construct the full theory of Special Relativity by using these postulates and a series of thought ("gendanken") experimental. This approach is quite elegant and intellectually pleasing and makes a very nice and tight exposition and thus coherent little books. However, here we are emphasizing both the experimental basis and applications and the importance of understanding relativity from more than one point of view.

In the next section we rederive the Lorentz transformations using the Minkowski geometrical view and the Poincare relativity principle (Einstein's postulate (1) but with a wider implication).

6 Minkowski Space-Time

The Minkowski (1908-1909) geometrical interpretation of Special Relativity is quite a technically powerful approach. The primary step is to assume that our world is described by a 3+1 dimension space-time continuum. There are four dimensions and space is Euclidean but the addition of time to be the fourth dimension makes space pseudo-Euclidean because the metric which defines distance has a different sign between time and space: There are two possible signatures for the signs: $\pm, \mp, \mp, \mp$ yielding the two possible metric equations:
The proper time convention:

$$(cd\tau)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{58}$$

The proper distance convention:

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \tag{59}$$

For most of this course and notes I use the proper time (first) convention since it has a positive value for the physical objects we consider.

Note that such a space is intrinsically different from a 4-D space with signature $+, +, +, +$. It is conceptually confusing to smooth this over by replacing ct by ict or just $i\tau = x_4$. Even if this is done for the stated reason that most people know the sine and cosine better than sinh and cosh.

6.1 Comments on 4-D Geometry for S.R.

The Minkowski metric and 4-D geometry makes quite an impact on how one can approach problems in Special Relativity.
<u>Importance</u>
1) Assists in developing the needed space-time intuitions
2) Avoids always singling out a particular axis ($x \parallel v_{\text{relative}}$).
3) 4-D language is suggestive and seldom misleading. e.g. ict is avoided! and it is more likely to account for all coordinates in appropriate frame.

4) 4-D vectors and invariants are powerful tools.
5) Is an essential approach of geometrical General Relativity.

With 4 axes one needs 4 numbers to specify an "event" in space-time. But directions are not equivalent. A meter stick can be rotated to measure y or z instead of x, but it cannot be rotated into a clock.

6.2 Invariant Interval

The (Minkowski) geometry of space-time is constructed so that the interval: $dx^2 + dy^2 + dz^2 - c^2dt^2$ is **invariant** under a Lorentz transformation. And the signature is invariant under all real transformations of coordinates.

In more general form the signature is written as a bilinear transformation or a matrix:

$$(ds)^2 = \sum_{\mu\nu} \eta_{\mu\nu} (dx_\mu)(dx_\nu) \tag{60}$$

where the Minkowski metric term $\eta_{\mu\nu}$ can be expressly written as

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{61}$$

Because the determinant of the signature is not equal to one but is -1, there are three different kinds of intervals:
(1) Space-like: Two space-time events separated such that

$$\Delta x^2 + \Delta y^2 + \Delta z^2 > c^2 \Delta t^2$$

$$\Delta s^2 > 0 \qquad \Delta\tau^2 < 0 \tag{62}$$

One can always find a Lorentz transformation to proper coordinates in which $\Delta t^2 = 0$. That means that one can find an inertial coordinate system in which two events which have a space-like interval happen simultaneously.
(2) Time-like: In this case events are separated such that $\Delta\tau^2 > 0$ (or $\Delta s^2 < 0$) because $c^2\Delta t^2 > \Delta x^2 + \Delta y^2 + \Delta z^2$. One can always find a Lorentz transformation to proper coordinates in which $\Delta x^2 + \Delta y^2 + \Delta z^2 = 0$
(3) Singular: In this case events separated such that $\Delta s^2 = \Delta\tau^2 = 0$ These events lie on the light cone, such as a light ray in vacuum.

The result is that space-like intervals can always be measured with a meter stick and time-like with a clock.

6.3 What leaves Δs^2 invariant?

(1) Moving origin in space. (translation in space) E.g.

$$x' = x + x_o$$

$$
\begin{aligned}
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
\tag{63}
$$

(2) Re-setting zero time (translation in time) x, y, z remain the same and $t' = t + t_o$.
(3) Rotation of spatial axes, E.g.

$$
\begin{aligned}
x' &= x cos\theta + y sin\theta \\
y' &= -x sin\theta + y cos\theta \\
z' &= z \\
t' &= t
\end{aligned}
\tag{64}
$$

(4) Lorentz Transformation:

$$
\begin{aligned}
x' &= \gamma \left(x - vt\right) \\
y' &= y \\
z' &= z \\
t' &= \gamma \left(t - xv/c^2\right)
\end{aligned}
\tag{65}
$$

Which is equivalent to

$$
\begin{aligned}
x' &= x cosh(\phi) + ct sinh(\phi) \\
y' &= y \\
z' &= z \\
ct' &= -x sinh(\phi) + ct cosh(\phi)
\end{aligned}
\tag{66}
$$

where $cosh(\phi) = \gamma \equiv 1/\sqrt{1 - v^2/c^2}$.

This is a Lorentz rotation of axes. It can be considered an imaginary rotation in the $x - t$ plane. Remember the hyperbolic trigonometry identity/definition.

$$cosh^2(\phi) - sinh^2(\phi) = 1$$

Consider $cosh(\phi) = cosh(i\phi), \quad isinh(\phi) = sin(i\phi)$ which gives

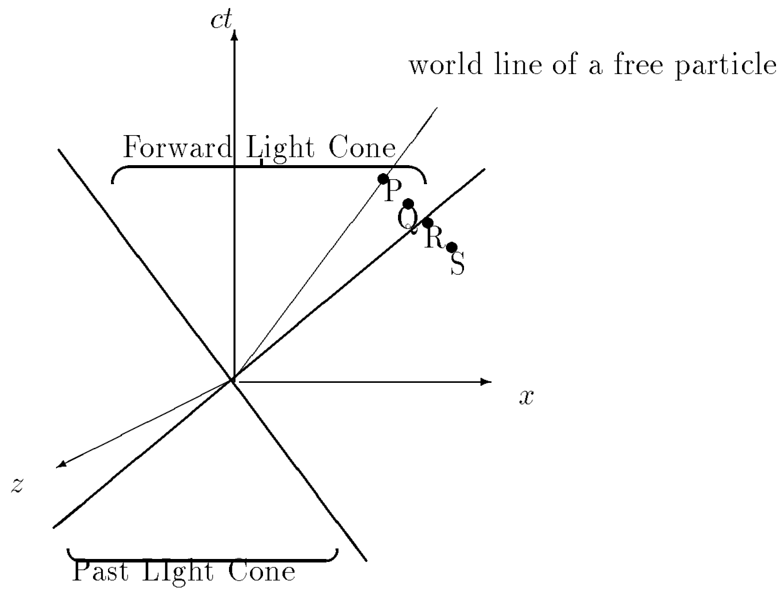
$$
\begin{aligned}
x' &= x cos(i\phi) + ict sinh(\phi) \\
y' &= y \\
z' &= z \\
ict' &= -x sin(i\phi) + ict cos(i\phi)
\end{aligned}
\tag{67}
$$

Consider a space-like interval

$$|dx| > |cdt|$$

Then $dx' = \gamma \left(dx - vdt\right), \qquad cdt' = \gamma \left(cdt - \frac{v}{c} dx\right)$ and one can always find a value of v/c with $|v/c| < 1$ for which $cdt' = 0$. Its magnitude is $|v/c| = |cdt/dx| < 1$. A similar argument works for time-like intervals.

In the following picture, OQ is time-like, OS is space-like, and OR is singular.



By a Lorentz transformation we may:
(1) Move Q to the t' axis.
or
(2) Move S to the x' axis.

But R will always be on a line of slope 1 in any S$'$

Q may be on the particle's world line, then we may find a frame in which x' stays zero, which is called the rest frame of the particle. In this frame clocks at rest measure the particle's proper time, $d\tau$. In other frames $dt = d\tau/\sqrt{1 - v^2/c^2} = \gamma\ d\tau$. $\overline{OQ}$ may be a meter stick. One can find S$'$ so that it lies on the x' axis and $x' = 0$.

Consider particles to be pieces of the stick. They are laid out in S$'$ to measure proper length λ. For other frames, $\ell' = \lambda\sqrt{1 - v^2/c^2} = \lambda/\gamma$. with the x direction of v and stick on the x axis.

For particles not free, that is with forces on them, we have the instantaneous rest frame.

6.4 Derivation of Lorentz Transformations

The Lorentz transformations result automatically from the metric and the assumption that in all inertial systems proper distances or times are invariants. That is any observer in any inertial system will calculate the same proper distance between two space-time events.

We start with two postulates:
(1) **Poincare' Relativity:** The Laws of Physics are the same in all inertial frames.
(2) **Minkowski Geometry/Metric** Space-time is a continuum in 3+1 dimensions

with metric

$$(cd\tau)^2 = -(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

where τ is the proper time and s is the proper distance. Proper time is invariant for all inertial systems.

Immediately we get time dilation

$$(cd\tau)^2 = (dt)^2 \left[c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 \right]$$

$$(d\tau)^2 = (dt)^2 \left[1 - \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2} \right] = (dt)^2 \left[1 - \frac{v^2}{c^2} \right]$$

$$d\tau = dt\,\sqrt{1 - v^2/c^2}$$

If proper time is invariant,the we can show Lorentz transformation is linear.

$$\begin{aligned}(c\Delta\tau)^2 &= (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2\end{aligned}$$

the second equation defines a numbering system for coordinates. But this same sum in the primed coordinate system must give the same proper time.

$$= (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$

The conversion from one coordinate system to another

$$dx'_\alpha = \sum_\beta \frac{\partial x'_\alpha}{\partial x_\beta} dx_\beta \equiv \frac{\partial x'_\alpha}{\partial x_\beta} dx_\beta$$

where the second right hand side defines the Einstein summation convention that a repeated index (in this case β) mean summation on that index. The Greek symbol index sums over four (4-D) going 0, 1, 2, 3 and Roman letters sum over three spatial coordinates going 1, 2, 3.

$$\begin{aligned}c^2 d\tau^2 &= \sum_\alpha dx_\alpha^2 = \sum_\beta \sum_\alpha \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial x'_\alpha}{\partial x_\delta} dx_\beta dx_\delta \\ &= \sum_\alpha dx_\alpha^2 = \sum_\beta \sum_\alpha \delta_{\beta\alpha} dx_\beta dx_\delta\end{aligned}$$

Therefore

$$\frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial x'_\alpha}{\partial x_\delta} = \delta_{\beta\alpha}$$

implying if one takes the derivative: $\frac{\partial}{\partial x_\epsilon}$ one finds

$$\frac{\partial^2 x'_\alpha}{\partial x_\beta \partial x_\epsilon} \frac{\partial x'_\alpha}{\partial x_\delta} + \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial^2 x'_\alpha}{\partial x_\delta \partial x_\epsilon} = 0$$

Now one can then shift through the indices: $\epsilon \rightarrow \beta \rightarrow \delta \rightarrow \epsilon$ and get generically

$$\frac{\partial^2 x'}{\partial x \partial x}\frac{\partial x'}{\partial x} = 0$$

and the determinant of $\partial x'/\partial x = \pm 1$ which implies

$$\frac{\partial^2 x'}{\partial x \partial x} = 0$$

and

$$x'_\alpha = A_\alpha + \sum_\beta A_{\alpha\beta} x_\beta$$

Which shows that the coordinate (Lorentz transformation) must be linear to preserve invariant the proper distance and time. Thus

$$dx'_\alpha = \sum_\beta A_{\alpha\beta} x_\beta$$

and

$$\sum_\alpha A_{\alpha\beta} A_{\alpha\delta} = \delta_{\beta\delta}$$

The solution to these equations is

$$A = \begin{bmatrix} cosh\psi & -sinh\psi & 0 & 0 \\ -sinh\psi & cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or equivalently

$$[ct', x', y', z',] = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

7 Lorentz Transformations

The Lorentz transformations may be obtained in one of several ways which includes (1) fitting to experimental observations, (2) using the two postulates of special relativity, or (3) assuming Minkowski (4-dimensional) space and finding what are the transformations that leave 4-D vectors lengths invariant.

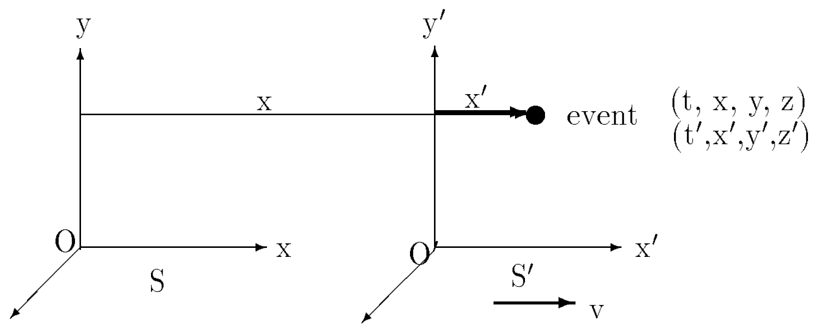
History: Lorentz transformation was derived by Lorentz before Einstein's work. Lorentz obtained them by considering invariance of Maxwell's Equations and the Michelson and Morely experimental results.

Significance: The Lorentz Transformations define the mathematical specification required to discuss a kinematic occurrence – a sequence of space-time events.

7.1 Experimental Lorentz Transformation Derivation

The Lorentz transformations are demanded and supported by experimental observations. The Lorentz transformation equations can readily be derived from length contraction and time dilation after taking a short detour to discuss clock synchronization.

Consider two frames of reference: S, the laboratory frame and S′ a frame of reference moving with velocity $\vec{v}$ in the $\hat{x}$ direction as shown in the following figure.



Arrange things so that at $t = 0$ and $t' = 0$ that the two origins O and O′ coincide.

Consider each reference system to be an actual lattice of meter sticks and clocks, e.g. each reference system is filled with these space and time measuring devices at every point.

Get clocks in S to agree. Identical clocks set by sending out a light pulse from origin O and also from the midpoint between any two clocks. Check times, reflect back to midpoint, if pulses arrive together, then clocks agree.

System S′ does the same with his clocks.

We have for the space-time event in the figure above

$$x = vt + x' \times \sqrt{1 - v^2/c^2} \tag{68}$$

where the second term takes into account length contraction of a moving frame. We can use our arguments about the transverse directions to show that they are unchanged and then have the spatial Lorentz transformations:

$$\begin{aligned} x' &= \frac{1}{\sqrt{1-v^2/c^2}}(x-vt) \\ y' &= y \\ z' &= z \\ t' &= t\sqrt{1-v^2/c^2} + \text{synchronization effect} \end{aligned} \tag{69}$$

7.1.1 Synchronizing Clocks in Moving Frame

Our approach to get our system of reference made of a grid of meter sticks and synchronized clocks requires that we synchronized the clocks. An approach to synchronizing the clocks is: bring the clock together, match their readings, then move into place. Move them slowly and gently so as not to disturb their operation.

Consider the simple case of two clocks brought together at the origin of the moving system S′. When they are together, from the laboratory frame S both clocks read same time and are going slow by a factor $\sqrt{1-(v/c)^2}$ as a result of **time dilation**. Now very slowly and gently move one clock back (in negative x'-direction; toward the laboratory system origin) a distance ℓ in elapsed time $\ell = \delta v\ \tau$. The clock at the origin has its rate slow by $\sqrt{1-v^2/c^2}$ relative to the laboratory frame. Clock moving back in negative x'-direction has its rate slowed by the factor $\sqrt{1-(v-\delta v)^2/c^2}$

$$f_A = \sqrt{1-v^2/c^2}\,f_o \qquad f_B = \sqrt{1-(v-\delta v)^2/c^2}\,f_o \tag{70}$$

The difference in the clocks' rates is

$$\begin{aligned} f_A - f_B &= f_o\left[\sqrt{1-v^2/c^2} - \sqrt{1-(v-\delta v)^2/c^2}\right] \\ &= \frac{f_o}{\sqrt{1-v^2/c^2}}\left[1-\frac{v^2}{c^2} - \left((1-\frac{v^2}{c^2})(1-\frac{(v-\delta v)^2}{c^2}\right)^{1/2}\right] \\ &= \frac{f_o}{\sqrt{1-v^2/c^2}}\left[1-\frac{v^2}{c^2} - \left((1-\frac{v^2}{c^2})(1-\frac{v^2}{c^2}+\frac{2\delta v v}{c^2}-\frac{\delta v^2}{c^2})\right)^{1/2}\right] \\ &= -f_a\frac{\delta v\ v/c^2}{1-v^2/c^2} = -f_o\frac{\delta v\ v/c^2}{\sqrt{1-v^2/c^2}} \end{aligned} \tag{71}$$

If it takes a time $\tau = \ell_o/\delta v$ to separate the clocks, the time difference between them is

$$\Delta t = \frac{f_A - f_B}{f_A}\tau = \left(\sqrt{1-v^2/c^2} - \sqrt{1-(v-\delta v)^2/c^2}\right)\tau$$

$$= \quad -\frac{\delta v \ v/c^2}{\sqrt{1-v^2/c^2}} \times \frac{\ell_o}{\delta v} = -\frac{\ell_0 \ v/c^2}{\sqrt{1-v^2/c^2}} = -\ell' v/c^2 \tag{72}$$

Note that the speed with which the clock moves drops out and the change in reading is proportional only to the distance displaced and the velocity of the moving system.

Clocks get out of synchronization (phase) by an amount proportional to their separation ℓ_o and v. If brought back together, the clocks will go into synchronization. **The clock that is farther behind in space is further ahead in time.**

Note that in the frame S′ the difference in rate of time kept between the clock at the origin and the one being moved back to its place is second order in v/c rather than first order:

$$f'_B = \frac{f'_A}{\sqrt{1-(\delta v)^2/c^2}} \simeq f'_A \times \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

So that by moving with a very, very slow velocity the integrated effect in the S′ frame can be made arbitrarily small while the effect as observed in the S frame is always $-\ell' v/c^2$ independent of δv. That is because the effect in frame S is first order in $\delta v/c$ and integrated over time equals the displacement.

The final Lorentz transformations are:

$$\begin{aligned} t' &= t\sqrt{1-v^2/c^2} - \frac{x'v}{c^2} = t\sqrt{1-v^2/c^2} - \frac{v^2}{c^2}\frac{1}{\sqrt{1-v^2/c^2}}(x-vt) \\ &= \frac{1}{\sqrt{1-v^2/c^2}}\left[t - \frac{vx}{c^2}\right] = \gamma\left[t - \frac{vx}{c^2}\right] \end{aligned} \tag{73}$$

Notice for $v << c$ get Galilean transforms and there is also a symmetry between the transformation equations.

$$\begin{array}{cc} t' = \gamma(t - vx/c^2) & t = \gamma(t' + vx'/c^2) \\ x' = \gamma(x - vt) & x = \gamma(x' + vt') \\ y' = y & y = y' \\ z' = z & z = z' \end{array} \tag{74}$$

7.2 Postulate Lorentz Transformation Derivation

We want to consider the transformation from one inertial coordinate frame S with coordinates (ct, x, y, z) more generally (x_0, x_1, x_2, x_3) to another inertial coordinate frame S' with coordinates (ct', x', y', z') more generally (x'_0, x'_1, x'_2, x'_3)

First we establish that the transformation must be linear. This can be shown many ways - for example, from Newton's first law and the idea of temporal and spatial homogeneity: An ideal standard clock is one that runs at a constant rate, e.g. ticking off a second at a fixed interval independent of the absolute time. We would like for its rate not to depend its position in space or time as an indication of spatial and temporal homogeneity. Consider a standard clock C moving through frame S, its motion being given by $x_i = x_i(t)$, where x_i $(i = 1, 2, 3)$ stand for the three spatial coordinates (x, y, z). Then $dx_i/dt = \text{constant}$. If τ is the time indicated by the clock C itself, homogeneity requires the constancy of $dt/d\tau$. Equal outcomes here and there, now and later, of the experiment that consists of timing the ticks of a standard clock moving at constant speed.

Together these results imply $dx_\mu/d\tau = \text{constant}$ and thus $d^2x_\mu/d\tau^2 = 0$, where we have written x_μ $(\mu = 0, 1, 2, 3)$ for $ct, x, y, z)$. In frame S' the same argument yields $d^2x'_\mu/d\tau^2 = 0$. We also have

$$\frac{dx'_\mu}{d\tau} = \sum_\nu \frac{\partial x'_\mu}{\partial x_\nu}\frac{dx_\nu}{d\tau}, \qquad \frac{d^2x_\mu}{d\tau^2} = \sum_\nu \frac{\partial x'_\mu}{\partial x_\nu}\frac{d^2x_\nu}{d\tau^2} + \sum_\nu\sum_\sigma \frac{\partial^2 x'_\mu}{\partial x_\nu \partial x_\sigma}\frac{dx_\nu}{d\tau}\frac{dx_\sigma}{d\tau}. \tag{75}$$

Thus for any free motion of such a clock the last term in the equation must vanish. This can only happen if $\partial^2x'_\mu/\partial x_\nu\partial x_\sigma = 0$: that is, if the transformation is linear.

An immediate consequence of linearity is that all the defining particles (that is, those at rest in the lattice) of any inertial S' move with identical, constant velocity through any other inertial frame S. Suppose that the coordinates of S and S' are related by

$$x_\mu = \left(\sum_\nu A_{\mu\nu}x'_\nu\right) + B_\mu \tag{76}$$

Then setting $x'_i = \text{constant}$ $(i = 1, 2, 3)$ for a particle fixed in S', we get $dt = A_{00}dt'$, $dx_j = A_{j0}dt'$, and thus $dx_j/dt = A_{i0}/A_{00} = \text{constant}$, as asserted. The defining particles of S' thus constitute, as judged in S, a rigid lattice whose motion is fully determined by the velocity of any one of its particles.

Another consequence of linearity (plus symmetry) is that the standard coordinates in two arbitrary inertial frames S and S' can always be chosen so as to be in standard configuration with respect to each other.

It is always possible to chose the line of motion of the spatial origin of S' as the x-axis of S, and to choose the zero points of time in S and S' so that they two origin clocks both read zero when they pass each other. Any two orthogonal planes intersecting along the x-axis can serve as the coordinate planes $y = 0$ and $z = 0$ of S. The two planes, fixed in S, plus the moving plane $x = vt$ (v being the velocity ofS' relative to S) correspond to plane sets of particles fixed in S'. Moreover, the planes $y = 0$ and $z = 0$ must also be regarded as orthogonal in S', otherwise the isotropy of S (in particular, its axial symmetry about the x-axis) would be violated. So we can takes these planes as the coordinate planes $y' = 0$ and $z' = 0$, respectively of S', otherwise the projection of that axis onto that plane would violate the isotropy of S. Hence we can take $x = vt$ as $x' = 0$. In what follows, we assume S and S' to be in standard configuration.

The Relativity Postulate implies that the transformation between any pair of inertial frames in standard configuration, with the same v, must be the same. Suppose we reverse the x- and z- axes of both S and S', by symmetry and reciprocity, this operation produces an identical pair of inertial frames with the roles of the 'first' and 'second' interchanged. So if we then interchange primed and unprimed coordinates, the transformation equations must be unchanged. In other words, the transformation must be invariant under what we shall call an xz reversal:

$$x \leftrightarrow -x', \quad y \leftrightarrow y', \quad z \leftrightarrow -z', \quad t \leftrightarrow t' \tag{77}$$

The same is true for an xy reversal.

Now by linearity, $y' = Ax + By + Cz + Dt + E$, where the coefficients are constants, with some dependence on v. Since, by our choice of coordinates, $y = 0$, must entail $y' = 0$, we have $y' = By$. Applying an xz reversal yields $y = By'$ and so $B = \pm 1$. However when $v \to 0$ one must continuously go to the identity transformation and to $y' = y$ and thus the only choice is $B = 1$. The argument for z and z' is similar, and we arrive at the 'trivial' members of the transformation:

$$y' = y, \qquad x' = z \tag{78}$$

just as in the Galilean/Newtonian case and for the same reasons.

Next suppose linearity so $x' = \gamma x + Fy + Gz + Ht + J$, where for tradition we have used γ as the coefficient for x. By our choice of coordinates, $x = vt$ must imply $x' = 0$, so that $\gamma v + H,\ F,\ G,\ J$ all vanish and

$$x' = \gamma\,(x - vt) = \gamma\,(x - \beta ct) \tag{79}$$

An xz reversal then yields

$$x = \gamma\,(x' + vt') = \gamma\,(x' + \beta ct') \tag{80}$$

At this stage Newton's axiom $t' = t$ would lead to $\gamma = 1$ and $x' = x - vt$; that is, to the Galilean transform. Instead we make the real use of Einstein's law of light propagation - the second postulate of Special Relativity. According to it, $x = ct$ and $x' = ct'$. Both frames S and S' have the speed of light as c and the two equations are simply descriptions of the same light signal in each reference frame. Substituting these expressions back into the transformation equations for x' and x we find the relations $ct' = \gamma t(c - v)$ and $ct = \gamma t'(c + v)$, whose product divided by tt', yields

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{81}$$

since $v \to 0$ must lead to $x' = x$ continuously, we must chose the positive root. This is the famous 'Lorentz factor' gamma (γ), which plays such an important role in Special Relativity. Previously it was found to satisfy experiment. Here it appears to satisfy the second postulate of the speed light being the same **finite** value in all inertial frames and in the context of the relativity postulate.

The elimination of the cross reference frame coordinate, e.g. x', finally leads to the most revolutionary of the four equations

$$t' = \gamma\left(t - vx/c^2\right) \tag{82}$$

Since in the above derivation we have used Einstein's light propagation postulate only on the x-axis, must still check whether the transformations respects it generality. First, the linearity of the transformation implies that any uniformly moving point transforms into an uniformly moving point. This, incidently recovers the invariance of Newton's first law, but, of course, it also applies to light signals. Next, one easily derives from the transformations the enormously important fundamental identity

$$c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2dt - dx^2 - dy^2 - dz^2 \tag{83}$$

The distance dr between neighboring points in a Euclidean frame S is given by the 'Euclidean' metric

$$cr^2 = dx^2 + dy^2 + dz^2 \tag{84}$$

From the identity we have $dr^2 = c^2dt^2$, which is characteristic of any effect traveling at the speed of light, implies that $dr'^2 = c^2dt'^2$ and vice versa. So the Euclidicity of the metric and the invariance of the speed of light are jointly respected by the Lorentz transform.

7.3 Minkowski SpaceTime Derivation

The (Minkowski) geometry of space-time is constructed so that the interval: $dx^2 + dy^2 + dz^2 - c^2dt^2$ is **invariant** under a Lorentz transformation. And the signature is invariant under all real transformations of coordinates.

In more general form the signature is written as a bilinear transformation or a matrix:

$$(ds)^2 = \sum_{\mu\nu} \eta_{\mu\nu} \left(dx_\mu\right)\left(dx_\nu\right) \tag{85}$$

where the Minkowski metric term $\eta_{\mu\nu}$ can be expressly written as

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{86}$$

for Euclidean (pseudo-Euclidean because of sign difference) coordinates.

We consider that inertial coordinate systems are those for which the four-dimensional length defined by the metric is invariant. Immediately, this gives us time dilation:

$$\begin{aligned} (cd\tau)^2 = -(ds)^2 &= (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \\ &= (dt)^2 \left[c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right] \\ &= (cdt)^2 \left[1 - \frac{v^2}{c^2}\right] \end{aligned}$$

$$d\tau \quad = \quad dt\,\sqrt{1 - v^2/c^2} \tag{87}$$

Thus we see immediately that the rest frame elapsed (rate) time τ will be dilated by the factor $\sqrt{1 - v^2/c^2}$.

If proper time is invariant, then we can show Lorentz transformation is linear. The conversion from one coordinate system to another

$$dx'_\alpha = \sum_\beta \frac{\partial x'_\alpha}{\partial x_\beta} dx_\beta \equiv \frac{\partial x'_\alpha}{\partial x_\beta} dx_\beta$$

where the second right hand side defines the Einstein summation convention that a repeated index (in this case β) mean summation on that index. The Greek symbol index sums over four (4-D) going 0, 1, 2, 3 and Roman letters sum over three spatial coordinates going 1, 2, 3.

$$\begin{aligned} c^2 d\tau^2 &= \sum_\alpha (dx'_\alpha)^2 = \sum_\alpha \sum_\beta \sum_\delta \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial x'_\alpha}{\partial x_\delta} dx_\beta dx_\delta \\ &= \sum_\alpha dx_\alpha^2 = \sum_\alpha \sum_\beta \sum_\delta \delta_{\beta\alpha} \delta_{\delta\alpha} dx_\beta dx_\delta \end{aligned} \tag{88}$$

Therefore

$$\sum_\alpha \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial x'_\alpha}{\partial x_\delta} = \sum_\alpha \delta_{\beta\alpha} \delta_{\delta\alpha} \tag{89}$$

implying if one takes the derivative: $\frac{\partial}{\partial x_\epsilon}$ one finds

$$\frac{\partial^2 x'_\alpha}{\partial x_\beta \partial x_\epsilon} \frac{\partial x'_\alpha}{\partial x_\delta} + \frac{\partial x'_\alpha}{\partial x_\beta} \frac{\partial^2 x'_\alpha}{\partial x_\delta \partial x_\epsilon} = 0$$

Now one can then shift through the indices: $\epsilon \to \beta \to \delta \to \epsilon$ and get generically

$$\frac{\partial^2 x'}{\partial x \partial x} \frac{\partial x'}{\partial x} = 0$$

and the determinant of $\partial x'/\partial x = \pm 1$ which implies

$$\frac{\partial^2 x'}{\partial x \partial x} = 0$$

and

$$x'_\alpha = A_\alpha + \sum_\beta A_{\alpha\beta} x_\beta$$

Which shows that the coordinate (Lorentz transformation) must be linear to preserve invariant the proper distance and time. Thus

$$dx'_\alpha = \sum_\beta A_{\alpha\beta} dx_\beta$$

and

$$\sum_\alpha A_{\alpha\beta}A_{\alpha\delta} = \delta_{\beta\delta}$$

The solution to these equations is

$$A = \begin{bmatrix} cosh\psi & -sinh\psi & 0 & 0 \\ -sinh\psi & cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or equivalently

$$[ct', x', y', z',] = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

The Lorentz transformation can be derived as the transformations that are velocity boosts from one inertial frame to another.

7.3.1 *ict* Derivation of L.T.

In this subsection we derive the Lorentz transformation the way that undergraduates were often introduced to the subject and this shows an alternate coordinate definition. First consider four dimensional Euclidean space with coordinates (x_1, x_2, x_3, x_4) which in Cartesian coordinates are x, y, z, ict). Then the length of a vector with one end on the origin and the other at point $P = (x_1, x_2, x_3, x_4)$ is given by the Pythorean (Euclidean/Cartesian) metric Define $x_4 = T \equiv ict$, then

$$\begin{aligned} d^2 &= \sum_{\alpha=1,4} x_\alpha^2 \\ &= x^2 + y^2 + z^2 + T^2 \\ &= x^2 + y^2 + z^2 + (ict)^2 \\ &= x^2 + y^2 + z^2 - (ct)^2 \end{aligned} \tag{90}$$

Thus the distance squared is just the same as before but with the opposite sign. If we want to consider all the transformations that leave the four-D distance invariant:

$$\begin{aligned} d^2 &= (d')^2 \\ x^2 + y^2 + z^2 - (ct)^2 &= (x')^2 + (y')^2 + (z')^2 - (ct')^2 \end{aligned} \tag{91}$$

The most general linear transformation for a Euclidean space (excluding translations) that leave vector lengths invariant is a simple rotation.

$$\begin{aligned} x' &= xcos\alpha + Tsin\alpha \\ T' &= -xsin\alpha + Tcos\alpha \end{aligned} \tag{92}$$

Consider the point(s) $x' = 0$. By definition of the relationship between the reference systems $x = vt = vT/(ic)$ for same point. The rotation angle is then

$$tan\alpha = i\frac{v}{c} \equiv i\beta \tag{93}$$

Thus α is an imaginary angle ($tan\alpha = iv/c \rightarrow \alpha^* = v/c$).

$$cos\alpha = 1/sec\alpha = 1/(1 + tan^2\alpha)^{1/2} = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma \tag{94}$$

$$sin\alpha = tan\alpha \times cos\alpha = i\frac{v}{c}\gamma = i\beta\gamma \tag{95}$$

Putting these into the rotation equations above

$$\begin{aligned} x' &= xcos\alpha + Tsin\alpha = \gamma x + Ti\beta\gamma = \gamma(x - vt) \\ T' &= -xsin\alpha + Tcos\alpha = -xi\beta\gamma + T\gamma = \gamma(T - i\beta x) \\ ct &= \gamma(ct - \beta x) \end{aligned} \tag{96}$$

7.4 General Lorentz transformations: the Lorentz group

One can explicitly verify that the Lorentz transformation

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \qquad & t' = \gamma(t - vx/c^2) \\ x' &= \gamma(x - \beta ct) \qquad & x' = \gamma(x - vt) \\ y' &= y & \\ z' &= z & \end{aligned} \tag{97}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$, leaves the space-time interval $ds = (cd\tau)$ invariant. By choosing space coordinates so that the relative velocity of two inertial frames is along the x direction, it follows that all Lorentz transformations leave the interval invariant.

We can explicitly write this out as a matrix equation:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \equiv \sum_\mu \sum_\nu \eta_{\mu\nu}dx^\mu dx^\nu \tag{98}$$

where a repeated index means a summation in the Einstein convention.

Specifically, in Cartesian coordinates

$$ds^2 = [cdt \quad dx \quad dy \quad dz] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{99}$$

The Lorentz transformation in matrix notation for Cartesian coordinates is

$$\Lambda_\alpha^\beta = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{100}$$

Since

$$\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{101}$$

it is clear that the Lorentz transformation leaves the interval $(ds')^2 = ds^2$ invariant.

Now we can define the Lorentz group as the group of matrix transformations that leave the interval ds invariant.

$$(ds')^2 = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma dx^\rho dx^\sigma \rightarrow = ds^2 \tag{102}$$

But ds^2 may be written as: $ds^2 = \eta_{\rho\sigma}dx^\rho dx^\sigma$ thus we may infer that

$$\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \tag{103}$$

Hence, any matrix Λ which leaves the metric invariant under the transformation represents a Lorentz transformation. These matrices form a group of transformations known as the Lorentz group. When combined with translation symmetry, $x^\mu \rightarrow (x')^\mu = x^\mu + \alpha^\mu$, with α^μ as the components of a constant 4-vector, it forms a larger group known as the Poincare group.

7.5 Lorentz Group

In Mathematics the Lorentz Group is the the group of all linear transformations of the vector space R^4 that leave the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2$ invariant. The Lorentz group is isomorphic to $O(1{,}3{,}R)$, a real form of the complex orthogonal group $O(4)$.

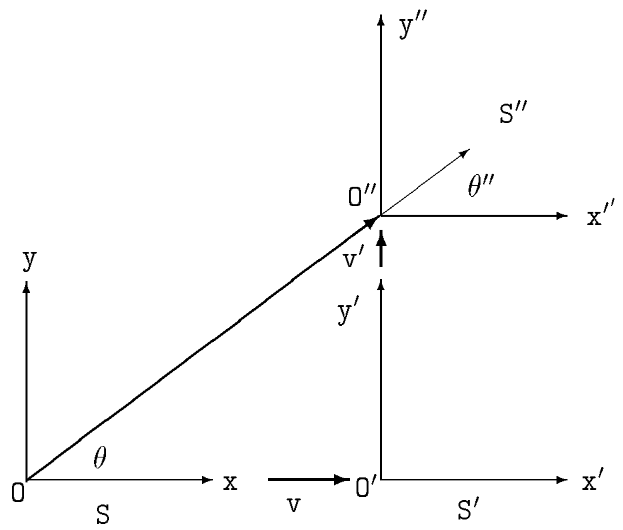
8 Thomas Precession

Thomas Precession is a kinematic effect discovered by L. T. Thomas in 1926 (L. T. Thomas *Phil. Mag.* **3**, 1 (1927)). It is fairly subtle and mathematically sophisticated but it has great importance in atomic physics in connection with spin-orbit interaction. Without including Thomas Precession, the rate of spin precession of an atomic electron is off by a factor of 2. Later we will see that there is a similar effect for gravitational fields.

The effect is connected with the fact that two successive Lorentz transformations in different directions are equivalent to a Lorentz transformation plus a three dimensional rotation. This rotation of the local frame of rest is the kinematic effect that causes the Thomas precession.

For the lecture we will not do the full mathematical treatment, since it is rather involved. Instead we will show by a simple example how the rotation and thus precession comes about.

Make two successive Lorentz transformations in orthogonal directions: from S to S$'$ with velocity v along the x axis, followed by a transformation from S$'$ to S$''$ with velocity v' along the y' axis, as shown by the following diagram.



The line from the origin O of S to the origin O'' of S$''$ making and angle θ in S and an angle θ'' in S$''$. We can calculate the angles in the two frames by applying the Lorentz transformations and evaluating them in each frame.

$$\begin{array}{cc}
x' = \gamma(x - vt) & x = \gamma(x' + vt') \\
t' = \gamma(t - vx/c^2) & t = \gamma(t' + vx'/c^2) \\
y' = y & y = y' \\
\\
y'' = \gamma'(y' - v't') & y' = \gamma'(y'' + vt'') \\
x'' = x' & x' = x'' \\
\text{where} & \\
\gamma = 1/\sqrt{1 - v^2/c^2} & \gamma' = 1/\sqrt{1 - (v'/c)^2}
\end{array} \tag{104}$$

Combing these equations one finds:

$$\begin{aligned}
y'' &= \gamma'[y - v'\gamma(x - vt)] \\
x'' &= \gamma(x - vt)
\end{aligned} \tag{105}$$

Now we can calculate the angle θ made by the line between origins. For a

Galilean transform one would have

$$tan\theta = \frac{y}{x} = \frac{v't}{vt} = \frac{v'}{v} \tag{106}$$

but Special Relativity shows us that 3-D velocities do not transform like 3-D vectors. So we must calculate carefully.

$$tan\theta = \frac{y}{x} = \frac{y'}{vt} = \frac{\gamma'(y'' + v't'')}{vt}|_{y''=0} = \frac{\gamma' v' t''}{vt} \tag{107}$$

$$t = \gamma(t' + vx'/c^2)|_{x''=x'=0} = \gamma\gamma'(t'' + v'y''/c^2)|_{y''=0} = \gamma\gamma' t'' \tag{108}$$

so that

$$tan\theta = \frac{\gamma' v' t''}{v\gamma\gamma' t''} = \frac{v'}{\gamma v} \tag{109}$$

Note that this answer is very near the Galilean result but with the factor of $1/\gamma$ which reminds us of aberration.

Now we calculate θ'':

$$tan\theta'' = \frac{y''}{x''} = \frac{\gamma'[y' - v't']}{x'} \tag{110}$$

where x'' and y'' are the coordinates of the origin O of system S in the S$''$ system. Thus

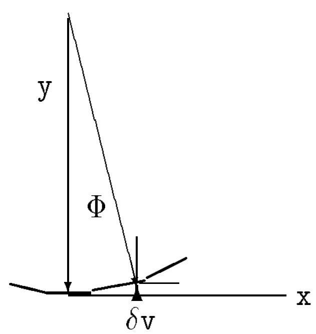
$$tan\theta'' = \frac{\gamma'[y - v']}{x'}|_{y=0} = -\frac{\gamma' v' t'}{x'} = -\frac{\gamma' v' t'}{\gamma(x - vt)}|_{x=0} = \frac{-\gamma' v' t'}{-\gamma v t} \tag{111}$$

$$t' = \gamma(t - vx/c^2)|_{x=0} = \gamma t; \tag{112}$$

$$tan\theta'' = \frac{\gamma' v'}{v} \tag{113}$$

This looks again similar to the Galilean angle except for the extra factor of γ'.

Now consider a particle on a curved path



At a certain time it is at the origin O of our system S. Put the x axis parallel to the path, and y axis toward the center of curvature. At $t = 0$, the rest frame S$'$ is moving in the x direction with velocity v. At a slightly later time its rest frame S$''$ is moving perpendicular to x' in the y direction with velocity $v' = \delta v$.

Define

$$\delta\theta = \theta'' - \theta = tan^{-1}\left(\frac{v'\gamma'}{v}\right) - tan^{-1}\left(\frac{v'}{\gamma v}\right) \tag{114}$$

For a very short time interval the motion is circular. That is fit the local curve with a tangent circle with appropriate radius of curvature.

$$\begin{array}{cc} v_x = \omega R cos\phi & v_y = \omega R sin\phi \\ v_x = v & v_y = \delta v = v' \end{array} \tag{115}$$

so

$$tan\phi = \frac{v'}{v}$$

$$\delta\theta = \theta'' - \theta = tan^{-1}(\gamma' tan\phi) - tan^{-1}\left(\frac{tan\phi}{\gamma}\right) \tag{116}$$

Choose ϕ to be very small;

$$\phi = \frac{\delta S}{R} = \frac{v\delta t}{R}$$

Then

$$\delta\theta \approx \frac{v\delta t}{R}\left(\gamma' - \frac{1}{\gamma}\right) \tag{117}$$

$$\omega_T = \frac{\delta\theta}{\delta t} \approx \frac{v}{R}\left(\gamma' - \frac{1}{\gamma}\right)$$

In a circle the acceleration is

$$a = \frac{v^2}{R} \qquad \text{so that} \qquad \frac{v}{R} = \frac{a}{v}$$

giving

$$\omega_T = \frac{a}{v}\left(\gamma' - \frac{1}{\gamma}\right)$$

Suppose we are in a non-relativistic region $v << c$, like an electron in an atom:

$$\gamma' - \frac{1}{\gamma} = \frac{1}{\sqrt{1-(v'/c)^2}} - \sqrt{1-(v/c)^2} \approx 1 + \frac{1}{2}(\frac{v'}{c})^2 - 1 + \frac{1}{2}(\frac{v}{c})^2 \sim \frac{1}{2}(\frac{v}{c})^2$$

since $tan\phi = v'/v << 1$. Putting this back into the expression for ω_T

$$\omega_T \approx \frac{a}{v}\frac{v^2}{2c^2} = \frac{va}{2c^2}$$

Thus $\theta'' > \theta$, thus a counter-clockwise rotation, implying

$$\vec{\omega}_T = \frac{\vec{v}\times\vec{a}}{2c^2} \tag{118}$$

The rigorous result is

$$\vec{\omega}_T = \frac{\gamma^2}{\gamma+1}\frac{\vec{v}\times\vec{a}}{2c^2} \tag{119}$$

9 Spin-Orbit Interaction of Electron with Nucleus in an Atom

Now we are set to apply this kinematic effect to spin precession in an atom. In its own rest frame the electron "sees" the nucleus flying by.

The electron's magnetic moment, $\vec{\mu}$, and spin angular momentum, $\vec{S}$, are related by

$$\vec{\mu} = \frac{e}{m_e c}\vec{S} \tag{120}$$

The torque on the magnetic moment is

$$\vec{\tau} = \frac{d\vec{S}}{dt} = \vec{\mu}\times\vec{B}' \tag{121}$$

where $\vec{B}'$ is the magnetic field in the e^- frame.

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v}_e}{c}\times\vec{E}\right) \tag{122}$$

Where $\vec{B}$ is the magnetic field and $\vec{E}$ is the electric field in the nucleus rest frame. $v/c << 1$ so that $\gamma \approx 1$,

$$\frac{d\vec{S}}{dt} = \vec{\mu}\times\left(\vec{B} - \frac{\vec{v}_e}{c}\times\vec{E}\right) \tag{123}$$

arises from the interaction energy

$$U' = -\vec{\mu}\cdot\left(\vec{B} - \frac{\vec{v}_e}{c}\times\vec{E}\right) \tag{124}$$

If $\vec{E}$ is due to a spherically symmetrical charge distribution – as for a one-electron atom or one outside a closed shell – then

$$e\vec{E} = -\vec{\nabla}V(r) - \frac{\vec{r}}{r}\frac{dV}{dr}. \tag{125}$$

Then

$$U' = -\frac{e}{m_e c}\vec{S}\cdot\vec{B} + \frac{e}{m_e c^2}\vec{S}\cdot\vec{v}\times\left(-\frac{\vec{r}}{r}\frac{dV}{dr}\right) \tag{126}$$

$$\vec{S}\cdot\vec{v}\times\vec{(}-r) = +\vec{S}\cdot\vec{f}\times\vec{v}$$

$$\begin{aligned} U' &= -\frac{e}{m_e c}\vec{S}\cdot\vec{B} + \frac{e}{m_e^2 c^2}\vec{S}\cdot(\vec{r}\times\vec{v})\frac{1}{r}\frac{dV}{dr} \\ &= -\frac{e}{m_e c}\vec{S}\cdot\vec{B} + \frac{e}{m_e c^2}\vec{S}\cdot\vec{L}\frac{1}{r}\frac{dV}{dr} \end{aligned} \tag{127}$$

since $m\vec{r}\times\vec{v} = \vec{L} \equiv$ angular momentum. This second term is the spin-orbit interaction.

Now, if the electron rest frame is **rotating** – Thomas angular velocity $\vec{\omega}$, $d\vec{S}/dt \neq \vec{\mu}\times\vec{B}'$. The general kinematic result from classical physics is:

$$\frac{\partial}{\partial t}|_{\text{rotation coordinates}} = \frac{\partial}{\partial t}|_{\text{inertial coordinates}} - \vec{\omega}\times \tag{128}$$

as an operator on <u>any</u> vector. So

$$\frac{\partial\vec{S}}{\partial t}|_{\text{rotation coordinates}} = \frac{\partial\vec{S}}{\partial t}|_{\text{inertial coordinates}} - \vec{\omega}\times\vec{S} \tag{129}$$

With this expression the interaction energy is changed to:

$$U = U' - \vec{S}\cdot\vec{\omega}_T \tag{130}$$

where $\vec{\omega}_T$ is proportional to the centripetal acceleration due to E_r.

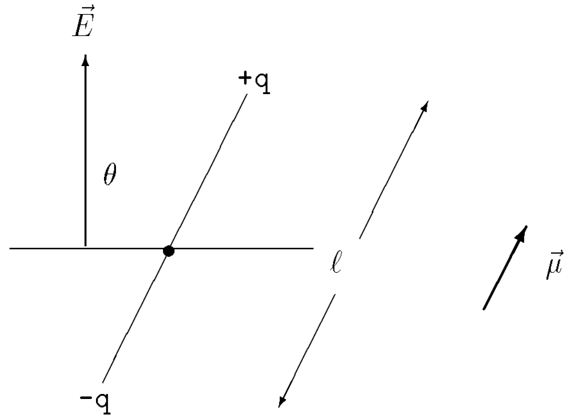
$$\begin{aligned} \vec{\omega}_T &\approx \frac{1}{2c^2}\vec{v}\times\vec{a} = \frac{1}{2c^2}\vec{v}\times\left(\frac{e\vec{E}}{m_e}\right) \\ &= \frac{1}{2mc^2}\vec{v}\times\left(-\frac{\vec{r}}{r}\frac{dV}{dr}\right) \\ &= \frac{1}{2m_e c^2}\left(\vec{r}\times\vec{v}\right)\frac{1}{r}\frac{dV}{dr} = \frac{\vec{L}}{2m_e^2c^2}\frac{1}{r}\frac{dV}{dr} \end{aligned} \tag{131}$$

Thus

$$\begin{aligned} U &= U' - \frac{1}{2m_e^2c^2}\vec{S}\cdot\vec{L}\frac{1}{r}\frac{dV}{dr} \\ &= -\frac{e}{m_e c}\vec{S}\cdot\vec{B} + (1-\frac{1}{2})\frac{1}{m_e^2c^2}\vec{S}\cdot\vec{L}\frac{1}{r}\frac{dV}{dr} \end{aligned} \tag{132}$$

The -1/2 is the famous one half. Including it, the observed fine-structure spacings in atomic spectra, due to electron spin, are correctly predicted.

This schematic gives a heuristic indication of how the torque arises.



The force on each charge (positive and negative) is $F = qE$. The magnetic moment is $\mu = g\ell$. The net torque is

$$\tau = qE\ell sin\theta = \mu E sin\theta$$

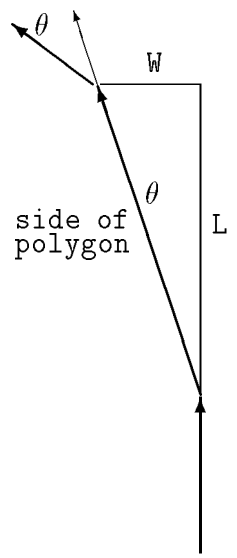
The energy relative to $\theta = \pi$ is

$$\Delta E = -2qE\frac{\ell}{2}cos\theta = -\vec{\mu} \cdot \vec{E}$$

10 A Simple Derivation of the Thomas Precession

The following derivation is based upon a suggestion by E.M. Purcell.

Imagine an aircraft flying in a large circular orbit. Approximate the orbit by a polygon of N sides, with N a very large number. As the aircraft traverses each of the N sides, it changes its angle of flight by the angle $\theta = 2\pi/N$ as shown in the figure.



After the aircraft has flown N segments, it is back at its starting point. In the laboratory frame, the aircraft has rotated through an angle of 2π radians. However in the aircraft's instantaneous rest frame, the triangles shown have a Lorentz-contraction along the direction it is flying but not transversely. Thus at the end of each segment, in the aircraft frame, the aircraft turns by a larger angle than the laboratory $\theta = 2\pi/N$, but by an angle $\theta' = \gamma\theta = W/(L/\gamma) = 2\pi\gamma/N$. After all N segments in the aircraft instantaneous rest frame the total angle of rotation is $2\pi\gamma$.

The difference in the reference frame is

$$\Delta\theta = 2\pi(\gamma - 1)$$

Since N has dropped out of the formula for the angle and angle difference, one can let it go to infinity and the motion is circular and the formula is for the rate of precession.

$$\frac{\omega_P}{\omega} = \frac{\Delta\theta/T}{2\pi/T} = \gamma - 1$$

This equation, despite the simplicity of the derivation, is the exact expression for the Thomas precession . The equation does not include the oscillating term because the derivation neglected the fact that the front and rear of the inertial airframe are not accelerated simultaneously.

11 Relativistic Dynamics of Particles

We will consider two approaches to reconciling Newtonian mechanics with relativity:

(1) The phenomenological approach based upon experimental results (e.g. length contraction, time dilation, clock synchronization) and generalize Newtonian mechanics to be consistent. We start with Newton's Principles (postulated three laws of mechanics) and generalize them to include Special Relativity.

(2) The second approach is to take literally Minkowski 3+1-D space and use four-dimensional vectors and 4-D vector algebra in the way we are used to doing 3-D vector algebra. In the previous sections we have seen how velocity and force (and other important quantities) do not transform as 3-dimensional vectors in relativity. We can, however, generalize them to 4-D vectors successfully.

First let us consider the phenomenological generalization of Newtonian mechanics based upon experimental observations. This will provide a comparison and motivation for the 4-vector approach.

11.1 Generalize Newton's Laws

11.1.1 Newton's Laws

First and Second Laws are a definition of force and implicitly the law of conservation of mass. The Third law is the law of conservation of momentum.

1. Results of the Lorentz Transformation
2. Generalized Conservation of Mass

   $\sum_i m_i =$ constant, not each m_i separately constant.
3. Conservation of Momentum

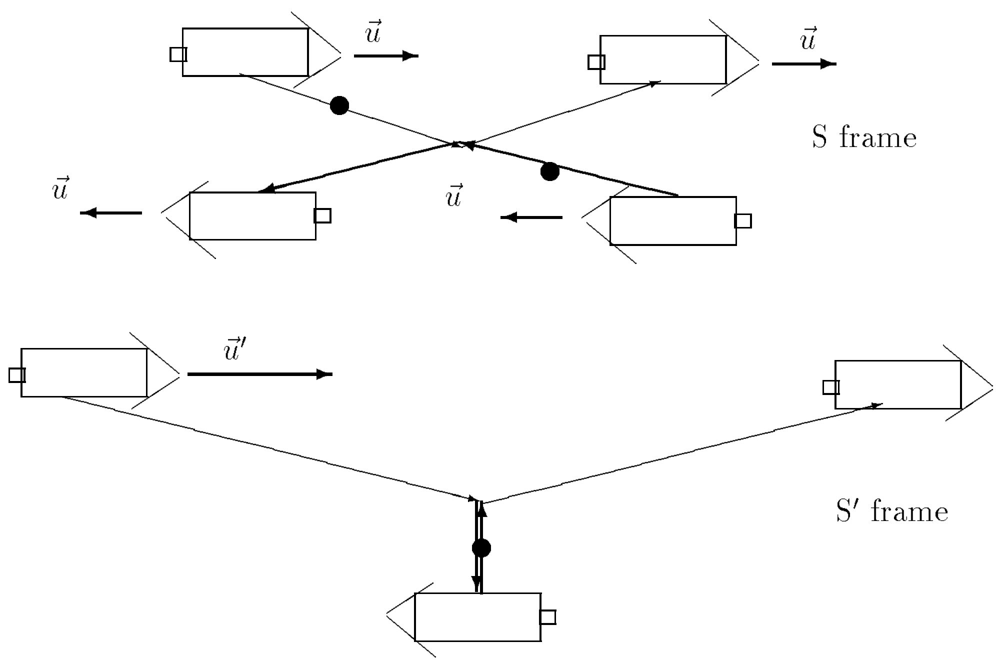
$$\sum m_i u_{xi} = \text{constant}$$
$$\sum m_i u_{yi} = \text{constant}$$
$$\sum m_i u_{zi} = \text{constant}$$

These postulates are as close as possible to those of Newton, but they produce different results.

11.2 The Mass of a Moving Particle

11.2.1 Space Billiards

Consider two identical space ships aligned to pass near each other with speed u in opposite directions at a distance d apart in frame S. At identical times ($t_i = -x_o/u = -d/(2u_y)$ in frame S each emits an identical steel ball with exactly opposite directions (in a direction perpendicular to their direction of motion in their rest frames) and with exactly equal transverse velocities u_y. The timing and positions are such that the two steel balls collide directly and elastically in the center (origin of frame S) between the ships and each ball rebounds to go back and be recaptured by the ship that emitted it at time $t_r = x_o/u = d/(2u_y)$. The location of the upper and lower ships is symmetrical by construction as is the collision.



Now consider the events in the frame S′ from the lower (in the picture) rest frame. That is, the inertial frame where the lower ship is at rest. By using the Lorentz transform it is easy to see that the events which were simultaneously in the observer O's frame S are no longer simultaneous in the frame S′. In fact in frame S′ the upper ship will emit its ball first, then the lower ship will emit its. The lower ship receives its ball back next and then the upper ship receives its ball back.

$$t' = \gamma\left(t + \frac{v}{c^2}x\right)$$

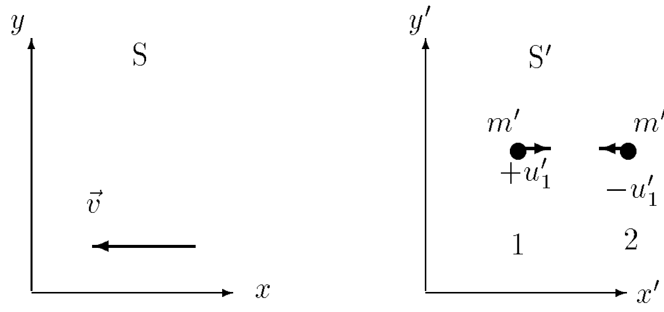
Spaceship	*event*	*Coordinates in S*	t' *in* S'	Δt
Upper	*emit*	$-x_o/u, -x_0$	$\gamma(-x_o/u - ux_o/c^2)$	
Lower	*emit*	$-x_o/u, +x_0$	$\gamma(-x_o/u + ux_o/c^2)$	$2\gamma v x_o/c^2$
Lower	*recapture*	$x_o/u, -x_0$	$\gamma(x_o/u - ux_o/c^2)$	
Upper	*recapture*	$x_o/u, +x_0$	$\gamma(x_o/u + ux_o/c^2)$	$2\gamma v x_o/c^2$

The balls still meet in the center by symmetry and because the Lorentz transformations leave transverse directions unchanged. The upper ball has a lower velocity and change in velocity upon scattering than the lower ball. If we want the y component of momentum conserved, $m_1\Delta u_{y1} + m_2\Delta u_{y2} = 0$ and since $\Delta u_{y1} < \Delta u_{y2}$, then the masses are not equal and $m_1 > m_2$. The more rapidly moving mass is greater, even though in frame S the two balls were identical and had identical mass by construction and symmetry.

This is exactly the case of transverse collisions discussed below.

11.2.2 Longitutional Collision

Consider the collision of two identical particles moving at each other with identical but oppositely directed velocities. These particles are aligned so that they will undergo an elastic collision and rebound in the opposite direction.



In S′ symmetry says that an elastic collision leads to a reversal of velocities of the colliding masses.

In S:

$$u_1 = \frac{u' + v}{1 + u'v/c^2}; \qquad u_2 = \frac{-u' + v}{1 - u'v/c^2}.$$

From the conservation of mass:

$$m_1 + m_2 = M = \text{constant}$$

From conservation of momentum

$$m_1 u_1 + m_2 u_2 = Mv = \text{constant}$$

By considering the motion at the instant of relative rest.

In the following algebra we show that $m = \gamma m_o$

$$m_1 \frac{u' + v}{1 + u'v/c^2} + m_2 \frac{-u' + v}{1 - u'v/c^2} = m_1 v + m_2 v$$

Subtracting the right hand side from both sides of the equation one has

$$m_1 \frac{u' + v - v - u'v^2/c^2}{1 + u'v/c^2} + m_2 \frac{-u' + v - v + u'v^2/c^2}{1 - u'v/c^2} = 0$$

$$m_1 \frac{u' - u'v^2/c^2}{1 + u'v/c^2} + m_2 \frac{-u' + u'v^2/c^2}{1 - u'v/c^2} = 0$$

$$\frac{m_1}{1 + u'v/c^2} = \frac{m_2}{1 - u'v/c^2}$$

$$\frac{m_1}{m_2} = \frac{1 + u'v/c^2}{1 - u'v/c^2}$$

The law of transformation of $\sqrt{1-u^2/c^2}$ is

$$\sqrt{1-u^2/c^2} = \frac{\sqrt{1-(u')^2/c^2}\sqrt{1-v^2/c^2}}{1+u'_x v/c^2}$$

Multiplying the mass ratio equation by $\sqrt{1-(u')^2/c^2}\sqrt{1-v^2/c^2}$ divided by itself give

$$\frac{m_1}{m_2} = \frac{1+u'v/c^2\sqrt{1-(u')^2/c^2}\sqrt{1-v^2/c^2}}{1-u'v/c^2\sqrt{1-(u')^2/c^2}\sqrt{1-v^2/c^2}}$$

One can then group the first term with its reciprocal to find

$$\frac{m_1}{m_2} = \frac{\sqrt{1-(u_2)^2/c^2}}{\sqrt{1-(u_1)^2/c^2}}$$

Noting that $+u'_x = +u'$ goes with u_1 and $-u'_x = -u'$ goes with u_2. Thus

$$m_1\sqrt{1-(u_1)^2/c^2} = m_2\sqrt{1-(u_2)^2/c^2} = m_o$$

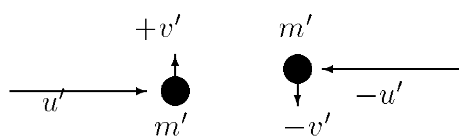
implying

$$m = \frac{m_o}{\sqrt{1-u^2/c^2}} = \gamma m_o \tag{133}$$

11.2.3 Transverse Collision

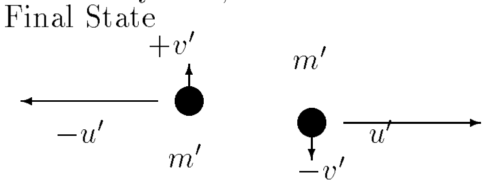
Consider a symmetric transverse collision

Inital State



This is a glancing collision with no change in the momenta in the x direction.
In the S′ system, one sees a reversal of v''s on collision.

Final State



In S system:

$$v'_1 = v'\frac{\sqrt{1-v^2/c^2}}{1+u'v/c^2}, \qquad v'_2 = -v'\frac{\sqrt{1-v^2/c^2}}{1-u'v/c^2}$$

The change in velocity of mass 1 is

$$\Delta v_1 = v_{1f} - v_{1i} = -2v' \frac{\sqrt{1 - v^2/c^2}}{1 + u'v/c^2}.$$

The change in velocity of mass 2 is

$$\Delta v_2 = v_{2f} - v_{2i} = +2v' \frac{\sqrt{1 - v^2/c^2}}{1 - u'v/c^2}.$$

Note that $\Delta v_1 \neq \Delta v_2$, so to conserver momentum p_y total, $m_1 \neq m_2$.

$$-2m_1 v' \frac{\sqrt{1 - v^2/c^2}}{1 + u'v/c^2} + 2m_2 v' \frac{\sqrt{1 - v^2/c^2}}{1 - u'v/c^2} = 0$$

$$\frac{m_1}{m_2} = \frac{1 + u'v/c^2}{1 - u'v/c^2} = \frac{\sqrt{1 - u_2^2/c^2}}{\sqrt{1 - u_1^2/c^2}}$$

So that by factoring and substituting one has

$$m_1 \sqrt{1 - u_1^2/c^2} = m_2 \sqrt{1 - u_2^2/c^2}$$

So for this to always be true

$$m = \frac{m_o}{\sqrt{1 - u^2/c^2}} = \gamma m_o$$

as derived for longitudinal case. We can generalize for any elastic collision with $\vec{p}_{total} = 0$ in S′ to get the same result.

11.2.4 "Internal" Mass

Consider an undeformed (unstrained) particle able to move freely. Let its mass be denoted by m_o when it is at rest.

In frame S′ during a symmetric longitudinal collision

$$M = \frac{m_o}{\sqrt{1 - u_1^2/c^2}} + \frac{m_o}{\sqrt{1 - u_2^2/c^2}}$$

$$= m_o \frac{1 - u'v/c^2}{\sqrt{1 - v^2/c^2}\sqrt{1 - (u')^2/c^2}} + m_o \frac{1 + u'v/c^2}{\sqrt{1 - v^2/c^2}\sqrt{1 - (u')^2/c^2}}$$

$$M = \frac{2m_o}{\sqrt{1 - (u')^2/c^2}\sqrt{1 - v^2/c^2}} = \gamma_{u'}\gamma_v \; 2m_o$$

At the instant of greatest deformation one might think $M = 2m_o/\sqrt{1 - v^2/c^2}$ because at that instant each particle has velocity $\vec{v}$ in S; but that is incorrect! $M > 2m_o/\sqrt{1 - v^2/c^2}$ because energy is added in deforming the particles.

11.2.5 Expressions for Force

We must choose a definition for force.

$$\vec{F} = \frac{d}{dt}(m\vec{u})$$

is more useful than $\vec{F} = m\frac{d\vec{u}}{dt}$, because then conservation of momentum implies that **action = reaction**.

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}}; \qquad \frac{dm}{dt} = m_o\frac{d}{dt}\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right) = \frac{m_o}{(1 - v^2/c^2)^{3/2}}\frac{u}{c^2}\frac{du}{dt}$$

So

$$\vec{F} = \frac{m_o}{\sqrt{1 - u^2/c^2}}\frac{d\vec{u}}{dt} + \frac{m_o}{\left(\sqrt{1 - u^2/c^2}\right)^3}\vec{u}\frac{u}{c^2}\frac{du}{dt}$$

So <u>force</u> is not parallel to <u>accelleration</u>!
We take components in S and S′:

$$F_x = m\dot{u}_x + \dot{m}u_x; \qquad F'_x = m'\dot{u}'_x + \dot{m}'u'_x$$

$$F_y = m\dot{u}_y + \dot{m}u_y; \qquad F'_y = m'\dot{u}'_y + \dot{m}'u'_y$$

$$F_z = m\dot{u}_z + \dot{m}u_z; \qquad F'_z = m'\dot{u}'_z + \dot{m}'u'_z$$

where $\dot{m}' = dm'/dt'$ and $\dot{u}'_x = du'_x/dt'$.

11.2.6 Transformation Equations for Force

The law of transformation of force is complicated by the fact that F_x, F_y, and F_z are **not** three of the four components of a relativistic four vector. Here are the relationships but we defer deriving them (see e.g. Rindler p. 91)

$$F'_x = \frac{F_x - \vec{F}\cdot\vec{u}v/c^2}{1 - u_x v/c^2} \tag{134}$$

$$F'_y = \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}F_y \tag{135}$$

$$F'_z = \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}F_z \tag{136}$$

11.2.7 Transformation Equations for Mass

$$m_o = m\sqrt{1-u^2/c^2} = m'\sqrt{1-(u')^2/c^2}$$

$$\sqrt{1-u^2/c^2} = \frac{\sqrt{1-(u')^2/c^2}\sqrt{1-v^2/c^2}}{1+u_x v/c^2}$$

So

$$m = m'\frac{1+u_x v/c^2}{\sqrt{1-v^2/c^2}}$$

Take the derivative with respect to time (d/dt)

$$\frac{dm}{dt} = \frac{1+u_x v/c^2}{\sqrt{1-v^2/c^2}}\frac{dm'}{dt'}\frac{dt'}{dt} + m'\frac{v/c^2}{\sqrt{1-v^2/c^2}}\frac{du'_x}{dt'}\frac{dt'}{dt}$$

The inverse Lorentz transformation is

$$t = \frac{t' + x'v/c^2}{\sqrt{1-v^2/c^2}}$$

so that

$$\frac{dt}{dt'} = \frac{1+u'_x/c^2}{\sqrt{1-v^2/c^2}}$$

Substituting for dt'/dt one finds

$$\frac{dm}{dt} = \frac{dm'}{dt'} + \frac{m'v}{c^2}\frac{1}{1+u_x v/c^2}\frac{du'_x}{dt'} \tag{137}$$

11.3 Work and Kinetic Energy

We define differential work

$$dW \equiv \vec{F}\cdot\vec{dr}$$

and also define the differential kinetic energy

$$dE_{\text{kinetic}} \equiv dW = m\frac{d\vec{u}}{dt}\cdot\vec{dr} + \frac{dm}{dt}\vec{u}\cdot\vec{dr}$$

Now

$$\frac{d\vec{u}}{dt}\cdot\vec{dr} = d\vec{u}\cdot\frac{d\vec{r}}{dt} = d\vec{u}\cdot\vec{u} = \vec{u}\cdot d\vec{u}$$

$$\frac{1}{2}d\left(\vec{u}\cdot\vec{u}\right) = \vec{u}\cdot d\vec{u} = \frac{1}{2}d\left(u^2\right) = udu$$

So

$$dE_{\text{kinetic}} = mudu + u^2 dm$$

$$= \frac{m_o u du}{\sqrt{1-u^2/c^2}} + \frac{m_o u^3/c^2 du}{\left(\sqrt{1-u^2/c^2}\right)^3} = \frac{m_o u du}{(1-u^2/c^2)^{3/2}}$$

$$E_{\text{kinetic}} = \int_{u=0}^{u} \frac{m_o u du}{(1-u^2/c^2)^{3/2}} = \frac{m_o c^2}{\sqrt{1-u^2/c^2}}|_{u=0}^{u}$$

$$E_{\text{kinetic}} = \frac{m_o c^2}{\sqrt{1-u^2/c^2}} - m_o c^2 \tag{138}$$

Notice that E_{kinetic} depends only on the final velocity squared and not on the way in which it is attained.

Of course for $v/c \ll 1$,

$$E_{\text{kinetic}} \to m_o c^2 \left[1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots - 1\right] = \frac{1}{2}mv^2 \left[1 + O(\frac{v^2}{c^2})\right]$$

In an elastic collision

$$\Delta E_{\text{kinetic}}(1) = \Delta E_{\text{kinetic}}(2)$$

from the equality of action and reaction.

11.4 Relations Among Mass, Energy, and Momentum

11.4.1 Conservation of Energy

From the relation $dE_{\text{kinetic}} = m_o u du / \left(1-u^2/c^2\right)^{3/2}$, $dE = c^2 dm$ and thus

$$\Delta E = c^2 \Delta m$$

So conservation of mass implies conservation of energy. Association of mass with energy from $\Delta E = c^2 \Delta m$, suppose

$$m = \frac{E}{c^2} \quad \text{and further} \quad E = mc^2 = \gamma m_o c^2$$

$$\vec{p} = \gamma m_o \vec{v}$$

11.4.2 Conservation of Momentum

With $dE = c^2 dm$, we have $\Delta(\text{K.E.}) = \mathrm{c}^2 (\mathrm{m} - \mathrm{m_o})$. So we postulate $E = mc^2$ and $m = E/c^2$ for any kind of E and m.

If energy E is being bodily transferred (transported) with velocity $\vec{u}$, the associated momentum is

$$\vec{G} = m\vec{u} = \frac{E}{c^2}\vec{u}.$$

But there are other ways to transfer energy, e.g. pushing at A and get out a B. so we have $\vec{g}$ defined as the density of momentum and $\vec{S}$ defined as energy flow, with $\vec{g} = \vec{S}/c^2$ for any mechanism of energy transfer.

Dimensions: $[G] = [Eu/c^2] = m/T$. $[g]$ = m/(L^{2}T; [S] = m/T^3] = Energy/(area * time).

12 Applications and Experimental Tests of Particle Dynamics

12.1 Mass of High Velocity Electrons

Bucher using β-rays and Hadka using cathode rays confirmed that $m = m_o/\sqrt{1-\beta^2}$ (where $\beta \equiv v/c$). In 1926 Gerlach put in the Handbuch der Physik

$$Ve = m_o c^2 \left[\frac{1}{\sqrt{1-v^2/c^2}} - 1\right] = m_o c^2 (\gamma - 1) \tag{139}$$

where V is the acceleration voltage.

All measurements really determine the ratio m/e; to get the relativistic result, they must assume e = constant. This results form requiring Maxwell's equations to be invariant under Lorentz transformation – Charge Conservation. We will revisit this issue later.

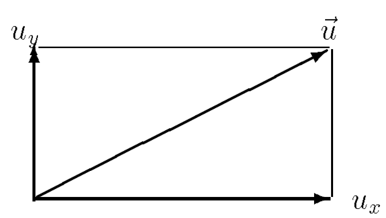
12.2 Relation Between Force and Acceleration

By our convention the vector force $\vec{F}$ is given by

$$\vec{F} = \frac{d}{dt}(m\vec{u}).$$

The force can be resolved into components parallel to $\vec{u}$ and $\vec{a} = d\vec{u}/dt$;

$$\vec{F} = m\frac{d\vec{u}}{dt} + \frac{dm}{dt}\vec{u} = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{d\vec{u}}{dt} + \frac{d}{dt}\left[\frac{m_o}{\sqrt{1-u^2/c^2}}\right]\vec{u}$$



$$\vec{u} = u_x\hat{i} + u_y\hat{j}; \qquad u^2 = u_x^2 + u_y^2$$

$$F_x = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{du_x}{dt} + \frac{d}{dt}\left[\frac{m_o}{\sqrt{1-u^2/c^2}}\right]u_x$$

$$F_y = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{du_y}{dt} + \frac{d}{dt}\left[\frac{m_o}{\sqrt{1-u^2/c^2}}\right]u_y$$

Consider an acceleration in the y direction:

$$\frac{du_x}{dt} = 0.$$

Then

$$\frac{du^2}{dt} = 2u_y\frac{du_y}{dt}$$

So that

$$F_x = \frac{m_o}{(1-u^2/c^2)^{3/2}}\frac{u_x u_y}{c^2}\frac{du_y}{dt}$$

$$F_y = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{du_y}{dt} + \frac{m_o}{(1-u^2/c^2)^{3/2}}\frac{u_y^2}{c^2}\frac{du_y}{dt}$$

$$= \frac{m_o}{(1-u^2/c^2)^{3/2}}\left[1-u^2/c^2+u_y^2/c^2\right]\frac{du_y}{dt}$$

$$= \frac{m_o}{(1-u^2/c^2)^{3/2}}\left[1-u_x^2/c^2\right]\frac{du_y}{dt}$$

So the ratio of Fx/Fy is

$$\frac{F_x}{F_y} = \frac{u_x u_y}{c^2-u_x^2}$$

We conclude that to accelerate in the y direction one must apply both F_y and $F_x = \frac{u_x u_y}{c^2-u_x^2}F_y$. To accelerate in the x direction one must apply F_x and $F_y = \frac{u_x u_y}{c^2-u_y^2}F_y$.

Under what conditions would the force $\vec{F}$ and acceleration $\vec{a}$ be in the same direction? If F_x and u_y are both zero, then F_y must be

$$F_y = \frac{m_o}{(1-u^2/c^2)^{1/2}}\frac{du_y}{dt}$$

.

Thus force and acceleration are parallel ($\vec{F} \parallel \vec{a}$) in y direction when perpendicular to the motion ($\perp \vec{u}$) and thus the velocity and acceleration are perpendicular ($\vec{a} \perp \vec{u}$). Then the "transverse mass" is $m_o/\sqrt{1-v^2/c^2}$.

If F_x and u_x are both zero:

$$F_y = \frac{m_o}{(1-u^2/c^2)^{1/2}}\frac{du_y}{dt} + = \frac{m_o}{(1-u^2/c^2)^{3/2}}\frac{u_y^2}{c^2}\frac{du_y}{dt}$$

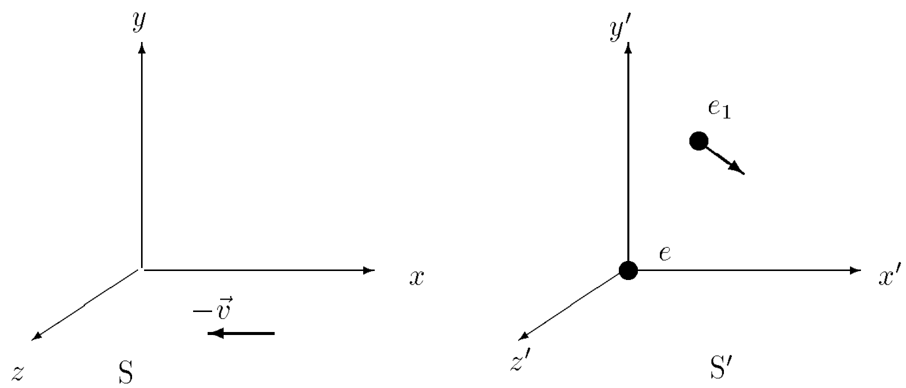
$$= \frac{m_o}{(1-u^2/c^2)^{3/2}}\frac{du_y}{dt}$$

Thus $\vec{F} \parallel \vec{a}$ in y direction $\parallel \vec{u}$ ($\vec{a} \parallel \vec{u}$) is longitudinal acceleration. Then the "longitudinal mass" is $m_o/(1-u^2/c^2)^{3/2}$ and we conclude $m_{\text{transverse}} < m_{\text{longitudinal}}$. The "longitudinal mass" is bigger because the force must also do work to raise the energy. (A transverse force does no work.)

Do not be confused. The "true" mass is always $m_o/\sqrt{1-u^2/c^2}$, which is conserved in a closed system.

12.3 Force Exerted by a Moving Charge e on a charge e_1

Pick a coordinate system in which one charge is fixed. E. g. e fixed in S′ with e at the origin of S and e_1 located at x, y, z at time considered.



In frame S, $u_{e_x} = v$, $u_{e_y} = u_{e_z} = 0$.
The force on e_1 in S′ is electrostatic.

$$F'_x = \frac{ee_1}{(x'^2+y'^2+z'^2)^{3/2}}x'$$

$$F'_y = \frac{ee_1}{(x'^2+y'^2+z'^2)^{3/2}}y'$$

$$F'_z = \frac{ee_1}{(x'^2+y'^2+z'^2)^{3/2}}z'$$

Now transform to frame S in which e moves.

$$F_x = \frac{ee_1}{s^3}\left(1-v^2/c^2\right)\{x + \frac{v}{c^2}(yu_y + zu_z)\}$$

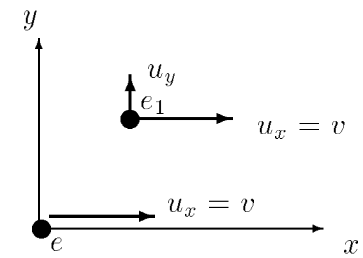
$$F_y = \frac{ee_1}{s^3}\left(1 - v^2/c^2\right)\left(1 - \frac{u_x v}{c^2}\right) y$$

$$F_z = \frac{ee_1}{s^3}\left(1 - v^2/c^2\right)\left(1 - \frac{u_x v}{c^2}\right) z$$

where $s^2 = x^2 + (1 - v^2/c^2)(y^2 + z^2)$.

This result is restricted to constant velocity v of e in x direction. If not constant, one must use the retarded potential.

Special Example: e fixed at origin, e_1 constrained to move in y direction.



The force on e_1 is:

$$F_x = \frac{ee_1}{s^3}\left(1 - v^2/c^2\right) y u_y$$

$$F_y = \frac{ee_1}{s^3}\left(1 - v^2/c^2\right)\left(1 - \frac{u_x v}{c^2}\right) y$$

$$\frac{F_x}{F_y} = \frac{u_x u_y}{c^2 - u_x^2},$$

which is just right to produce acceleration in the y direction only!

12.4 Nuclear Reactions

These were the first accurate tests of relations between mass and energy, also momentum. The experiments of Oliphant, Kinsey, and Rutherford (1933) and Bainbridge (1933) are cited.

$$Li^7 + H^1 \rightarrow Be^8 \rightarrow He^4 + He^4 + 17 \text{ MeV}$$

$$Li^6 + H^2 \rightarrow Be^8 \rightarrow He^4 + He^4 + 17 \text{ MeV}$$

Solar Energy: Around this time Hans Bethe realized that the primary source of solar energy is the reaction

$$4H^1 \rightarrow He^4 + 30 \text{ MeV}$$

Uranium Fission: Han and Sparissman found

$$U^{235} + n^1 \rightarrow F_1 + F_2 + \text{a few neutrons} + 200 \text{ MeV}$$

where F_1 and F_2 are the fission fragments which are randomly distributed but with average atomic number around 90 and 140 respectively.

<u>"Monochromatic" Neutrons</u>

$$Li^7 + H^1 \rightarrow Be^7 + n - 1.6 \text{ MeV}$$

is a reaction which absorbs energy.

12.5 Light Pressure

Let $\mathcal{E} \equiv$ energy density. Then $\mathcal{E}/c^2$ is the equivalent mass density of that energy. I.e. energy density has an effective inertia density. The momentum density is then $\mathcal{E}c/c^2 = \mathcal{E}/c \equiv \pi$ and $\pi c = p =$ the change of momentum per unit area per second if light is absorbed. Since $\pi c = \mathcal{E}$, then $p = \mathcal{E}$ in absorption.

For reflection $p = 2\mathcal{E}$.

In a cavity (as in hohlraum) $p = \mathcal{E}/3$.

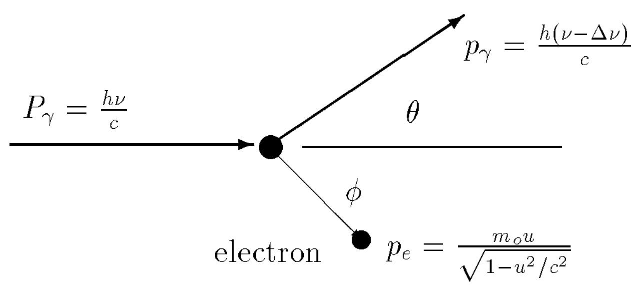
12.5.1 Mass, Energy, Momentum for Photons

$$E = h\nu; \qquad m = h\nu/c^2$$

$$G = h\nu/c; \qquad \vec{G} = \frac{h\nu}{c^2}\vec{c}$$

12.5.2 Compton Effect

Arthur Compton provided the relativistic treatment of a photon scattering on a free electron.



Conservation of energy yields:

$$h\Delta\nu = m_o c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]$$

Conservation of momentum in x-direction yields the equation:

$$\frac{h\nu}{c} = \frac{h(\nu - \Delta\nu}{c)} cos\theta + \frac{m_o u}{\sqrt{1 - v^2/c^2}} cos\phi$$

Conservation of momentum in y-direction yields the equation:

$$\frac{h\nu + \Delta\nu}{c} sin\theta = \frac{m_o u}{\sqrt{1 - v^2/c^2}} sin\phi$$

The solution to these equations is:

$$\Delta\lambda = \frac{2h}{m_o c} sin^2 \left(\frac{\theta}{2}\right) \tag{140}$$

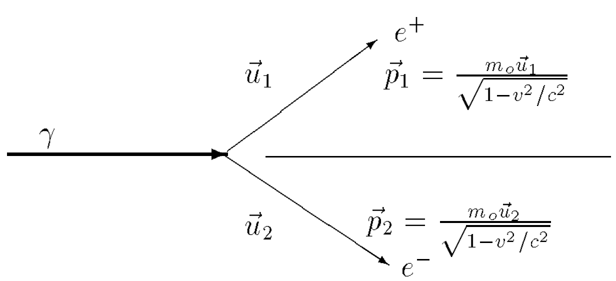
The importance to Quantum Mechanics was not only the direct evidence of photon scattering but the Compton effect is used to explain the "γ-ray microscope" interpretation of the Uncertainty Principle. The uncertainty in coordinate above is

$$\Delta x_{\min} \sim \frac{h}{mc}$$

To make it go to zero requires an infinite mass m and photon energy.

12.5.3 Pair Production

An energetic photon may have sufficient energy to convert into an election-positron $(e^- e^+)$ pair.



This process cannot occur in vacuum because it cannot conserve both energy and momentum. It must occur in the vicinity of a mass which can these by absorbing some of the E and $\vec{p}$.

12.5.4 Positron Annihilation

The reaction $e^+ + e^- \rightarrow \gamma$, that is, annihilation of a positron with an electron to a single energetic photon does not occur. However, the reaction $e^+ + e^- \rightarrow \gamma + \gamma$ is commonly observed.

12.5.5 Zitterbewegung and Antiparticles

To make quantum mechanics and Special Relativity plus causality consistent requires the existence of an identical (CPT) antiparticle for every particle. Consider the world line of an electron described by a Gaussian wave packet. Clearly parts of the packet are outside the light cone of the central ray (or any other portion). That is parts of the wave packet have space-like separation and are out of causal contact. However, the geometrical optics or particle path approach can understand that an electron does not necessarily travel in a straightline. The electron can have a jittering path which is bounded by the wave packet and set by the Uncertainty Principle.

Consider the following scenario: By the Uncertainty Principle virtual pairs consisting of an electron and positron can appear in the vacuum. The original electron can annihilate with the positron and if it has just the right energy (off the mass shell $\tilde{p} \cdot tildep \neq (m_o c^2)^2$) then the annihilation photons have zero energy and do not appear while the other member of the pair, the new electron continues along a parallel path to the original electron. This description conserves energy and momentum but leaves the electron path displaced by an amount set by the scale allowed by the Uncertainty Principle $\Delta x = \hbar c / m_e c^2$ or the Compton wavelength of the electron.

This scenario can be made consistent with causality in Special Relativity provided that the antiparticle behaves exactly like the particle going backwards in time. One can draw a world line for the electron that explains it geometric jittery path or its finite non-locality.

12.6 Some Practical Examples of the Use of Invariants

In this section we show some practical examples of how to use invariants as derived either laboriously or through the use of 4-vectors of Minkowski space. This will motivate the next section in which we will learn the properties of vectors and tensors in 3+1-D space.

12.6.1 Mass, $\vec{\beta}$, γ

First consider the relation between mass, energy, and three momentum or equivalently the norm of the 4-momentum

$$\left(m_o c^2\right)^2 = E^2 - p^2 c^2 \tag{141}$$

Conservation of energy and momentum (4-momentum) for a collection of particles in a given frame leads to the invariant for the total mass M.

$$\left(M c^2\right)^2 = \left(\sum_i E_i\right)^2 - \left(\sum_i p_i c\right)^2 \tag{142}$$

where M is the center-of-mass equivalent mass.

$$\vec{\beta} = \frac{\vec{p}}{E} \qquad \vec{\beta}_{CM} = \frac{\vec{p}_{CM}}{E_{CM}} \tag{143}$$

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{E}{m_o c^2} \qquad \gamma_{CM} = \frac{E_{CM}}{M_{CM} c^2} \tag{144}$$

12.6.2 Energy, momentum, and velocity of one particle in rest frame of another

The energy momentum, and velocity of one particle in rest frame of another can be calculated readily by making use of the concept of invariants. If one is given the 4-momenta of two particles in any frame the energy of particle two from the rest frame of particle one E_{21} can be found by a simple relationship. Let $\tilde{p}_1$ and $\tilde{p}_2$ are the momentum four vectors of particles one and two. The dot or inner product between the two 4-vectors is defined as

$$\tilde{p}_1 \cdot \tilde{p}_2 = E_1 E_2 / c^2 - \vec{p}_1 \cdot \vec{p}_1$$

where $\vec{p}_1$ and $\vec{p}_1$ are the relativistic 3-momenta of particles one and two respectively. We can derive the relationship using the principle that this dot product should be invariant to frame of reference. Thus

$$\tilde{p}_1 \cdot \tilde{p}_2 = \tilde{p}_1' \cdot \tilde{p}_2'$$

Take the system S$'$ to be the frame in which particle one is at rest. Then $\tilde{p}_1' = (m_1 c, 0, 0, 0)$, where m_1 is the rest mass of particle one and thus

$$\tilde{p}_1 \cdot \tilde{p}_2 = \tilde{p}_1' \cdot \tilde{p}_2' = m_1 E_{21}$$

Dividing through by m_1 gives the relationship

$$E_{21} = \frac{\tilde{p}_1 \cdot \tilde{p}_2}{m_1} \tag{145}$$

Once we have the energy E_{21} of particle two in the rest frame of particle one, it is easy to find its three-momentum amplitude making use of the relationship between mass, energy, and three-momentum amplitude.

$$E_{21}^2 = |p_{21} c|^2 + (m_2 c^2)^2 \quad \rightarrow \quad |p_{21} c|^2 = E_{21}^2 - (m_2 c^2)^2$$

where m_2 is the rest mass of particle two.

$$|p_{12}|^2 = \frac{(\tilde{p}_1 \cdot \tilde{p}_2)^2}{(m_1 c)^2} - \frac{(m_2 c)^2}{c^2} = \frac{(\tilde{p}_1 \cdot \tilde{p}_2)^2 - m_1^2 m_2^2 c^4}{m_1^2 c^2} \tag{146}$$

And for the relative velocity we have

$$\beta_{21}^2 = \frac{|p_{21}|^2}{E_{21}^2} = \frac{(\tilde{p}_1 \cdot \tilde{p}_2)^2 - m_1^2 m_2^2 c^4}{(\tilde{p}_1 \cdot \tilde{p}_2)^2} = 1 - \frac{m_1^2 m_2^2 c^4}{(\tilde{p}_1 \cdot \tilde{p}_2)^2} = 1 - \left(\frac{m_2 c^2}{E_{21}}\right)^2 \tag{147}$$

12.6.3 Energy, momentum, and velocity of a particle in the center of momentum frame

Use the same formulae as above but for particle one use the center-of-momentum particle (fictional) as particle one.

5 Four Vectors

A natural extension of the Minkowski geometrical interpretation of Special Relativity is the concept of four dimensional vectors. One could also arrive at the concept by looking at the transformation properties of vectors and noticing they do not transform as vectors unless another component is added. We define a four-dimensional vector (or four-vector for short) as a collection of four components that transforms according to the Lorentz transformation. The vector magnitude is invariant under the Lorentz transform.

5.1 Coordinate Transformations in 3+1-D Space

One can consider coordinate transformations many ways: If $x_1, x_2, x_3, x_4 = x, y, z, ict$, then ordinary rotations (in $x_1 - x_2$ plane around x_3)

$$\begin{aligned} x_1' &= x_1 cos\theta + x_2 sin\theta \\ x_2' &= -x_1 sin\theta + x_2 cos\theta \end{aligned} \qquad \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

But in $x_1 - x_4$ plane:

$$\begin{aligned} x_1' &= x_1 cos\alpha + x_4 sin\alpha \\ x_4' &= -x_1 sin\alpha + x_4 cos\alpha \end{aligned} \qquad \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix}$$

where the angle α is defined by

$$cos\alpha = 1/\sqrt{1 - v^2/c^2} = 1/\sqrt{1 + tan^2\alpha} = \gamma$$

$$sin\alpha = i\frac{v/c}{\sqrt{1 - v^2/c^2}} = \frac{tan\alpha}{\sqrt{1 + tan^2\alpha}}$$

$$tan\alpha = iv/c = i\beta.$$

And thus one has the trignometic identity:

$$cos^2\alpha + sin^2\alpha = \gamma^2\left(1 - \beta^2\right) = 1$$

$$\begin{aligned} x_1' &= \gamma\left[x_1 + (ict)(i\beta)\right] = \gamma\left[x_1 - \beta ct\right] \\ x_4' &= \gamma\left[x_4 - i\beta x_1\right] \\ ict' &= \gamma\left[ict - i\beta x_1\right] \\ ct' &= \gamma\left[ct - \beta x_1\right] \\ t' &= \gamma\left[t - \beta x_1/c\right] \end{aligned}$$

So the extension to 3+1-D includes Lorentz transformations, if angles are imaginary.

Really, we are considering the set of all 4×4 orthogonal transformations matrices in which one angle may be pure imaginary.

In general all angles may be complex, combining real rotations in 2-space with imaginary rotations relative to t.

An alternate way of writing this is

$$x' = x cosh\phi - ct sinh\phi$$

$$ct' = -x sinh\phi + ct cosh\phi$$

where $\phi = cosh^{-1}\gamma$.

$$x' = x cos(i\phi) + ict sin(i\phi)$$

$$ict' = -x sin(i\phi) + ict cos(i\phi)$$

and

$$\alpha = i\phi = i cosh^{-1}\gamma, \qquad tan\alpha = i\beta = iv/c$$

Still another notation is (with $x_4 = ict$)

$$x_1' = \gamma\,(x_1 + i\beta x_4)$$

$$x_4' = \gamma\,(x_4 - i\beta x_1)$$

The transformation matrix is then

$$\begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

Still yet another notation is with $x_0 = ct$

$$x_0' = \gamma\,(x_0 - i\beta x_1)$$

$$x_1' = \gamma\,(x_1 + i\beta x_0)$$

The transformation matrix is then

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

5.1.1 Generalized Lorentz Transformation

For spatial coordinates the Lorentz transform fits the linear form

$$(x^\mu)' = \sum_{\nu=1}^{4} \Lambda^\mu_\nu x^\nu \tag{148}$$

subject to the condition that the proper length

$$(cd\tau)^2 = -(ds)^2 = \sum_\mu (x^\mu)' = \sum_\mu x^\mu = (ct)^2 - |\vec{x}|^2 \tag{149}$$

is an invariant. This condition requires that the coefficients Λ^μ_ν form an orthogonal matrix:

$$\begin{aligned} \sum_\alpha \Lambda^\mu_\alpha \Lambda^\nu_\alpha &= \delta^{\mu\nu} \\ \sum_\alpha \Lambda^\alpha_\mu \Lambda^\alpha_\nu &= \delta_{\mu\nu} \\ \sum_\alpha \Lambda^\alpha_\mu \Lambda^\nu_\alpha &= \delta^\nu_\mu \end{aligned} \tag{150}$$

where the Kronecker delta is defined by $\delta_{\mu\alpha} = \delta_{\mu\nu} = \delta^\nu_\mu = 1$ when $\mu = \nu$ and 0 otherwise.

The invariance group can be enlarged to be the *Poincaré group* by the addition of translations:

$$(x^\mu)' = \sum_{\nu=1}^{4} \Lambda^\mu_\nu x^\nu + a^\mu \tag{151}$$

The full group includes: translations, 3-D space rotations, and the Lorentz boosts.

5.2 The Inner Product of 3+1-D Vectors

The definition of the inner product (dot product) must be modified in 3+1 dimensions.

$$\tilde{A} \cdot \tilde{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_4 B_4$$

if $x_4 = ict$. But with our usual convention

$$\tilde{A} \cdot \tilde{B} = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3$$

or with the opposite signature metric one has

$$\tilde{A} \cdot \tilde{B} = -A_0 B_0 + A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$\tilde{A} \cdot \tilde{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 - A_4 B_4$$

if $x_4 = ct$ which is often the convention for the opposite sign convention. It is an exercise to show that the inner product is unchanged under a Lorentz transformation. Can be done simply by substitution. This can be extended to the general class of Lorentz transformations.

5.3 Four Velocity

So we have the position 4-vector $\tilde{x} = (x_0, x_1, x_2, x_3)$ and the displacement 4-vector $\tilde{dx} = (dx_0, dx_1, dx_2, dx_3)$. What other 4-vectors are there? That is what other 4-vectors are natural to construct? What we mean by a four-vector is a four-dimensional quantity that transforms from one inertial frame to another by the Lorentz transform which will then leave its length (norm) invariant.

Consider generalizing the 3-vector velocity $(v_x, v_y, v_z) = (dx/dt, dy/dt, dz/dt)$ what can we do to make this into a 4-vector naturally? One clear problem is that we are dividing by a component dt of a vector so that the ratio is clearly going to Lorentz transform in a complicated way. We need to take the derivative with respect to a quantity that will be the same in all reference frames, e.g. $d\tau$ the differential of the proper time, and add a fourth component to make the 4-vector. It is clear that the derivative of the 4-vector position (ct, x, y, z) with respect to the proper time τ will be a 4-vector for Lorentz transformations since (ct, x, y, z) transform properly and $d\tau$ is an invariant. So we can define the 4-velocity as

$$u_\alpha = \frac{dx_\alpha}{d\tau}; \qquad \tilde{u} = \left(\frac{dct}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) \tag{152}$$

Note that

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = dt^2 \left(c^2 - \frac{dx}{dt}^2 - \frac{dy}{dt}^2 - \frac{dy}{dt}^2\right)$$

$$= dt^2 \left(c^2 - v_x^2 - v_y^2 - v_z^2\right) = dt^2 \left(c^2 - v^2\right)$$

or the time dilation formula we got before

$$\frac{d\tau}{dt} = \sqrt{1 - v^2/c^2}; \quad \text{and} \quad \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

So we can now explicitly write out the 4-velocity using the chain derivative rule:

$$u_\alpha = \frac{dx_\alpha}{d\tau} = \frac{dx_\alpha}{dt}\frac{dt}{d\tau}$$

$$\tilde{u} = (u_0, u_1, u_2, u_3) = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z) = \gamma\,(c, v_x, v_y, v_z)$$

Thus three components of the 4-velocity are the three components of the 3-vector velocity times γ.

$$\frac{d}{d\tau}(\tilde{u} \cdot \tilde{u}) = \frac{d\tilde{u}}{d\tau} \cdot \tilde{u} + \tilde{u} \cdot \frac{d\tilde{u}}{d\tau} = 2\tilde{u} \cdot \frac{d\tilde{u}}{d\tau} = \frac{dc^2}{d\tau} = 0$$

Thus

$$\tilde{u} \cdot \tilde{a} = 0 \tag{153}$$

Note also that the norm - the magnitude or vector invariant length - of the four-velocity is not only unchanged but it is the same for all physical objects (matter plus energy). For 3+1 dimensions the norm or magnitude is found from the inner product or dot product which has the same signature as the metric (see just above) so that

$$\tilde{u} \cdot \tilde{u} = u_0^2 - u_1^2 - u_2^2 - u_3^2 = \gamma^2 \left(c^2 - v_x^2 - v_z^2 - v_z^2\right) = c^2 \frac{1 - v^2/c^2}{1 - v^2/c^2} = c^2$$

Thus every physical thing, including light, moves with a 4-velocity magnitude of c and the only thing that Lorentz transformations do is change the direction of motion. A particle at rest is moving down its time axis at speed c. When it is boosted to a fixed velocity, it still travels through space-time at speed c but more slowly down the time axis as it is also moving in the spatial directions.

One should also note that as the spatial speed (three-velocity) approaches c, all components of the 4-velocity u_α are unbounded as $\gamma \to \infty$. One cannot then define a Lorentz transformation that moves to the rest frame. Thus all massless particles will have no rest frame.

5.3.1 Law of Transformation of a 4-Vector

We can write the transformation in our standard algebraic Lorentz notation

$$A'_0 = \gamma \left(A_0 - \beta A_1\right) \qquad \gamma = 1/\sqrt{1 - \beta^2}$$

$$A'_1 = \gamma \left(A_1 - \beta A_0\right) \qquad \beta \equiv \frac{V}{c}$$

$$A'_2 = A_2; \qquad A'_3 = A_3$$

where β and γ refer to the relative velocity V of the frames.

5.3.2 Law of Transformation of a 4-Velocity

$$u'_1 = \gamma \left(u_1 - \beta u_0\right)$$

where β and γ are for the relative velocity of the frames and not of the particle. But in the formula for the 4-velocity

$$\tilde{u} = (u_0, u_1, u_2, u_3) = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z) = \gamma\left(c, v_x, v_y, v_z\right)$$

The γ is for the particle! So we should have labeled it γ_p and the β and γ for the frame transform β_f and γ_f. Then we have

$$\gamma'_p v'_x = \gamma_f \left(\gamma_p v_x - \beta_f \gamma_p c\right)$$

So we can get out a formula for v'_x

$$v'_x = \frac{\gamma_f \gamma_p}{\gamma'_p}(v_x - V) = \frac{\sqrt{1-\beta'^2_p}}{\sqrt{1-\beta^2_p}\sqrt{1-\beta^2_f}}(v_x - V)$$

This is our old friend on the law of transformation of $\sqrt{1-u^2/c^2}$

$$\sqrt{1-u^2/c^2} = \frac{\sqrt{1-(u')^2/c^2}\sqrt{1-V^2/c^2}}{1+u'_x V/c^2}$$

and

$$\sqrt{1-(u')^2/c^2} = \frac{\sqrt{1-u^2/c^2}\sqrt{1-V^2/c^2}}{1+u_x V/c^2}$$

which is simply

$$\frac{1}{\gamma'_p} = \frac{1}{\gamma_p \gamma_f (1 - u_x V/c^2)}$$

So

$$v'_x = \frac{v_x - V}{1 - u_x V/c^2}$$

as derived earlier by the differential route.

Continuing onward

$$u'_2 = u_2; \quad \text{or} \quad \gamma'_p v'_y = \gamma_p v_y$$

so that

$$v'_y = \frac{\gamma_p}{\gamma'_p} v_y$$

$$\frac{\gamma_p}{\gamma'_p} = \frac{1}{\gamma_f (1 - u_x V/c^2)}$$

and

$$v'_y = \frac{\sqrt{1-V^2/c^2}}{1 - u_x V/c^2} v_y$$

which is the same relationship as before from the differential Lorentz transform. Similarly for v'_z and v'_t:

$$u'_o = \gamma_f (u_0 - \beta_f u_1)$$

Explicitly this is

$$\gamma'_p c = \gamma_f (\gamma_p c - \gamma_p u_x V/c) = \gamma_p \gamma_f c \left(1 - u_x V/c^2\right)$$

So

$$\gamma'_p = \gamma_p \gamma_f \left(1 - u_x V/c^2\right)$$

which is our relation from the transformation of γ's and its reciprocal used above.

5.4 Four Momentum

What is the natural extension of the 3-vector momentum to 4-momentum. The answer is clear from dimensional/transform analysis and from our experimental approach on how masses transformed. The 4-momentum is simply:

$$p_\alpha = m_o u_\alpha;\ \tilde{p} = (p_0, p_1, p_2, p_3) = \gamma m_o\,(c, v_x, v_y, v_z) \tag{154}$$

The three spatial components are just the Newtonian 3-momentum with the mass of the particle replaced by γm_o.

We can see that the 4-momentum also has an invariant norm by making use of our results for the 4-velocity:

$$\tilde{p}\cdot\tilde{p} \equiv p_0^2 - p_1^2 - p_2^2 - p_3^2 = E^2/c^2 - p^2 = m_o^2\tilde{u}\cdot\tilde{u} = m_o^2 c^2$$

Thus the invariant length of the 4-momentum vector is just the rest mass of the particle times c.

$$\tilde{p} = m_0\tilde{u} = m_0\gamma(c, v_x, v_y, v_z) = (m_0\gamma c, \gamma m_0 v_x, \gamma m_0 v_y, \gamma m_0 v_z) = (mc, mv_x, mv_y, mv_z) = (E/c, \vec{p})$$

Thus the magnitude squared of the four-momentum is

$$\tilde{p}\cdot\tilde{p} = (m_0\gamma)^2(c^2 - v_x^2 - v_y^2 - v_z^2) = m_0^2\gamma^2(1 - v^2/c^2)c^2 = (m_0 c)^2$$

or we can write this equation as

$$\left(\frac{E}{c}\right)^2 - p^2 = (m_0 c)^2$$

$$E^2 = p^2c^2 + \left(m_0 c^2\right)^2 \tag{155}$$

This is particularly simple to use when the units of mass are $Energy/c^2$ and the units of momnetum are $Energy/c$. Fore example, a proton has a rest mass $m_p = 938$ MeV/c^2 and an electron $m_e = 0.511$ MeV/c^2.

5.5 The Acceleration Four-Vector

In a similar way one may derive the acceleration four-vector. Again we differentiate with respect to the proper time τ.

$$a_\alpha = \frac{du_\alpha}{d\tau} \tag{156}$$

The four-vector acceleration will have a part parallel to the acceleration three-vector and a part parallel to the velocity three-vector.

Exercise: Prove that the inner product of the 4-acceleration and the 4-velocity are zero; $\tilde{a}\cdot\tilde{u}=0$ as they must be if the norm of the four-velocity is to remain constant c.

We have also constructed the 4-acceleration to be a 4-vector so that $\tilde{a}\cdot\tilde{a}$ is an invariant. Evaluate it in the rest frame $\tilde{a}\cdot\tilde{a}=|\tilde{a}|^2$

$$\tilde{a}\cdot\tilde{a}=|\tilde{a}_{\text{rest frame}}|^2$$

in any frame. This can be very useful in various calculations and we will use it later to treat radiation from and accelerating charged particle.

Acceleration 4-vector transforms by the relations:

$$a_0'=\gamma_f\left(a_0-\beta_f a_1\right), \qquad a_2'=a_2,$$
$$a_1'=\gamma_f\left(a_1-\beta_f a_0\right), \qquad a_3'=a_3,$$

This is the best starting place from which to derive the detailed Lorentz transformation equations for acceleration.

5.6 The Four Vector Force

We now consider the four-vector force, which we define the following way:

$$\tilde{F}\equiv\frac{d\tilde{p}}{d\tau} \tag{157}$$

$$\tilde{F}\equiv\frac{d\tilde{p}}{d\tau}=\frac{d\tilde{p}}{dt}\frac{dt}{d\tau}=\gamma\frac{d\tilde{p}}{dt}$$

$$\tilde{F}\equiv(F_0,F_1,F_2,F_3)=\gamma\left(W/c,F_{N1},F_{N2},F_{N3}\right)\rightarrow\gamma\left(\vec{F}_N\cdot\vec{\beta},F_{N1},F_{N2},F_{N3}\right) \tag{158}$$

where $\vec{F}_N$ is the three-dimensional Newtonian force, e.g. $\vec{F}_N=(F_{N1},F_{N2},F_{N3})$

Note that the four force can be space-like, time-like and null. If a frame can be found where the three-force on an object is zero but the object is exchanging internal energy with the environment, then the four-force is time-like. The converse is space-like.

Then the 4-vector force $\tilde{F}$ has the same transformation law as all 4-vectors:

$$\tilde{F}_0'=\gamma_f\left(F_0-\beta_f\tilde{F}_1\right)$$
$$\tilde{F}_1'=\gamma_f\left(F_1-\beta_f\tilde{F}_0\right)$$
$$\tilde{F}_2'=\tilde{F}_2; \qquad \tilde{F}_3'=\tilde{F}_3$$

So we can now conveniently transform any of the familiar vectors used in mechanics, but not electric and magnetic fields, and pseudovectors obtained from cross-products, such as angular momentum and angular velocity. We will treat these later.

The 4-vector force transforms are much easier than the 3-D fource transforms which involve a γ^3. See the homework problem for the transformaiton of acceleration to grasp how much more complicated it is.

5.7 4-D Potential

It is convenient to do physics in terms of potential and find the resulting force as the derivative, e.g. the gradient, of the potential. Classical physics examples are:

$$\begin{aligned} F_G &= -m\vec{\nabla}\Phi_G && \text{Newtonian Gravitation} \\ F_E &= -q\vec{\nabla}\Phi_E && \text{Electrostatics} \end{aligned} \tag{159}$$

Once we have a 4-D potential, then we need to learn how to take derivatives in 4-D spaces.

One approach is to make the simplest possible frame-independent (scalar) estimate of the interaction of two particles. This manner of thinking eventually leads one to the interaction Lagrangian as a the product of the two currents (electrical, matter, strong, weak, gravitational).

$$L = \alpha \tilde{j}_1 \cdot \tilde{j}_2 \tag{160}$$

where α is the coupling constant and the next term is the inner (4-D dot) product of the current of particle 1 and the current of particle 2. When the two currents are in contact (zero proper distance separation), there is an interaction. When they are not in proper distance contact, there is no interaction. This means that all interaction is on the proper distance null (the light cone). Thus there is no action at a proper distance. It is manifestly invariant as the inner product of two 4-D vectors.

From this Lagrangian we can generate the 4-D potential of the effect of all other currents (or a single current) $\tilde{j}_2$ on our test particle which has current $\tilde{j}_1$.

$$\tilde{A}(\tilde{x}_1) = \int\int\int\int \alpha f(s_{12}^2)\tilde{j}_2(\tilde{x}_2)dV_2 dt_2 = \int\int\int\int \alpha f(c^2(t_1-t_2)^2 - r_{12}^2)\tilde{j}\,dV\,dt \tag{161}$$

where $s_{12}^2 \equiv |\tilde{x}_1 - \tilde{x}_2|^2 = c^2(t_1-t_2)^2 - r_{12}^2$ is the invariant separation between $\tilde{x}_1$ and $\tilde{x}_2$ dV is the 3-D spatial volume and dt is the time. $f(s_{12}^2)$ is a function which is zero every where but peaks when the square of the 4-vector distance s_{12}^2 between the source (2) and the point of interest (1) is very small. The integral over $f(s_{12}^2)$ is also normalized to unity. The Dirac delta function is the limiting case for $f(s_{12}^2)$. Thus $f(s_{12}^2)$ is finite only for

$$s_{12}^2 = c^2(t_1-t_2)^2 - r_{12}^2 \approx \pm\epsilon^2 \tag{162}$$

Rearranging and taking the square root

$$c(t_1-t_2) \approx \sqrt{r_{12}^2 \pm \epsilon^2} \approx r_{12}\sqrt{1 \pm \frac{\epsilon^2}{r_{12}^2}} \approx r_{12}(1 \pm \frac{\epsilon^2}{2r_{12}^2}) \tag{163}$$

So

$$(t_1-t_2) \approx \frac{r_{12}}{c} \pm \frac{\epsilon^2}{2cr_{12}} \tag{164}$$

which says that the only times t_2 that are important in the integral of $\tilde{A}$ are those which differ from the time t_1, for which one is calculating the 4-potential, by the delay r_{12}/c ! – with negligible correction as long as $r_{12} \gg \epsilon$. Thus the Bopp theory approaches the Maxwell theory as long as one is far away from any particular charge.

By performing the integral over time one can find the approximate 3-D volume integral by noting that $f(s_{12}^2)$ has a finite value only for $\Delta t_2 = 2 \times \epsilon^2/2r_{12}c$, centered at $t_1 - r_{12}/c$. Assume that $f(s_{12}^2 = 0) = K$, then

$$\tilde{A}(\vec{x}_1) = \int \tilde{j}(t_2, \vec{x}_2) f(s_{12}^2) dV_2 dt_2 \approx \frac{K\epsilon^2}{c} \int \frac{\tilde{j}(t - r_{12}/c, \vec{x}_2)}{r_{12}} dV_2 \tag{165}$$

which is exactly the 3-D version, if we pick K so that $K\epsilon^2 = 1$.

5.8 Derivative in 4-Space

The 3-D vector gradient operator is DEL:

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \tag{166}$$

which behaves as a 3-D vector.

This can be generalized to 4-D:

$$\Box = (\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) \tag{167}$$

How does it transform?

$$\Box' = (\frac{\partial}{\partial x'_0}, \frac{\partial}{\partial x'_1}, \frac{\partial}{\partial x'_2}, \frac{\partial}{\partial x'_3}) \tag{168}$$

Operate first on a scalar function $\phi(x_0, x_1, x_2, x_3)$

$$\frac{\partial \phi(x_0, x_1, x_2, x_3)}{\partial x'_\nu} = \sum_\mu \frac{\partial \phi}{\partial x_\mu} \frac{\partial x_\mu}{\partial x'_\nu} = \sum_\mu \frac{\partial \phi}{\partial x_\mu} R_{\nu\mu} \tag{169}$$

where $R_{\nu\mu}$ is the rotation matrix/tensor defined by

$$\begin{aligned} x'_\mu &= \sum_\nu a_{\mu\nu} x_\nu \\ x_\nu &= \sum_\mu (a^{-1})_{\nu\mu} x'_\mu \end{aligned} \tag{170}$$

$A^{-1} = a^\dagger$ († means transpose), if a is orthogonal.

$$x_\nu = \sum_\mu a_{\mu\nu} x'_\mu \tag{171}$$

$$\frac{\partial \phi}{\partial x'_\mu} = \sum_\nu a_{\mu\nu} \frac{\partial \phi}{\partial x_\nu}$$

$$x'_\mu = \sum_\nu a_{\mu\nu} x_\nu \tag{172}$$

so that

$$\Box'_\mu = \sum_\nu a_{\mu\nu} \Box_\nu \tag{173}$$

and $\Box$ is a Lorentz 4-vector.

5.9 Operate with $\Box$

Operate with $\Box$ on a Lorentz 4-vector, to get the dot (inner) product:

$$\begin{aligned} \Box \cdot \tilde{x} &= \frac{\partial ct}{\partial ct} + \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \\ &= 1 + 1 + 1 + 1 = 4 = \textbf{invariant} \end{aligned} \tag{174}$$

Now operate on velocity 4-vector $\tilde{u}$:

$$\begin{aligned} \Box \cdot \tilde{u} &= \frac{\partial \gamma c}{\partial ct} + \frac{\partial \gamma v_x}{\partial x} + \frac{\partial \gamma v_y}{\partial y} + \frac{\partial \gamma v_z}{\partial z} \\ &= \frac{\partial}{\partial t}\frac{1}{\sqrt{1-\beta^2}} + \frac{\partial}{\partial x}\frac{v_x}{\sqrt{1-\beta^2}} + \frac{\partial}{\partial y}\frac{v_y}{\sqrt{1-\beta^2}} + \frac{\partial}{\partial z}\frac{v_z}{\sqrt{1-\beta^2}} \\ &= \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1-\beta^2}}\right) + \vec{\nabla} \cdot \left(\frac{\vec{v}}{\sqrt{1-\beta^2}}\right) \end{aligned} \tag{175}$$

This equation is an expression related to continuity.

5.9.1 Hydrodynamics

Conservation of fluid matter is expressed by the equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{176}$$

If one integrates this equation over a fixed volume containing mass M

$$\frac{\partial}{\partial t}\int_{vol} \rho\, dx dy dz + \int_{vol} \vec{\nabla} \cdot (\rho \vec{v}) dx dy dz = M \tag{177}$$

The first term is the mass contained in the volume and the second part is the divergence theorem and yields:

$$\frac{\partial M}{\partial t} + \int_{surface} \rho \vec{v} \cdot \hat{n}\, dS = 0 \tag{178}$$

$\frac{\partial M}{\partial t}$ = - outward transport of mass and equals the inward transport of mass.

Since our expression for $\Box \cdot \tilde{u}$ is

$$\Box \cdot \tilde{u} = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1-\beta^2}}\right) + \vec{\nabla} \cdot \left(\frac{\vec{v}}{\sqrt{1-\beta^2}}\right) \tag{179}$$

the role of density is played by $\gamma = 1/\sqrt{1-\beta^2}$.

5.9.2 Wavevector $\tilde{k}$

The 3-D wave vector $\vec{k}$ is simply $\vec{k} = \hat{v}/\lambda$ The natural 4-vector is

$$\tilde{k} = \left(\omega, \frac{\hat{n}}{\lambda}\right)$$

Note that

$$\tilde{p} = \hbar\,(\omega, k_x, k_y, k_z)$$

$$\tilde{p} \cdot \tilde{x} = \phi = \hbar(\omega t - \vec{k} \cdot \vec{x})$$

5.10 The Metric Tensor

Now before moving to make electromagnetism consistent with our relativistic mechanics, we need to generalize the concepts of the distance, vectors, vector algebra and tensors as they work in 3+1 D space.

The metric tensor defines the measurement properties of space-time. (Metric means measure – Greek: metron = a measure.)

Riemannian space in general, flat space in particular

$$(ds)^2 = \sum_{i,j} g^{ij} dx_i dx_j \tag{180}$$

by definition $g_{ij} = g_{ji}$ since the measure must be symmetric under interchange of coordinate multiplication order.

In the general case: Riemannian space

$$(ds)^2 = \sum_{i,j} g_{ij} dx^i dx^j = \text{scalar invariant} \tag{181}$$

(Note the superscripts. Section of covariant and contravariant vectors explains this.)

If g_{ij} is **diagonal**, the coordinates are **orthogonal**.

Physical interpretation: $g_{ii} = h_i^2$, where h_i is defined by the components of the vector line element, $ds_i = h_i dx_i$. An example of this is spherical polar coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{182}$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{183}$$

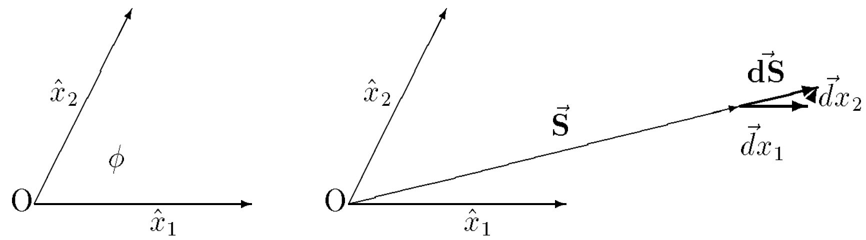
For the 3 + 1 dimension Minkowski space-time

$$ds^2 = d(c\tau)^2 = d(ct)^2 - dx^2 - dy^2 - dz^2 \tag{184}$$

$$g_{\nu\mu} \equiv \eta_{\nu\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{185}$$

In general the symbol $\eta_{\nu\mu}$ is used to denote the Minkowski metric. Usually it is displayed in rectangular coordinates (ct, x, y, z) or (x_0, x_1, x_2, x_3) but could be expressed in spherical (ct, r, θ, ϕ) or cylindrical (ct, r, θ, z) equally well.

The off-diagonal $g_{ij} = \sqrt{h_i h_j}(\vec{ds}_i \cdot \vec{ds}_j)$ for $i \neq j$. An example is skew coordinates in two dimensions.



By the law of cosines

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + 2dx_1 dx_2 cos\phi \\ &= g_{11}dx_1^2 + g_{22}dx_2^2 + g_{12}dx_1dx_2 + g_{21}dx_1dx_2 \end{aligned} \tag{186}$$

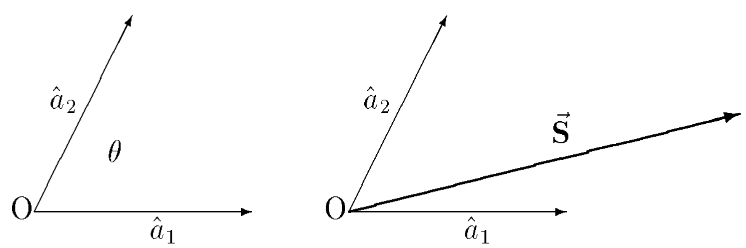
$$\begin{gathered} ds_1 = dx_1, \quad ds_2 = dx_2 \\ g_{11} = h_1^2 = 1, \quad g_{22}h_2^2 = 1 \end{gathered} \tag{187}$$

$$g_{12} = g_{21} = \sqrt{h_1 h_2} cos\phi = cos\phi \tag{188}$$

$$g_{ij} = \begin{bmatrix} 1 & cos\phi \\ cos\phi & 1 \end{bmatrix} \tag{189}$$

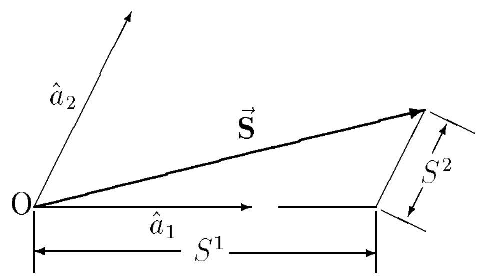
5.11 Contra & Covariant Vectors

First we consider a simple example to illustrate the significance of contravariant and covariant vectors. Consider two non-parallel unit vectors $\hat{a}_1$ and $\hat{a}_2$ in a plane with $\hat{a}_1 \cdot \hat{a}_2 = cos\theta \neq 1$.



A displacement from O to P can be represented by a vector, $\vec{\mathbf{S}}$. Its components in the directions of $\hat{a}_1$ and $\hat{a}_2$ can be denoted S^1 and S^2:

$$\vec{\mathbf{S}} = S^1\hat{a}_1 + S^2\hat{a}_2 \tag{190}$$

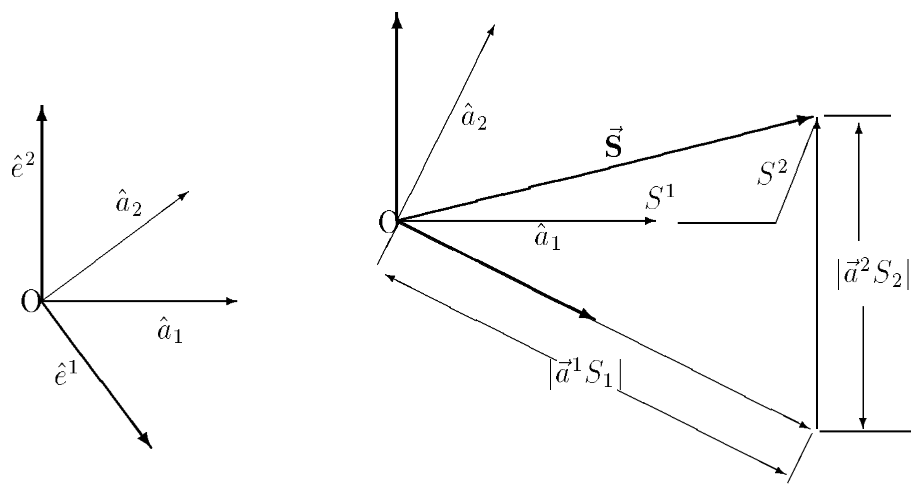


Another set of basis vectors $\vec{a}^1$ and $\vec{a}^2$, respectively, may be defined, being perpendicular to $\hat{a}_1$ and $\hat{a}_2$ and having lengths found the following way: Let $\hat{a}_3$ be a unit vector normal to the plane, proportional to $\hat{a}_1 \times \hat{a}_2$. Then

$$\vec{a}^1 = \frac{\hat{a}_2 \times \hat{a}_3}{\hat{a}_1 \times \hat{a}_2 \cdot \hat{a}_3} = \frac{\hat{e}^1}{sin\theta} \tag{191}$$

$$\vec{a}^2 = \frac{\hat{a}_3 \times \hat{a}_1}{\hat{a}_1 \times \hat{a}_2 \cdot \hat{a}_3} = \frac{\hat{e}^2}{sin\theta} \tag{192}$$

We denote the triple scalar product by $[\ \ ]_{123}$.



The displacement vector, $\vec{\mathbf{S}}$ may also be expressed by its components S_1 and S_2 as follows:

$$\vec{\mathbf{S}} = S_1\vec{a}^1 + S_2\vec{a}^2. \tag{193}$$

The relations among S^1, S^2, S_1, and S_2 may be found by elementary geometry: They are:

$$v_1 = v^1 + v^2 cos\theta \tag{194}$$

$$v_1 = v^1 cos\theta + v^2 \tag{195}$$

$$v^1 = (v_1 - v_2 cos\theta)/sin^2\theta \tag{196}$$

$$v^2 = (-v_1 cos\theta + v_2)/sin^2\theta. \tag{197}$$

Using the original pair of unit vectors,

$$\begin{aligned} S^2 &= (S^1)^2 + (S^2)^2 + 2(S^1)(S^2)cos\theta \\ &= \sum_{i,j=1}^{2} g_{ij} S^i S^j \end{aligned} \tag{198}$$

with the metric tensor

$$g_{ij} = \begin{bmatrix} 1 & cos\theta \\ cos\theta & 1 \end{bmatrix} \tag{199}$$

Defined to be symmetric.

The tensor g^{ij} is defined by

$$\sum_{j=1}^{2} g^{ij} g_{jk} = \delta^i_k. \tag{200}$$

It is easy to find that

$$g^{ij} = \frac{1}{sin^2\theta} \begin{bmatrix} 1 & -cos\theta \\ -cos\theta & 1 \end{bmatrix}. \tag{201}$$

From this relation one finds that

$$S_i = \sum_j g_{i,j} S^j \tag{202}$$

and

$$S^i = \sum_j g^{i,j} S_j. \tag{203}$$

The components S^i are **contravariant** and the components S_i are **covariant**.

The square of the length of $\vec{\mathbf{S}}$ is (as given above)

$$|\vec{\mathbf{S}}|^2 = \sum_{i,j} g_{ij} S^i S^j = \sum^{i,j} g^{ij} S_i S_j, \tag{204}$$

but is given more compactly by

$$S^2 = \sum_j S_j S^j \tag{205}$$

Other relations of interest are:

$$g^{ij} = \frac{\text{Signed Minor of } g_{ij}}{\text{Det } g_{ij}} = \frac{\text{Cofactor of } g_{ij}}{g} \tag{206}$$

For this example Det $g_{ij} = g = sin^2\theta$; the cofactor of g_{ij} is $(-1)^{i+j} g_{ji} = (-1)^{i+j} g_{ij}$ because g_{ij} and g^{ij} are symmetric.

Returning to the original sets of basis vectors

$$\vec{a}^1 = \frac{\hat{a}_2 \times \hat{a}_3}{\hat{a}_1 \times \hat{a}_2 \cdot \hat{a}_3} = \frac{\hat{a}_2 \times \hat{a}_3}{[\ \ ]_{123}} \tag{207}$$

and others by cycling indices, by substitution one has:

$$\hat{a}_1 = \frac{\hat{a}^2 \times \hat{a}^3}{\hat{a}^1 \times \hat{a}^2 \cdot \hat{a}^3} = \frac{\hat{a}^2 \times \hat{a}^3}{[\ \ ]^{123}} \tag{208}$$

$$[\ \ ]^{123} = \frac{1}{[\ \ ]_{123}} = \frac{1}{sin\theta} \tag{209}$$

Also one has

$$\mathrm{Det}(g^{ij}) = \frac{1}{\mathrm{Det}(g_{ij})} = \frac{1}{sin^2\theta}. \tag{210}$$

5.12 Electric Charge

We now consider the implications for electric charge. We define electric charge density as the charge per volume, ρ. We have a law of conservation of charge: Charge cannot be created or destroyed. Thus

$$\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v}) = 0. \tag{211}$$

So the charge-current density Lorentz 4-vector

$$\tilde{j} \equiv \rho_0 \tilde{u} = (\rho c, \rho v_x, \rho v_y, \rho v_z) = (j_0, j_1, j_2, j_3) \tag{212}$$

(where $\rho = \gamma\rho_0$) and

$$\Box \tilde{j} = 0 \qquad j^{\mu}_{,\mu} = 0 \tag{213}$$

is the equation for the conservation of charge. $\tilde{j}$ is the 4-vector charge current.

Now consider the vector and scalar potentials of the electromagnetic fields.

$$\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \qquad \text{where} \qquad \vec{A} = \frac{1}{c}\int\int\int \frac{\vec{j}\ dV}{r} \\ \vec{E} &= -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \qquad \text{where} \qquad \Phi = \int\int\int \frac{\rho\ dV}{r} \end{aligned} \tag{214}$$

The Lorentz 4-vector potential is

$$\tilde{A} = (\Phi, A_x, A_y, A_z) = (A_0, A_1, A_2, A_3) \qquad \text{where} \qquad A^{\mu} = \frac{1}{c}\int\int\int \frac{j^{\mu}\ dV}{r} \tag{215}$$

Then the inner product gives

$$\begin{aligned}
\Box \cdot \tilde{A} &= \Box_\mu A^\mu \\
&= \frac{\partial \Phi}{\partial ct} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
&= \frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0
\end{aligned} \tag{216}$$

This is the equation of Lorentz gauge invariance.

5.12.1 Box on $\tilde{A}$ is a four vector

It is clear that $\tilde{j} = \rho_0 \tilde{u}$ is a four vector since $\tilde{u}$ was constructed to be one and we constructed $\tilde{j}$ as a scalar (rest frame charge density) times that four vector. However, I merely asserted that $\tilde{A}$ was a four vector. That is true only if dV/r is invariant under Lorentz transforms. We have this as an exercise for the student to show that is true. The following are hints: Show that $dV' = (1 + \beta cos\theta)\gamma dV$ and that $r' = r\gamma (1 + \beta cos\theta)$ and thus $dV'/r' = dV/r$.

5.13 Lorentz Force Law

The 3-D vector form of the force law is

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{217}$$

We need to write this in 4-D vector form to show that it is Lorentz invariant. The relativistic force law must involve the particle velocity and the simplest form is linear in the 4-D velocity. The 4-D vector form then would be

$$\tilde{F} = \frac{q}{c}\tilde{\tilde{F}} \cdot \tilde{u}, \qquad F_\mu = \frac{q}{c} F_{\mu\nu} u^\nu \tag{218}$$

To obtain the 4-D expression for the electromagnetic fields we need second rank tensors, i.e. $F_{\mu\nu}$.

Since we want the force F_μ to be rest-mass preserving, we have the requirement that $F_\mu u^\mu = 0$ and thus $F_{\mu\nu} u^\mu u^\nu = 0$. Since this must hold for all u^μ, the $F_{\mu\nu}$ must be antisymmetric.

A cartesian flat-space second rank tensor has components C_{ij}. The tensor is the sum of a symmetric tensor S_{ij} and an antisymmetric tensor A_{ij}:

$$\begin{aligned}
C_{ij} &= \frac{1}{2}(C_{ij} + C_{ji}) + \frac{1}{2}(C_{ij} - C_{ji}) \\
&= S_{ij} + A_{ij}
\end{aligned} \tag{219}$$

$$S_{ij} = S_{ji}; \qquad A_{ij} = -A_{ji} \tag{220}$$

The property of being symmetric or of being antisymmetric is preserved under orthogonal transformations.

Now construct the antisymmetric tensor in a generalized curl

$$F_{\mu\nu} = \Box_\mu A_\nu - \Box_\nu A_\mu = \partial^\mu A_\nu - \partial^\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu} \tag{221}$$

Note that

$$F_{00} = F_{11} = F_{22} = F_{33} = 0$$

$$F_{23} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = (\vec{\nabla}\times\vec{A})_x = B_x$$

Similarly, $F_{31} = B_y$, $F_{10} = B_z$.

$$F_{10} = \frac{\partial A_0}{\partial x_1} - \frac{\partial A_1}{\partial x_0} = \frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial ct} = -E_x$$

and similarly $F_{20} = -E_y$ and $F_{30} = -E_z$. So the full tensor is

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \tag{222}$$

$F_{\mu\nu}$ is the electromagnetic field tensor.

The contravariant form of the electromagnetic field tensor is

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{223}$$

One can raise and lower indices by use of the metric tensor.

$$F_{\mu\nu} = \sum_\gamma \sum_\delta g_{\mu\gamma} F^{\gamma\delta} g_{\delta\nu} \tag{224}$$

In 3-D Maxwell's equations are:

$$\begin{aligned} \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} &= \rho\frac{\vec{v}}{c} = \frac{\vec{j}}{c} \\ \vec{\nabla}\cdot\vec{E} &= \rho \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} &= 0 \\ \vec{\nabla}\cdot\vec{B} &= 0 \end{aligned} \tag{225}$$

Now we take the 4-D divergence of the electromagnetic field tensor

$$\Box \cdot \tilde{\tilde{F}} = \tilde{j}/c \tag{226}$$

which reduces to the first two Maxwell equations. The continuity equation is simply

$$j^{\mu}_{,\mu} = 0. \tag{227}$$

Since there were actually two possible ways to unify the electric and magnetic fields into a single entity, we now define the dual electromagnetic field tensor:

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix} \tag{228}$$

The second set of Maxwell's equations can be simply written as

$$\sum_{\nu} \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{229}$$

Or, if one does not wish to resort to the dual electromagnetic field tensor, then the second set of Maxwell's equations can be simply written as

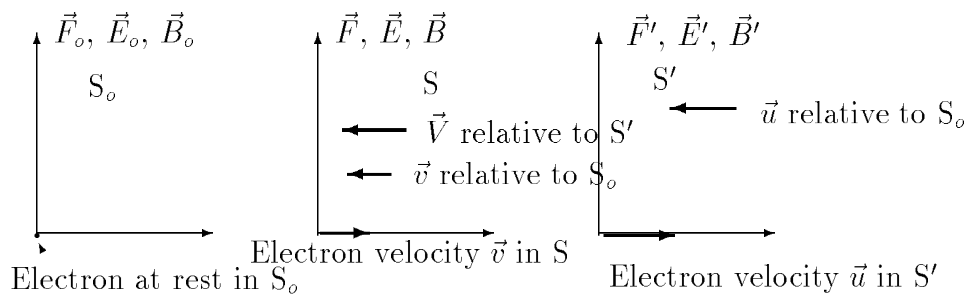
$$\partial^{\alpha} F^{\beta\gamma} + \partial^{\beta} F^{\gamma\alpha} + \partial^{\gamma} F^{\alpha\beta} = 0 \tag{230}$$

a generalized curl.

5.14 Transformation of the EM Fields

One can derive the transformation of the electromagnetic field by using the Lorentz force law $\vec{F} = q(\vec{E} + \vec{V} \times \vec{B})$ as a definition of the $\vec{E}$ and $\vec{B}$ (and by the transformation of second rank tensors as shown below.) To derive the $\vec{E}$ and $\vec{B}$ requires using three reference frames in order to see how both transform.

Do use the Lorentz force law we need a test electron or charge to probe the force and thus how the fields must transform. We consider the field acting on an electron located at the origin of three reference frames in relative motion.



The electron is at rest relative to reference frame S$_o$, moving with velocity $\vec{v}$ with respect to reference frame S, and moving with velocity $\vec{u}$ with respect to reference frame S$'$. We arrange the coordinate systems so that the velocities all lie along the

x axes. Thus the relative velocity $\vec{V}$ of the frames S and S$'$ is given by the velocity addition formula as

$$V = \frac{u+v}{1+uv/c^2}$$

We can write simple expression for the Lorentz force components in frames S, S$'$, and S$_o$, respectively:

$$\begin{array}{ccc} S & S' & S_o \\ F_x = eE_x & F'_x = eE'_x & F_{ox} = eE_{ox} \\ F_y = e(E_y - vB_z) & F'_y = e(E'_y - uB'_z) & F_{oy} = eE_{oy} \\ F_z = e(E_z + vB_y) & F'_z = e(E'_z + uB'_y) & F_{oz} = eE_{oz} \end{array}$$

Note that in S$_o$ the electron is not moving so that the magnetic field does not produce a force.

The equations for the transformation of force (for $u'_x = 0$) give

$$\begin{array}{cc} F_x = F_{ox} & F'_x = F_{ox} \\ F_y = F_{oy}\sqrt{1-v^2/c^2} & F'_y = F_{oy}\sqrt{1-u^2/c^2} \\ F_y = F_{oy}\sqrt{1-v^2/c^2} & F'_z = F_{oz}\sqrt{1-u^2/c^2} \end{array}$$

Then we have

$$\begin{array}{cc} E_x = E_{ox} & E'_x = E_{ox} \\ E_y - vB_z = E_{oy}\sqrt{1-v^2/c^2} & E'_y - uB'_z = E_{oy}\sqrt{1-u^2/c^2} \\ E_z + vB_y = E_{oz}\sqrt{1-v^2/c^2} & E'_z - uB'_y = E_{oz}\sqrt{1-u^2/c^2} \end{array}$$

We can see at once that $E_x = E'_x$. From the velocity addition law we have

$$\frac{v}{c} = \frac{u/c + V/c}{1 + (u/c)(V/c)}$$

and thus

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1 + \frac{u}{c}\frac{V}{c}}{\sqrt{1-u^2/c^2}\sqrt{1-V^2/c^2}}$$

Thus

$$\frac{E_y - vB_z}{\sqrt{1-v^2/c^2}} = E_{oy} = \frac{E'_y - uB'_z}{\sqrt{1-u^2/c^2}}$$

so that

$$\left[E_y - \frac{u+V}{1+uV/c^2}B_z\right] \times \left[\frac{1+\frac{u}{c}\frac{V}{c}}{\sqrt{1-u^2/c^2}\sqrt{1-V^2/c^2}}\right] = \frac{E'_y - uB'_z}{\sqrt{1-u^2/c^2}}$$

If these equations are to hold true for all values of u, then since the terms which contain u must be equal and those that do not must also be equal:

$$E'_y = \frac{E_y - VB_z}{\sqrt{1 - V^2/c^2}}$$

$$B'_z = \frac{-(V/c)E_y + B_z}{\sqrt{1 - V^2/c^2}}$$

Similarly by equating the expression for E_{oz} one finds

$$E'_z = \frac{E_z + VB_y}{\sqrt{1 - V^2/c^2}}$$

$$B'_y = \frac{(V/c)E_z + B_y}{\sqrt{1 - V^2/c^2}}$$

This gives the transformation law for 5 of the six components of the electromagnetic field. We are missing B_x since we started with a stationary electron in frame S_o. This can be found by considering an electron moving at right angles to B_x and recalling that the force is unchanged in the x direction. Thus $B'_x = B_x$.

Now do the derivation of field transformation from the transformation of a second rank tensor and apply that to $F_{\mu\nu}$.

$$F'_{\mu\nu} = \sum_\alpha \sum_\delta a_{\mu\alpha} a_{\nu\delta} F_{\alpha\delta} \tag{231}$$

applied to either the electromagnetic field tensor $\tilde{\tilde{F}}$ or its dual gives

$$\begin{array}{cc} E'_x = E_x & B'_x = B_x \\ E'_y = \gamma(E_y - \beta B_z) & B'_y = \gamma(B_y + \beta E_z) \\ E'_z = \gamma(E_z + \beta B_y) & B'_z = \gamma(B_z - \beta E_y) \end{array} \tag{232}$$

5.15 The Equations of Motion for a Charge Particle

The 3-D Lorentz force law

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \tag{233}$$

We can turn this into 4-D vector equation by first replacing $dt = \gamma d\tau$ and 3-vector velocity $\vec{v}$ by the 4-vector velocity $\tilde{u}$.

$$F_\mu = \frac{dp_\mu}{d\tau} = qF_{\mu\nu}u^\nu \tag{234}$$

5.16 The Energy-Momentum Tensor

First a brief review to provide motivation for the study and understanding of tensors:
(1) Electromagnetism described by a tensor field (4 by 4)
(2) Gravity represented by a tensor field (4 by 4)
(3) elastic phenomena in continuous media mechanics (classical 3 x 3)
(4) metric tensor for generalized coordinates

First we found a 4-vector equation of motion for a single particle:

$$\frac{d\tilde{p}^{\Box}}{d\tau} = \tilde{F}^{\Box} \qquad \frac{d\tilde{p}}{d\tau} = \tilde{F} \qquad \frac{dp^{\alpha}}{d\tau} = F^{\alpha} \tag{235}$$

Next we found the equation of motion for a single particle in an electromagnetic field as:

$$\frac{dp^{\alpha}}{d\tau} = m_0 \frac{du^{\alpha}}{d\tau} = \tilde{\tilde{F}}^{\alpha\beta} u_{\beta} \tag{236}$$

Later we will find that the equation of motion for a single particle in a weak gravitation field is

$$\frac{dp_{\mu}}{d\tau} = m_0 \frac{du_{\mu}}{d\tau} = \frac{1}{2}\kappa h_{\alpha\beta,\mu} m_o u^{\alpha} u^{\beta} \tag{237}$$

The last equation the second rank tensor $h_{\alpha\beta}$ is obvious but there is another simple second rank tensor there $m_o u^{\alpha} u^{\beta}$. This is an important tensor. The next paragraph supplies a little more motivation to study this important and one of the simplest that one could think to form.

In classical mechanics one has the concept that the integral of the force times distance is the work done (energy gained) and that the gradient of the potential is the force.

$$W = \Delta E = \int \vec{F} \cdot d\vec{x} \qquad \vec{F} = -\vec{\nabla} V \tag{238}$$

All this points to the need to develop the same concept in 4-D.

$$\Delta E = \int \vec{F} \cdot d\vec{x} = \int \frac{d\vec{p}}{dt} \cdot d\vec{x} = \int \frac{d\vec{x}}{dt} \cdot d\vec{p} = \int \vec{u} \cdot d\vec{p} \tag{239}$$

From the last part of the equality one finds that the integral to get the "4-potential" will involve $p^{\alpha} u^{\beta}$. The tensor $p^{\alpha} u^{\beta}$ is labeled the energy-momentum tensor. We can write out explicitly the tensor for a particle.

$$\begin{aligned} T^{\alpha\beta} &= p^{\alpha} u^{\beta} = m_o u^{\alpha} u^{\beta} \\ &= m_o c^2 \gamma^2 \begin{bmatrix} 1 & \beta_x & \beta_y & \beta_z \\ \beta_x & \beta_x^2 & \beta_x\beta_y & \beta_x\beta_z \\ \beta_y & \beta_x\beta_y & \beta_y^2 & \beta_y\beta_z \\ \beta_z & \beta_x\beta_z & \beta_y\beta_z & \beta_z^2 \end{bmatrix} \end{aligned} \tag{240}$$

since $(u^{\alpha}) = \gamma c(1, \beta_x, \beta_y, \beta_z)$.

The quantity, $\gamma^2 m_o c^2 = \gamma E$, seems a bit strange but not so when we consider a collection of particles or a continuum in density of material, ρ. $\rho = \gamma^2 \rho_o$ since one factor of γ comes from the mass increase and another factor of γ comes from the volume contraction due to length contraction along the direction of motion.

$$T^{\alpha\beta} = \rho c^2 \begin{bmatrix} 1 & \beta_x & \beta_y & \beta_z \\ \beta_x & \beta_x^2 & \beta_x\beta_y & \beta_x\beta_z \\ \beta_y & \beta_x\beta_y & \beta_y^2 & \beta_y\beta_z \\ \beta_z & \beta_x\beta_z & \beta_y\beta_z & \beta_z^2 \end{bmatrix} \tag{241}$$

and now we see that the energy-momentum tensor components are the transport of energy-momentum-component in α-direction into the β-direction.

Consider an interesting case: a large ensemble of non-interacting (elastic scattering only) particles – an ideal gas. For an ideal gas, $<\beta_i>= 0$ and $<\beta_i\beta_j>= 0$, for $i \neq j$, and $< v_x^2 >=< v_y^2 >=< v_z^2 >$, so that the energy-momentum tensor is diagonal

$$T^{\alpha\beta}_{\text{ideal gas}} = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & \rho < v_x^2 > & & \\ 0 & & \rho < v_y^2 > & \\ 0 & & & \rho < v_z^2 > \end{bmatrix} = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & & \\ 0 & & P & \\ 0 & & & P \end{bmatrix} \tag{242}$$

where ϵ is the full energy density due to the mass density, and $P = \rho < v_i >^2$ which is easily derived for an ideal gas ($PV = nkT = nm < v_i^2 >$).

We can write a simple formula for the energy-momentum tensor for a perfect fluid in a general reference frame in which the fluid moves with 4-D velocity u^μ as

$$T^{\mu\nu} = \left(\rho_0 + p/c^2\right) u^\mu u^\nu - p g^{\mu\nu} \tag{243}$$

which reduces to the equation above in its rest frame.

We want the full generalized relation between the energy-momentum tensor, $T^{\alpha\beta}$, and the 4-force to be:

$$\tilde{F} = \Box \cdot \tilde{\tilde{T}} \tag{244}$$

$$f^\mu = \sum_\nu \frac{\partial T^{\mu\nu}}{\partial x_\nu} \equiv \sum_\nu T^{\mu\nu}_{,\nu} \equiv T^{\mu\nu}_{,\nu} \tag{245}$$

where the last term represents the repeated indices summation convention. One uses $_{,\text{index}}$ indicates partial derivative with respect to x_{index} and repeated index to indicate summation on that index to make the equations easier to write and view.

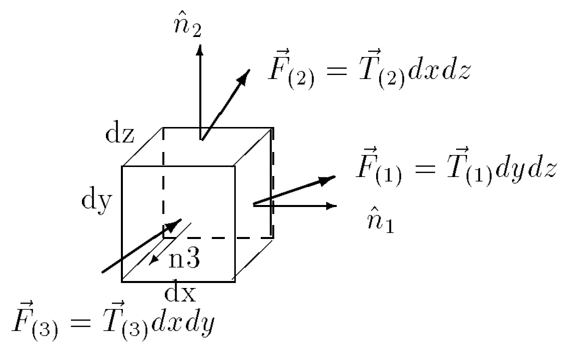
5.17 The Stress Tensor

Now we can consider the case of a medium or field that can have non-zero off-diagonal components. First it is good to review the concept of stress. Stress is defined as force per unit area, (same a pressure which is a particularly simple stress),

Imagine a distorted elastic solid or a viscous fluid such as molasses in motion. Imagine a surface (conceptual/mathematical) in the medium (The surface can and will be curved or distorted.) with a plus and a minus side and unit normal vector for every point on it. A differential area element dA, with normal $\hat{n}$ will exert forces on each of its sides. The forces are equal and opposite by Newton's second law, since the mass of the element is zero. $\vec{F}_{\text{total}} = m\vec{a} = 0$, so $\vec{F}_{+\text{ on }-} + F_{-\text{ on }+} = 0$

The force per unit area on the small element of the surface is the stress. It is a vector, not necessarily known. It underlies the dynamics of continuous media.

Consider a small piece of material at the surface



We define stress which stretches as positive and stress which compresses as negative.

Clearly each of the three axes has a vector force associated with it so that we have a second rank tensor field associated with the stress. We define the stress tensor, $E_{ij} \equiv T_{(i)j}$. Normal Stress is when the vector $T_{(i)}$ is co-directional with the normal $\pm\hat{n}_{(i)}$.

If $E_{ij} = C\delta_{ij}$, C is the hydrostatic pressure, if $C > 0$.

Simple Tension Consider $E_{ij} = C\hat{n}_i\hat{n}_j$, then $T_{(j)} = \vec{\vec{E}}_{ij} \cdot \hat{n}_i = C\hat{n}_i\hat{n}_j \cdot \hat{n}_i = C\hat{n}_j$ thus is co-directional with $\pm\hat{n}_i$. If $\hat{m}_i$ has directional orthogonal to n_i, then $\vec{T}_{(j)} = C\hat{n}_i\hat{n}_j \cdot \hat{m}_i = 0$.

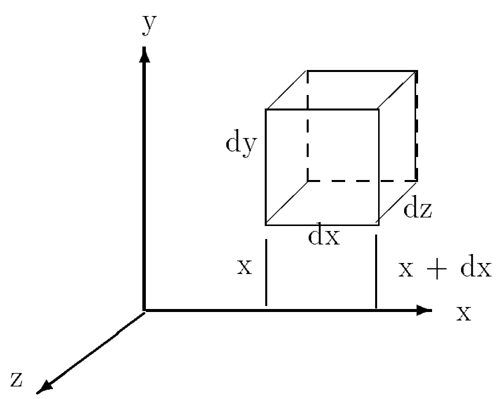
If C is negative ($C < 0$), the stress is simple compression.

Shearing Stress is specified by $\vec{\vec{E}}_{ij} = C(\hat{n}_i\hat{m}_j + \hat{n}_j\hat{m}_i)$

We will see by example the following generalization: **A simple tension in one direction and a single compression along an orthogonal direction is equivalent to a shearing stress along along shearing stress along the direction bisecting the angle between the two directions.**

In anticipation of later integration to 4-D we can call the stress tensor $E_{ij} = T_{ij} \equiv$ Force per area on the surface along the i-axis along the surface with normal in the j-direction by the material on the side with smaller x_j. Since action must equal reaction $-T_{ij} =$ force by material on the side of larger x_j.

Now return to our infinitesimal cube of the medium, with sides lined up along the cartesian coordinate planes:



The force on the back face is

$$F_x(x) = +T_{xx}(x)dydz = T_{xx}(x)dydz \tag{246}$$

The force on the front face: F_x is exerted on it toward inside in the x-direction is

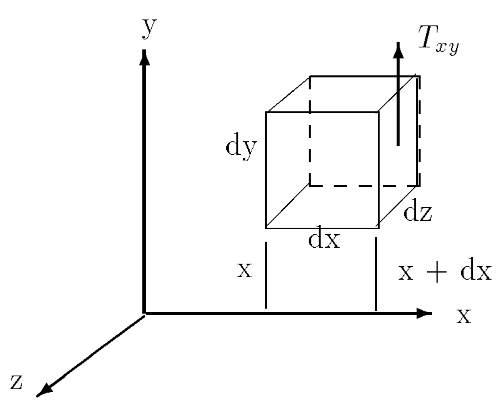
$$F_x(x+dx) = -T_{xx}(x+dx)dydz = -\left(T_{xx}(x) + \frac{\partial T_{xx}}{\partial x}dx\right)dydz \tag{247}$$

The net force on the cube is F_x is exerted on it toward inside in the x-direction is

$$F_x = -\frac{\partial T_{xx}}{\partial x}dxdydz \tag{248}$$

If $T_{xx} > 0$, inside pushes on the outside, pressure: compressive stress. If $T_{xx} < 0$, inside pulls on the outside, tension: tensile stress.

T_{xy} and T_{yx} are shear stresses.



Similarly to the treatment above the net force in the y-direction, F_y, on the front and back face is

$$F_y = -\frac{\partial T_{xy}}{\partial x}dxdydz \tag{249}$$

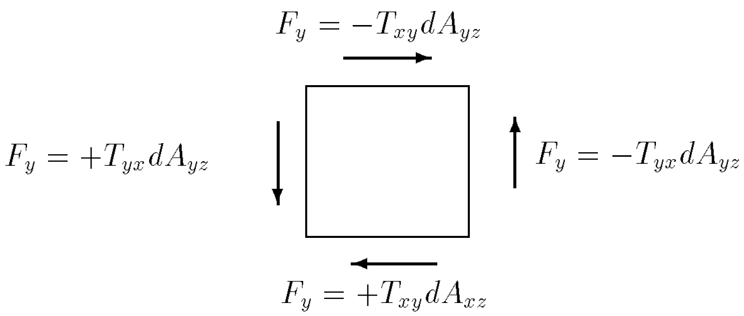
and

$$F_z = -\frac{\partial T_{xz}}{\partial x} dxdydz \tag{250}$$

Thus the total F_x on the material inside is

$$\begin{aligned} F_x \text{ total} &= -\left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{zx}}{\partial x}\right) dxdydz \\ F_i &= -\sum_{i=1}^{3} \frac{\partial T_{ij}}{\partial x^i} dV \end{aligned} \tag{251}$$

Now consider F_y on the two faces perpendicular to x and F_x on the two faces perpendicular to y as exerted from the outside.



The sign changes because from the surface the force is toward the inside. Now calculate the net torque. The two x faces have a counter-clock-wise torque:

$$\text{torque from x} - \text{face} = \text{Force } \times \text{ moment arm} = (T_{xy} dydz) dx/2 \tag{252}$$

$$\text{torque from y} - \text{face} = -(T_{yx} dxdz) dy/2 \tag{253}$$

To the net torque is

$$\tau = (T_{xy} - T_{yx}) dxdydz/2 = I\frac{d\omega}{dt} \tag{254}$$

where $I \propto mr^2 \sim \rho(dxdydz)r^2$ is the moment of inertia and $d\omega/dt$ is the angular acceleration so that

$$T_{xy} - T_{yx} \propto \rho r^2 \frac{d\omega}{dt} \tag{255}$$

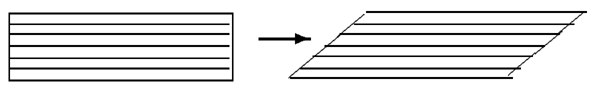
as we consider an infinitesimal cube, $r^2 \to 0$ so that

$$T_{xy} = T_{yx} \tag{256}$$

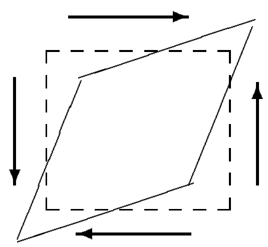
which means the stress tensor must be symmetric. The stress tensor is symmetric, so only six independent components.

5.18 Consideration of Shear

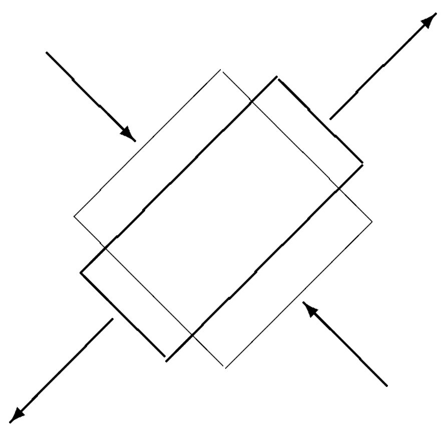
Simple shear displacement is like sliding a deck of cards.



A pure shear displacement keeps the center at the same place and is what our four forces try to do:



If the little cube is cut differently, e.g. cut at 45° to the previous cube, a different effect occurs:



Thus pure shear is a superposition of tensile and compressive stresses of equal size at right angles to each other.

Let us follow our example of shear a little further:

$$T_{ij} = \begin{bmatrix} 0 & T_{xy} & 0 \\ T_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{257}$$

We can look at the transformation properties by considering on the 2×2 portion. Now rotate the axes 45°. How do the tensor components change?

$$S'_{ij} = \sum_k \sum_l a_{ik} a_{jl} S_{kl} \tag{258}$$

where a_{ik} is the matrix for the coordinate transformation, rotation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{259}$$

For 45°, the rotation matrix is:

$$[A_{ij}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{260}$$

so that

$$\begin{aligned} T_{11} &= (a_{11})^2 T_{11} + a_{12}a_{11}T_{21} + a_{11}a_{12}T_{12} + a_{12}a_{21}T_{22} \\ &= \frac{1}{2}(T_{11} + T_{21} + T_{12} + T_{22}) = T_{21} = T_{12} \end{aligned} \tag{261}$$

$$\begin{aligned} T_{12} &= a_{11}a_{21}T_{11} + a_{11}a_{22}T_{12} + a_{12}a_{21}T_{21} + a_{12}a_{22}T_{22} \\ &= \frac{1}{2}(-T_{11} + T_{12} - T_{21} + T_{22}) = 0 \end{aligned} \tag{262}$$

$$\begin{aligned} T_{22} &= a_{21}a_{21}T_{11} + a_{21}a_{22}T_{12} + a_{22}a_{21}T_{21} + a_{22}a_{22}T_{22} \\ &= \frac{1}{2}(T_{11} - T_{12} - T_{21} + T_{22}) = -T_{21} = -T_{12} \end{aligned} \tag{263}$$

So that for the 45° rotation we have

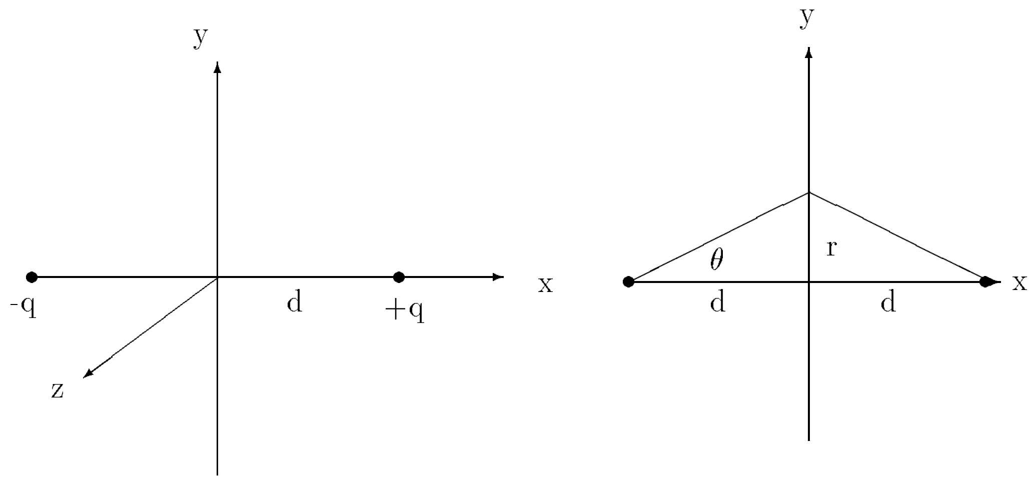
$$T'_{ij} = \begin{bmatrix} T_{21} & 0 \\ 0 & -T_{21} \end{bmatrix} \tag{264}$$

Thus we have shown that a pure shear stress rotated by 45° is equivalent to equal amounts of tension and compression stress at right angles to each other with the pure shear bisecting the angle they make.

5.19 Electric and Magnetic Stress

In this section we see that using the Faraday lines of force concept that both the electric and magnetic field lines can be under tension or compression and thus by the argument just above under shear stress.

First consider two opposite charges, magnitude q, a distance $2d$ apart, located symmetrically opposite the origin on the x-axis. The force between them is $F = q^2/(4d^2)$ according to the Coloumb law. We can imagine putting a metal plate (perfect conductor) in the $y-z$ plane and know that an image charge will form and have the same force on it and thus the plate. This makes sense in terms of the Faraday lines of force. We can calculate the total integrated mean square value of the electric field in the $y-z$ plane.



The only non-zero component is $E_x = 2qcos\theta/r^2 = 2qd/r^3$ where $r^2 = \rho^2 d^2$.

$$\int E_x^2 dA = 4q^2d^2 \int_0^\infty \frac{2\pi\rho d\rho}{(\rho^2+d^2)^3} = 4\pi q^2 d^2 \int_{\rho=0}^\infty \frac{d[\rho^2+d^2]}{(\rho^2+d^2)^3} = 4\pi q^2 d^2 \frac{1}{2(\rho^2+d^2)^2}|_{\rho=\infty}^{\rho=0} = \frac{2\pi q^2}{d^2} \tag{265}$$

The actual force between the charges is $q^2/(4d^2)$, so that the force per unit area in field must be $\frac{E^2}{8\pi}$ which is a tensile stress and is along the lines of electric field.

Now consider the same situation but with both charges having the same sign. In this case the lines bend and become tangent to the $y-z$ plane and are clearly in compression. By symmetry the only non-zero component of the electric field is that that goes radially (in the $\hat{\rho}$ direction).

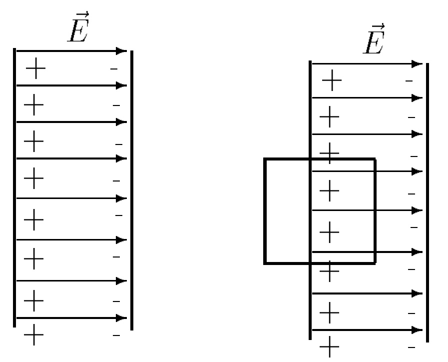
$$E_\rho^2 = \frac{2q}{d^2} sin\theta = \frac{2q}{d^2}\frac{\rho}{r} = \frac{2q\rho}{r^3}$$

where $E_\rho^2 = E_y^2 + E_z^2$. Again we can compute the total integrated mean square electric field strength in the $y-z$ plane:

$$\int E_\rho^2 dA = 4q^2 \int_0^\infty \frac{\rho^2}{r^6} 2\pi\rho d\rho = 4\pi q^2 \int_{d^2}^\infty \frac{(r^2)-d^2}{(r^2)^3} d(r^2) = 4\pi q^2 \left[\frac{1}{r^2} - \frac{d^2}{2(r^2)}\right]|_\infty^{d^2} = \frac{2\pi q^2}{d^2} \tag{266}$$

Thus again we find the compressive stress perpendicular to the electric field lines is $E^2/8\pi$.

Consider another simple case of tension along the lines of electric field, which is the familar simple capacitor.

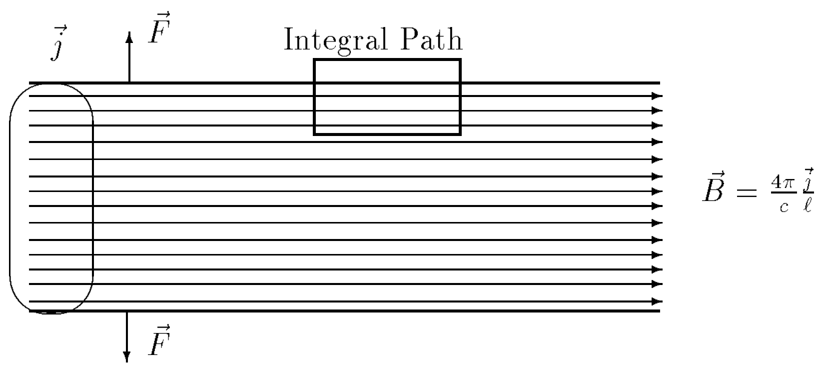


Clearly the lines of force, electric field lines are under tension. We can consider the charge on each of the capacitor faces to have a surface charge density equal to σ. Then by Gauss's law we can construct the usual pill box which has a uniform electric field passing though the face with area A and not on the sides or outside face. Thus in Gaussian units $4\pi\sigma = E$ (in Heavyside-Lorentz units, $\sigma = E$) and the force between the plates per unit area is

$$\frac{F}{A} = \frac{E\sigma}{2} = \frac{E^2}{8\pi} \tag{267}$$

(or in Heavyside-Lorentz units, $E^2/2$).

Now we turn to magnetic stress. First consider a very long solenoid or a current sheet.



The magnetic field is parallel to the solenoid and

$$\int \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c}\vec{j}$$

so that $B = 4\pi j/c\ell$. The Lorentz force on the current is

$$\vec{F} = q(\vec{v}\times\vec{B}) = \vec{j}\times\vec{B}$$

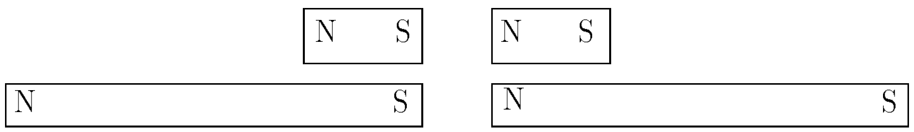
The force per unit area is equal to the average of the magnetic field at each edge of the solenoid or for an ideal solenoid this is half the internal magnetic field. We then have pressure stress

$$P_{\text{magnetic}} = \frac{cB^2}{8\pi} \tag{268}$$

The factor c depends upon the units one uses. Thus we see that like the electric field, the magnetic field can have compression perpendicular to the magnetic field lines.

Now we observe tension along magnetic field lines. Consider two magnets placed with poles near each other. If the poles are opposite, the magnets are attracted – tension in the direction of the lines. If the poles are the same, the magnets are repulsed – compression perpendicular to the lines.

We can see that this reduces to exactly the same case as for the charges calculated above by considering two long magnets.



As the magnets get longer and longer, each pole acts exactly as if it is an isolated charge and the math is the same.

Now we see that we need to have a momentum-energy tensor or more properly stress-energy tensor for electromagnetism.

5.20 Stress-Energy Tensor

We need to generalize this to 4-vectors and Lorentz invariance. This will require the use of second rank tensor - the stress-energy tensor.

In relativistic mechanics for continuous media the energy-momentum or stress-energy tensor, $T^{\alpha\beta}$, is usually defined as:

$$T^{ij} = \rho u^i u^j - E^{ij}; \qquad T^{i0} = T^{0i} = \rho u^i; \qquad T^{00} = \rho \tag{269}$$

where ρ is the density and E^{ij} is the Cartesian stress tensor usually defined as the tensor that describes the surface forces on a differential cube around the point in question. The normal surface force is pressure but there can be terms for tension/compression and shearing stress.

Then the equations of motion of a continuous medium is

$$\sum_\alpha \frac{\partial T^{\alpha\beta}}{\partial x_\alpha} \equiv T^{\alpha\beta}_{,\alpha} = f^\beta \tag{270}$$

where f^β is the 4-force density. That is the net force on material in a volume V is

$$F^\beta = \int\int\int_V f^\beta d^3V = \int\int_{surface} T^{\alpha\beta} dA_\beta \tag{271}$$

where the last equality comes from invoking Stoke's theorem.

In the case of electromagnetism in the 3-dimensional form the parallel equations are

$$\vec{F} = \int\int\int_V \left(\vec{E} + \vec{v} \times \vec{B}\right) \rho d^3V = \int\int\int_V \left(\rho\vec{E} + \vec{j} \times \vec{B}\right) d^3V \tag{272}$$

Thus the force density $\vec{f}$ is

$$\vec{f} = \rho\vec{E} + \vec{j} \times \vec{B} \tag{273}$$

Now we want to replace ρ and $\vec{j}$ by the fields via Maxwell's equations.

$$\rho = \vec{\nabla} \cdot \vec{E}, \qquad \vec{j} = \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Thus

$$\vec{f} = (\vec{\nabla} \cdot \vec{E})\vec{E} + (\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\times\vec{B}$$

Through suitable use of Maxwell's equations this can be recast to

$$\vec{f} = (\vec{\nabla} \cdot \vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E}) + (\vec{\nabla} \cdot \vec{B})\vec{B} - \vec{B}\times(\vec{\nabla}\times\vec{B}) - \frac{1}{2}\vec{\nabla}\left(E^2 + B^2\right) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

This is not a particularly elegant expression but is symmetrical in $\vec{E}$ and $\vec{B}$. The approach can be simplified by introducing the **Maxwell Stress Tensor**,

$$T_{ij} = \left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \tag{274}$$

For example the indices i and j can refer to the coordinates x, y, and z, so that the Maxwell Stress Tensor has a total of nine components (3×3). E.g. with ϵ_0 and μ_0 explicitly stated instead of the units we usually use with c

$$T_{ij} = \begin{bmatrix} \frac{1}{2}\epsilon_0\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right) & \epsilon_0\left(E_xE_y\right) + \frac{1}{\mu_0}\left(B_xB_y\right) & \epsilon_0\left(E_xE_z\right) + \frac{1}{\mu_0}\left(B_xB_z\right) & \\ \epsilon_0\left(E_xE_y\right) + \frac{1}{\mu_0}\left(B_xB_y\right) & \frac{1}{2}\epsilon_0\left(E_y^2 - E_z^2 - E_x^2\right) + \frac{1}{2\mu_0}\left(B_y^2 - B_z^2 - B_x^2\right) & \epsilon_0\left(E_xE_y\right) + \frac{1}{\mu_0}\left(B_xB_y\right) & \epsilon_0\left(E_xE_z\right) + \frac{1}{\mu_0}\left(B_xB_z\right) \\ \epsilon_0\left(E_xE_z\right) + \frac{1}{\mu_0}\left(B_xB_z\right) & \epsilon_0\left(E_yE_z\right) + \frac{1}{\mu_0}\left(B_yB_z\right) & \frac{1}{2}\epsilon_0\left(E_z^2 - E_x^2 - E_y^2\right) + \frac{1}{2\mu_0}\left(B_z^2 - B_x^2 - B_y^2\right) & \end{bmatrix}$$

And thus the force per unit volume is then

$$\vec{f} = \vec{\nabla} \cdot \vec{\vec{T}} - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} \tag{275}$$

And by Stoke's Law

$$\vec{F} = \int\int_{surface} \vec{\vec{T}} \cdot d\vec{A} - \frac{1}{c^2}\frac{d}{dt}\int\int\int_V \vec{S}d^3V \tag{276}$$

This turns out to be a much more compact equation in 4-D vector notation.

For 4-dimensions the force law is $f^\mu = F^{\mu\nu} j_\nu$.

We want the full generalized relation between the energy-momentum tensor, $T^{\alpha\beta}$, and the 4-force to be:

$$\tilde{F} = \Box \cdot \tilde{\tilde{T}} \tag{277}$$

$$f^\mu = \sum_\nu \frac{\partial T^{\mu\nu}}{\partial x_\nu} \equiv \sum_\nu T^{\mu\nu}_{,\nu} \equiv T^{\mu\nu}_{,\nu} \tag{278}$$

where the last term represents the repeated indices summation convention. One uses $_{,\text{index}}$ indicates partial derivative with respect to x_{index} and repeated index to indicate summation on that index to make the equations easier to write and view.

For example,

$$\begin{aligned} f_x &= T_{xx,x} + T_{xy,y} + T_{xz,z} \\ \text{force}_\text{x} &= \Delta\text{pressure} + \Delta\text{shear stress} \end{aligned} \tag{279}$$

For electromagnetism the force equation is

$$f_\mu = F_{\mu\nu} j_\nu = F_{\mu\nu} F_{\nu\sigma,\sigma} \tag{280}$$

since $F_{\nu\sigma,\sigma} = j_\nu$. Thus we have

$$T_{\mu\nu,\nu} = F_{\mu\nu} F_{\nu\sigma,\sigma} \tag{281}$$

A tensor satisfying this equation is

$$T_{\mu\nu} = -\frac{1}{4\pi}\left[F_{\mu\alpha}F^\alpha_\nu - \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{282}$$

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\alpha}F^\nu_\alpha - \frac{1}{4}\delta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] \tag{283}$$

$$T^{\mu\nu} = -\frac{1}{4\pi}\left[F^{\mu\alpha}F^\nu_\alpha - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] \tag{284}$$

First consider the Maxwell stress tensor,

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \tag{285}$$

$$T_{xx} = \frac{\epsilon_0}{2}\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right) \tag{286}$$

$$T_{xy} = \epsilon_0\left(E_xE_y\right) + \frac{1}{\mu_0}\left(B_xB_y\right) \tag{287}$$

and so on. Bear in mind that the stress tensor is symmetric. It is also possible to add some additional terms.

$$T^{00} = \frac{1}{8\pi}\left(E^2 + B^2\right) + \frac{1}{4\pi}\vec{\nabla}\cdot(\Phi\vec{E}) \tag{288}$$

$$T^{0i} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)_i + \frac{1}{4\pi}\vec{\nabla}\cdot(A_1\vec{E}) \tag{289}$$

$$T^{i0} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)_i + \frac{1}{4\pi}\vec{\nabla}\times(\Phi\vec{B}) - \frac{\partial}{\partial x_0}(\Phi E_i) \tag{290}$$

The added terms uses the free field $\vec{j} = 0$ Maxwell equations and included for completeness. If the fields are reasonably localized, then T^{00} is the field energy density, and the $T^{0i} = cP^i_{field}$ is the components of the field momentum density or the Poynting vector $\vec{S}$. Thus a simplified form is

$$T_{\mu\nu} = \begin{bmatrix} \frac{1}{8\pi}\left(E^2+B^2\right) & \vec{S} \\ \vec{S} & \text{Maxwell Stress Tensor} \end{bmatrix} \tag{291}$$

$$T_{\mu\nu} = \frac{1}{4\pi}\begin{bmatrix} \frac{E^2+B^2}{2} & E_yB_z - E_zB_y & E_zB_x - E_xB_z & E_xB_y - E_yB_x \\ E_yB_z - E_zB_y & \frac{(E_x^2-E_y^2-E_z^2)+(B_x^2-B_y^2-B_z^2)}{2} & E_xE_y + B_xB_y & E_xE_z + B_xB_z \\ E_zB_x - E_xB_z & E_xE_y + B_xB_y & \frac{(E_y^2-E_x^2-E_z^2)+(B_y^2-B_x^2-B_z^2)}{2} & E_yE_z + B_yB_z \\ E_xB_y - E_yB_z & E_xE_z + B_xB_z & E_yE_z + B_yB_z & \frac{(E_z^2-E_x^2-E_y^2)+(B_z^2-B_x^2-B_y^2)}{2} \end{bmatrix} \tag{292}$$

5.21 Bopp Theory

In classical electromagnetic theory there are two additional factors that must be taken into account: (1) the finite speed of light which means that the charge distribution can change and the change only propagates at the speed of light and (2) the 1/r form of the potential means that any point charge has infinite energy. To take into account the motion of charges one must end up using retarded potentials. In 3-D one has:

$$A_i(t,\vec{x}_1) = \frac{1}{c}\int \frac{j_i(t - r_{12}/c, \vec{x}_2)}{r_{12}} dV_2 \tag{293}$$

Bopp suggested a simpler form of the 4-vector potential which he thought might handle both problems:

$$A_\mu(\vec{x}_1) = \int\int\int\int j_\mu(t_2,\vec{x}_2) f(s_{12}^2) dV_2 dt_2 \tag{294}$$

Where $f(s_{12}^2)$ is a function which is zero every where but peaks when the square of the 4-vector distance s_{12}^2 between the source (2) and the point of interest (1) is very small. The integral over $f(s_{12}^2)$ is also normalized to unity. The Dirac delta function is the limiting case for $f(s_{12}^2)$. Thus $f(s_{12}^2)$ is finite only for

$$s_{12}^2 = c^2(t_1 - t_2)^2 - r_{12}^2 \approx \pm\epsilon^2 \tag{295}$$

Rearranging and taking the square root

$$c(t_1 - t_2) \approx \sqrt{r_{12}^2 \pm \epsilon^2} \approx r_{12}\sqrt{1 \pm \frac{\epsilon^2}{r_{12}^2}} \approx r_{12}(1 \pm \frac{\epsilon^2}{2r_{12}^2}) \tag{296}$$

So

$$(t_1 - t_2) \approx \frac{r_{12}}{c} \pm \frac{\epsilon^2}{2cr_{12}} \tag{297}$$

which says that the only times t_2 that are important in the integral of A_μ are those which differ from the time t_1, for which one is calculating the 4-potential, by the delay r_{12}/c ! – with negligible correction as long as $r_{12} \gg \epsilon$. Thus the Bopp theory approaches the Maxwell theory as long as one is far away from any particular charge.

By performing the integral over time one can find the approximate 3-D volume integral by noting that $f(s_{12}^2)$ has a finite value only for $\Delta t_2 = 2 \times \epsilon^2/2r_{12}c$, centered at $t_1 - r_{12}/c$. Assume that $f(s_{12}^2 = 0) = K$, then

$$A_\mu(\vec{x}_1) = \int j_\nu(t_2,\vec{x}_2) f(s_{12}^2) dV_2 dt_2 \approx \frac{K\epsilon^2}{c}\int \frac{j_\nu(t - r_{12}/c, \vec{x}_2)}{r_{12}} dV_2 \tag{298}$$

which is exactly the 3-D version shown above if we pick K so that $K\epsilon^2 = 1$.

This manner of thinking eventually leads one to the interaction Lagrangian as a the product of the two currents (electrical, matter, strong, weak, gravitational).

5.22 The Principle of Covariance

The laws of physics are independent of the choice of space-time coordinates.

Special Relativity applies this only to the choices of Euclidean (pseudo-Euclidean), non-rotating coordinate moving with constant velocities with respect to each other. That is by definition inertial frames.

General Relativity applies this to all conceivable space-time coordinates: rotating, accelerating, distorting, non-Euclidean, non-orthogonal, etc.

Einstein said that this principle is an inescapable axiom, since coordinates are introduced only by thought and cannot affect the workings of Nature.

Therefore the Principle of Covariance cannot have Physical Content to determine the laws of any part or field of physics.

Tensors are essential because all tensor equations of proper form are manifestly covariant; their functional form does not change when coordinates are changed. (Proper form means that both sides of the equation result in tensors of the same rank and, if the equation matches the classical limit formula, then it is the **only** correct form. *get the stuff in these parentheses, precisely right.*)

The form of a tensor equation provides no guide for selecting a particular "fixed" or "at rest" coordinate system. However, its content may provide this.

Covariance Language has heuristic invariance:

(1) It guides in proceeding, without telling where to go.

(2) It helps to prevent errors from staying with particular coordinates (through oversight or error).

(3) One should take as a first approximation to physical laws those which are *simple* in tensor language, but not necessarily simple in a particular coordinate system.

6 POINT OF VIEW

In this chapter we consider relativistic effects from different points of view. In essentially all the cases we have done before, we have assumed that we had a complete reference frame of meter sticks and clocks so that we could determine lengths and times at any place in space-time. This I refer to as the physicist's god-like view provided by his reference frame and ancillary tools. This concept of reference frames comes to us from Galileo and Newton.

Most mere mortals, such as astronomers and individuals, have more limited access to data about remote objects. In general, especially for astronomy, the observer either sits at a point in space-time and images light coming to his instrument – eye, telescope, camera, etc. – or sits at a point in space and observes the light arriving as a function of time.

The result of being limited to a single point of view, instead of the physicist's god-like plan view is to observe very different relativistic behavior than we have considered so far. One can observe cases of a moving clock running faster. Radio astronomers observe many objects moving superluminally (that is with velocities faster than light), and fast moving objects appear very differently than a resting object at the same place. Sometimes one can not see the front of an approaching object but can see the back.

We consider some of these effects in the following sections.

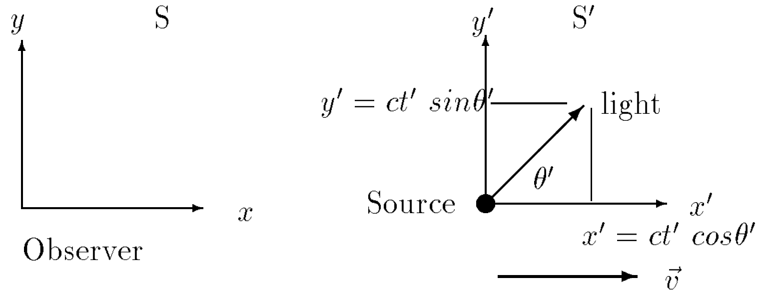
7 The Relativistic Doppler Effect

From the point of view of a single observer confine to a location in space, a moving clock can run either faster or slower than an identical clock at rest with respect to the observer depending upon its velocity (direction and speed of motion). We consider the case of a clock that is a light source with a particular frequency and work out the relativistic Doppler shift. The frequency can be considered the beats of the clock.

We work the problem out by considering two different inertial frames and use the Lorentz transformations in order to determine what a single-place observer would see.

7.1 Ray Optics Approach

First, go to the frame S$'$ where the source is at rest and emits light at frequency $\nu' = \nu_o$. Now consider a pulse light going in the direction θ' relative to the x'-axis.



Now consider the frame S, where the source is moving in the x direction with velocity (speed) v, and consider the path of the light in this frame. We can use the Lorentz transformations to calculate the location of a light pulse emitted at time $t' = 0$ and trace its path as a light ray.

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} = \frac{ct'cos\theta' + vt'}{\sqrt{1 - v^2/c^2}} = \frac{ct'\,(cos\theta' + v/c)}{\sqrt{1 - v^2/c^2}}$$

$$y = y' = ct'sin\theta'$$

By taking the ratio of y over x when can find $tan\theta$

$$tan\theta = \frac{y}{x} = \frac{sin\theta'}{cos\theta' + v/c}\sqrt{1 - v^2/c^2} = \frac{1}{\gamma}\frac{sin\theta'}{cos\theta' + v/c} \tag{299}$$

This is the full relativistic aberration of light formula derived by ray optics argument. This is the same result as found using the Lorentz contraction and ether approach.

Now using the Lorentz transform for t and then t' we can derive a formula for the relative rate at which clocks appear to run.

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - v^2/c^2}} = \frac{t' + \frac{v}{c}t'cos\theta'}{\sqrt{1 - v^2/c^2}} = t'\frac{\left(1 + \frac{v}{c}cos\theta'\right)}{\sqrt{1 - v^2/c^2}} = \gamma t'\left(1 + \frac{v}{c}cos\theta'\right)$$

Similarly and symmetrically

$$t' = t\frac{\left(1 - \frac{v}{c}cos\theta\right)}{\sqrt{1 - v^2/c^2}} = \gamma t\left(1 - \frac{v}{c}cos\theta\right)$$

Taking the derivative of t with respect to t' and vice versa and inverting we find the relations

$$\frac{dt}{dt'} = \gamma\left(1 + \frac{v}{c}cos\theta'\right) = \left[\gamma\left(1 - \frac{v}{c}cos\theta\right)\right]^{-1} \tag{300}$$

Note that it matters whether one uses the angle θ or θ' because of the aberration of angles. The frequency of clock ticks would be:

$$\nu = \nu'\gamma\left(1 + \frac{v}{c}cos\theta'\right) = \nu'\left[\gamma\left(1 - \frac{v}{c}cos\theta\right)\right]^{-1} \tag{301}$$

7.2 Phase of Plane Wave Approach

Now we can calculate the direction and wavelength or frequency of light observed by considering the phase of a plane wave traveling in the same direction θ' in the frame S$'$ where the light source is at rest. Remember the relationship between wavelength λ, frequency ν, and the speed of light c: $\lambda_o \nu_o = c$

$$\Phi = 2\pi \left[\nu_o t' - \frac{x' cos\theta' + y' sin\theta'}{\lambda_o} \right]$$

apply the Lorentz transforms expressing x', t' in terms of x and t and $y' = y$ to obtain:

$$\Phi = 2\pi \left[\nu_o \gamma \left(t - \frac{v}{c^2} x \right) - \frac{cos\theta'}{\lambda_o} \gamma (x - vt) - \frac{sin\theta'}{\lambda_o} y \right]$$

Now in the laboratory or observer rest frame coordinates

$$\Phi = 2\pi \left[\nu t - \frac{cos\theta}{\lambda} x - \frac{sin\theta}{\lambda} y \right] = 2\pi \left[\nu \gamma \left(t' + \frac{v}{c^2} x' \right) - \frac{cos\theta}{\lambda} \gamma (x' + vt') - \frac{sin\theta}{\lambda} y \right]$$

Since we realize that the phase must be the same in the two frames, we can compare the previous equations and obtain the coefficients for t, x, and y which must be the same. I.e. for t

$$\nu = \gamma \nu_o + \frac{cos\theta'}{\lambda_o} \gamma v = \gamma \nu_0 \left(1 + \frac{v}{c} cos\theta' \right)$$

Collecting the coefficients for t' yields

$$\nu_o = \gamma \nu \left(1 - \frac{v}{c} cos\theta \right)$$

These are the relativistic Doppler effect for frequency

$$\nu = \gamma \nu' \left(1 + \frac{v}{c} cos\theta' \right) = \nu' / \left[\gamma \left(1 - \frac{v}{c} cos\theta \right) \right] \tag{302}$$

These are the same equations we got for the ratio of clock running rates using the geometrical ray tracing.

We can also find aberration of angles, started by setting the coefficients for x and y equal from the two equations for the phase.

$$\frac{cos\theta}{\lambda} = \gamma \frac{cos\theta'}{\lambda_o} + \gamma \nu_o \frac{v}{c^2}$$

$$\frac{sin\theta}{\lambda} = \frac{sin\theta'}{\lambda_o}$$

where we make use of the relationship $\lambda' \nu' = \lambda_o \nu_o = c = \lambda \nu$. The ratio of these equations gives

$$tan\theta = \frac{sin\theta'}{\gamma (cos\theta' + v/c)}$$

is the same aberration from ray optics above. This is natural since one is geometrical (ray) optics and the other wave but rays propagate normal to wave fronts.

7.3 Special Cases

7.3.1 Doppler shift parallel to direction of observation

Consider the special case when the source is approaching or receding directly. That is to say that the velocity of the source is parallel to the line of sight. Then both versions of the formula yield the following relationship

$$\nu = \nu' \sqrt{\frac{1+\beta}{1-\beta}}$$

This is left as an exercise to the reader to show this and to show that the equation is exactly symmetrical on reversal of the frames

$$\nu' = \nu \sqrt{\frac{1+\beta'}{1-\beta'}} = \nu \sqrt{\frac{1-\beta}{1+\beta}}$$

7.3.2 Doppler shift perpendicular to direction of observation

The case of motion perpendicular to the direction of observation (in the observation frame). is quite simple:

$$\nu = \nu'/\gamma \qquad \nu' = \gamma\nu$$

This is called the transverse Doppler shift and is simply a result of time dilation as one would anticipate.

7.3.3 Fresnel's Velocity Dragging Coefficient

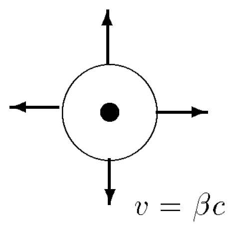
$$u = u' + vcos\theta(1 - u'^2/c^2) = c/n + \kappa vcos\theta$$

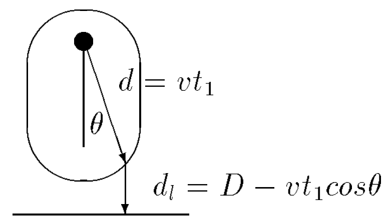
8 Superluminal

Radio astronomers routinely observe objects that they classify as superluminal. Operationally this means that a radio astronomers uses his radio telescope (often an interferometer array) to make an image of an object at multiple times and the time rate of change of the angular diameter of the astronomical object times the estimated distance to the object gives a result that implies a velocity transverse to the line of sight which is greater than the speed of light, sometimes by up to five times.

There are a number of potential explanations for these observations but nearly all can be ruled out easily by companion observations.

Consider the following scenario where the source is at rest with respect to the observer (radio astronomer) and has sent out an relativistic expanding shell of light emitting matter.





A radio astronomy telescope images the incoming wavefront which means that it accepts photons which have arrived at the telescope at the same time. Hence we need to find the locus of points on the expanding wave front which have the same total travel time to the radio telescope. This means that the sum, t_{total}, of the time $t_1 = R/v$ taken for the point on the expanding sphere to reach the point at radius $R = vt_1$ where it emits the light plus the time $t_2 = (D - Rcos\theta)/c$ it takes light to travel from the point of emission to the radio telescope. Note that D is the distance from the original expanding source to the radio telescope.

$$t_{total} = R\left(\frac{1}{v} - \frac{cos\theta}{c}\right)$$

$$R = \frac{vt}{1 - \beta cos\theta}$$

note that for $\beta \ll 1$, this radius is $R \simeq vt(1 + \beta cos\theta)$.

Note also that this is an alternate definition of an ellipse with eccentricity $e = \beta$. Usually an ellipse is geometrically defined a the locus of points for which the sum of the distance from two points is a constant. However, a more general definition of a conic section is the locus of points whose distance between a point and a line, called the directrix (in this case the wavefront), is in a constant ratio e. In this case $e = v/c$. If e is less than 1, the resulting figure is an ellipse. If e is equal 1, the resulting figure is an parabola. If e is greater than 1, the resulting figure is a hyperbola. The eccentricity e of an ellipse varies between 0 and 1 and the value of e indicates the degree of departure from circularity. (Focus is at a distance of ae from the center and the directrix is at a distance a/e from the center of the ellipse.)

The apparent diameter set by the symmetric pair of such points is twice $Rsin\theta$.

$$\text{Diameter} = 2Rsin\theta = 2vt\frac{sin\theta}{1 - \beta cos\theta}$$

The velocity perpendicular to the line of sight is

$$v_\perp = \frac{vsin\theta}{1 - \beta cos\theta}$$

We can find the maximum apparent diameter (still assuming the expanding shell is opaque and emitting light) by taking the derivative of the diameter with respect to θ setting that to zero and finding the maximum apparent diameter at time t_o.

$$\frac{d\text{Diameter}}{d\theta} = 2vt\left(\frac{cos\theta}{1-\beta cos\theta} - \frac{\beta sin^2\theta}{(1-\beta cos\theta)^2}\right)$$

$$= \frac{2vt}{(1-\beta cos\theta)^2}\left(cos\theta - \beta\right)$$

The maximum clearly occurs at

$$cos\theta = \beta; \quad sin\theta = \sqrt{1-\beta^2}; \quad \theta = cos^{-1}\beta$$

At the maximum

$$R = \frac{2vt}{1-\beta cos\theta} = \frac{2vt}{1-\beta^2} = \gamma^2 vt$$

The diameter is then

$$\text{Diameter} = 2vt\frac{sin\theta}{1-\beta cos\theta} = 2vt\frac{\sqrt{1-\beta^2}}{1-\beta^2} = \frac{2vt}{\sqrt{1-\beta^2}} = 2\gamma vt$$

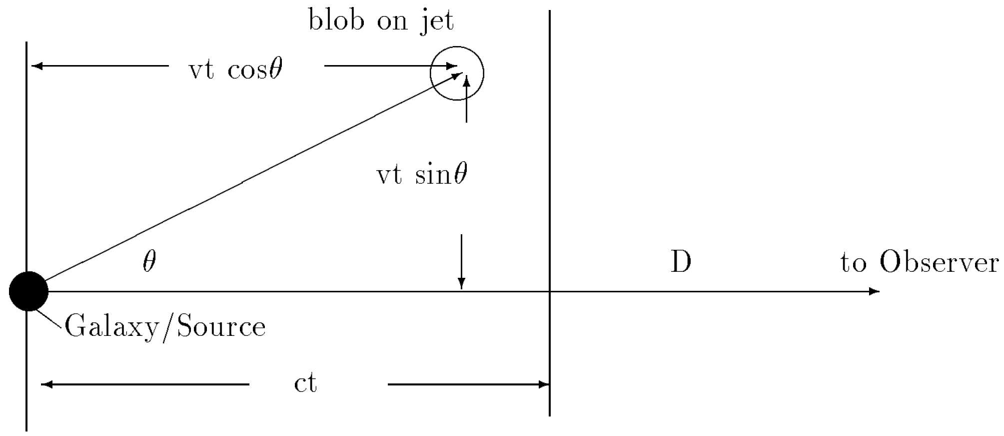
$$v_\perp = 2\gamma v$$

The subtended angle is $\simeq 2\gamma vt/D$ and the apparent velocity is γ times the expanding sphere velocity.

The most visible radio objects are double-lobe radio sources which have back-to-back relativistic jets. In practice one generally only able to measure well relativistic jet that is coming towards the observer because the Doppler effect both changes the observed temperature and intensity. The intensity of the portion coming towards the observer is typically increased by the factor 8γ and the portion moving away decreased by the same factor. See the following exercise:

8.1 Superluminal Motion Exercise

Astronomers observe a large number of radio sources that move with apparent superluminal speed. That is the rate of change of angular separation between components times the distance to the radio source gives a velocity well in excess of the speed of light ($v_{observed} = D \times d\alpha/dt$). Consider the following problem and diagram to help understand how an astronomer could measure apparent superluminal speed, if there is a relativistic beam coming from the source.



Neglect the source (host galaxy) motion relative to the observer and consider the motion of only a single blob on a radio jet. The blob moves at velocity v with respect to the galactic nucleus (and observer) beginning at time $t = 0$. Also assume that the blob and nucleus continuously emit radio waves so that they can be observed.

Consider the radio emission received as a function of time by the observing radio astronomer very far (distance D) away. Show that the observer sees the blob coincident with the galaxy source at time $t_0 = D/c$ corresponding to $t = 0$. Show also that the observer sees the blob with transverse displacement $vt\ sin\theta$ from the galactic nucleus at the time

$$t_r = t + (D - vt\ cos\theta)/c$$

Show that the elapsed time for the observer was

$$t_r - t_0 = t(1 - \beta cos\theta)$$

where $\beta = v/c$.

The apparent transverse velocity of the blob relative to the nucleus $v_{apparent-transverse}$ equals the transverse displacement divided by the time difference observed for the displacement to occur. Show that this leads to the formula:

$$\beta_{apparent-transverse} = \frac{\beta sin\theta}{1 - \beta cos\theta}$$

Plot this formula for the following values: $\beta = 0.5,\ 1$ (a special case) and $\gamma = 2, 3, 4, 5, 7, 16$.

Show that the maximum transverse velocity happens for $cos\theta = \beta$ (and thus $sin\theta = \sqrt{1-\beta^2} = 1/\gamma$), as derived in class for an expanding spherical shell, and that the maximum apparent transverse velocity is

$$\beta_{apparent-transverse-max} = \beta/\sqrt{1-\beta^2} = \gamma\beta$$

and that your graphs agree with this.

Note that for the critical angle and $\gamma >> 1$, the transverse speed is roughly $v_{apparent-transverse-max} \approx \gamma c$.

8.2 Too Rapid Time Variability

The minimum size for an astronomical object is often estimated by use of our earlier finding that no causal impulse can travel with a speed faster than the speed of light. Thus if an object is observed to vary its brightness very significantly in a given time period Δt, then it must be no larger than $d = \Delta t$ in extent. This is a good rule for non-relativistic objects. However, if the object, e.g. a jet, is moving towards the observer with relativistic speeds, then this can be compressed by a factor $\gamma(1+\beta cos\theta'))$. which can be as much as $2\gamma_{\text{object}}$.

This effect has been observed (R. A. Remillard, B. Grossan, H. V. Brandt, T. Ohashi, K. Hayashida, F. Makio, & Y Tanaka, Nature 1991 vol 350 p 589-592) in the rapid variability of an energetic X-ray flare in the quasar PKS0558-504. The quasar X-ray flux was observed to increase by 67% in three minutes while there was no significant change in the spectrum. Since we know the mass of the black hole from the limit of accretion efficiency, we know it's size. From the minimum (light) travel time across the source, we know the minimum variability time scale. The observed time is shorter, by about a factor of 16; therefore, we must have relativistic beaming.

Another interesting example of variability, however, is the time dilation of supernova light curves. Nearby Type 1A supernova are observed to have a very standard brightness and time dependence of the light curve. (This can be made even a tighter standard by the correlation between the intensity and light curve width in time.) When observed at great distances, the light from a Type 1A supernova is observed to be reddened by an amount that is consistent with a Doppler frequency shift and the light curve time taken is stretched by the same amount predicted by the relativistic Doppler shift formula. Most observed distant supernova have frequency shift factors ranging from 1.2 to 1.9. As we will see later this is evidence that the Universe is actually expanding and one can understand this stretching from a General Relativistic point of view also.

9 Appearance of Rapidly Moving Objects

Surprisingly, if an observer looks at or photographs a small fast-moving object ($\beta \approx 1$), which approaches him at even a relatively small angle, he cannot see the front of the object but can see the bottom and back. Likewise, it is impossible to see the Lorentz-Fitzgerald contraction by this technique. Instead of looking shortened along the direction of motion, an object will appear rotated. This is a combined effect of the aberration of light and the fact that our instruments (eye and camera) use the incoming wavefront from the object.

In 1959 James Terrell (J. Terrel 1959 Phy. Rev. 116, 1041) realized that the visual appearance of an object would moving at high speeds would not reveal the Lorentz contraction in the direction of motion as commonly expected. That same year Roger Penrose (R. Penrose 1959 Proc. Cambridge Philosophical Soc. 55, 137) proved that a sphere would always appear to be a sphere rather than a Lorentz-contracted ellipsoid. These and some other results were brought to physicists' general attention by a Physics Today article of Victor F. Weisskopf (1960).

The key point is that when we see of photograph an object, we record light quanta (wavefronts) emitted by the object, when they arrive simultaneously at the retina or at the photographic film. This implies that these light quanta (portions of the wavefront) were **not** emitted simultaneously by all points of the object. The points further away have emitted their part of the picture earlier than the closer points of the object. Hence, if the object is in motion, the eye or the camera gets a "distorted" picture of the object, since the object has been at different locations, when the different parts of it have emitted the light seen in the picture.

In special relativity, this distortion has the remarkable effect of canceling the Lorentz contraction so that small solid-angle objects appear undistorted and only rotated.

9.1 Appearance of a Moving Stick

We do a very simple case first. Consider a moving stick of length $\ell_o = \ell'$ in its rest frame S$'$ which is aligned with the x' axis. In frame S where you the observer is idealized as a point at the origin which can take photographs. In frame S the stick has length $\ell = \ell_o/\gamma = \ell_o\sqrt{1 - v^2/c^2}$ and is moving with velocity $+v$ along the x axis.

Consider the junior physics lab experiment where the student is asked to determine the apparent length of the stick from a point the center of the laboratory frame. Student A - Jim Photographer - sets up a camera and a self-illuminated stick and his partner, Student B - Lena Timer sets up a radar or laser ranger and a meter stick with retro-reflectors on each end.

9.1.1 Sell-Illuminated Stick

First consider the stick as a cartoon meter stick - a frame which defines the edges of the meter stick and the frame is glowing. The rest of the meter stick is transparent (not there). A view or photograph from the center of the frame S shows one rectangle (outline of far end) inside another (outline of the near end) and the corners of the two rectangles connected by lines (edges of the length of the cartoon stick). If the stick were not moving, the relative size of the rectangles is set by the ratio $D/(\ell+D)$ of their respective ends distances from the origin. But the stick is moving, thus contracted, but also the light from the more distant end must start toward the camera sooner than the light from the near end in order to arrive at the camera at the same time. This second effect is present classically and causes distortions in pictures of rapidly moving objects.

Consider first the stick moving toward (approaching) the origin. The light from the far end of the stick must catch up with the front end of the stick to continue on with the light just then emitted from the front end of the stick. In the approaching direction the light must travel the length of the stick plus the distance the stick has moved from the time the light leaves the far end of the stick until the time it reaches the front of the stick.

$$\text{distance light travels} = \text{stick length} + \text{distance moved}$$

$$c\Delta t_1 = \ell + v\Delta t_1$$

$$\Delta t_1 = \frac{\ell}{c-v}$$

$$\ell_a = \ell + v\Delta t_1 = \ell\left(1 + \frac{v}{c-v}\right) = \ell\frac{1}{1-\beta} = \ell_o\sqrt{\frac{1+\beta}{1-\beta}}$$

Thus the stick appears longer even though it is length contracted.

When the stick is receding, the light leaving the far end (now the front of the stick) must reach the near end (now the back of the stick) at the time the light leaves the near end of the stick. So the light must, once again be emitted first from the far end of the stick, but it has to travel less distance to the front because the stick is moving towards the light.

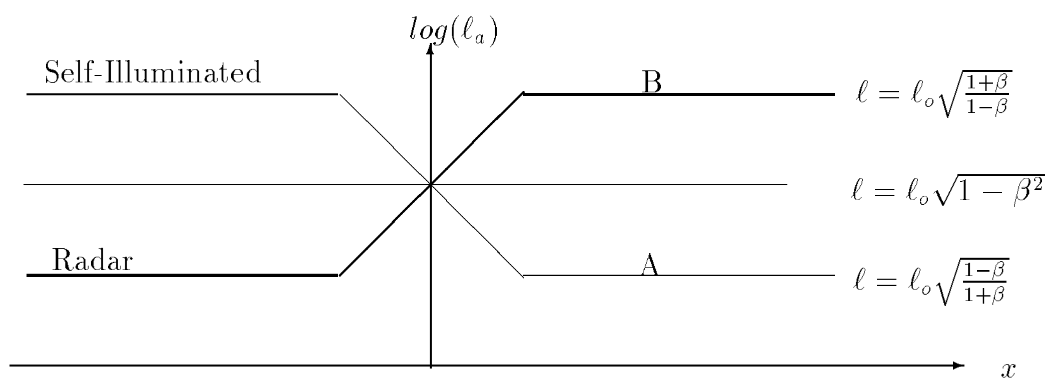
$$\text{distance light travels} = \text{stick length} - \text{distance moved}$$

$$c\Delta t_1 = \ell - v\Delta t_1$$

$$\Delta t_1 = \frac{\ell}{c+v}$$

$$\ell_a = \ell - v\Delta t_1 = \ell\left(1 - \frac{v}{c+v}\right) = \ell\frac{1}{1+\beta} = \ell_o\sqrt{\frac{1-\beta}{1+\beta}}$$

Thus the apparent length is now shorter as the stick recedes into the distance. Student A takes a lot of photographs and measures distances and ratios finally he plots up the apparent length as a function of position and finds:



9.1.2 RADAR or LIDAR-Illuminated Stick

Student B knows measuring times is easy and already has her results plotted. In her apparatus the radar or laser pulse first hits the near end of the stick and reflects back to her receiver where she records the time. The pulse then reflects from the far end of the moving stick and returns to her receiver and she records the time. The difference in times divided by $2c$ gives her the apparent observer-illuminated stick length.

We can calculate the extra time to get to the far edge (back edge of approaching stick) and find the the light pulse has to travel less than the laboratory length of the stick because the stick has moved forward to meet it. It is just the symmetric opposite case of the receding self-illuminated stick. The radar apparent length of an approaching stick is

$$\ell_a = \ell_o \sqrt{\frac{1-\beta}{1+\beta}}$$

For the receding stick the light going to the back edge to reflect has to travel the length of the stick plus the distance the stick has traveled and so the radar apparent length of the receding stick is

$$\ell_a = \ell_o \sqrt{\frac{1+\beta}{1-\beta}}$$

which is longer than the apparent length of the approaching stick.

Who is right? They both are. This is an illustration about the care one needs to take in defining the question.

Because Student B's technique was so much faster, she had plenty of time after taking the data to puzzle over the results and realizes that a lot of the effect is to be expected simply because of the finite speed of light - a necessary component of her measurement. The finite speed of light makes the approaching stick reflections closer by the factor $1-\beta$ and the receding stick's reflections further apart by the factor $1+\beta$. She corrects for this effect and finds the length of the stick is always $\ell = \ell_o\sqrt{1-\beta^2}$. She claims she has "observed" the length of the stick and it is contracted by just the Lorentz factor $\sqrt{1-\beta^2}$. The lab instructor is impressed and knows the "right" answer from the Michelson-Morely experiment and the Lorentz contraction.

Student A is miffed but also shows he is really sharp also, even if he has done the observations the hard way. He argues: "Yes, there is a classical effect, that does cause the stick to appear distorted." However if we were asking, if we can observe the Lorentz contraction by eye or camera, then a more careful analysis shows that we cannot "see" it directly but have to correct our calculations to do so. The image is actually distorted in such a way that the Lorentz contraction is hidden. Consider the following argument about the true appearance of a rapidly moving object.

9.1.3 Sell-Illuminated Small Cube

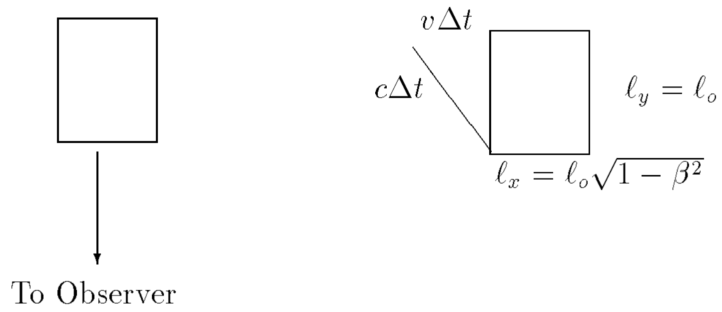
Consider a small cube moving towards the observer or camera with very large velocity. Arrange for it to pass over head by a small but reasonable amount. This is both for reality and to avoid the problem the zero in the coordinate system. We will see that aberration of light will cause the cube to appear rotated and the finite travel time of light and the rotation together just compensate for the Lorentz contraction. Thus the object appears completely normal but rotated.

If an observer looks or photographs a fast-moving object ($\beta \sim 1$) which approaches him at a small angle α of observation then, if $\alpha \gtrsim \sqrt{1-\beta^2}$, the observer no longer sees the front side of that object, but can see the backside. We can appreciate this qualitatively and then quantitatively. First consider the aberration of light.

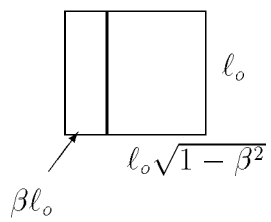
In the rest frame of the object radiation can be considered emitted isotropically. In the observer's rest frame, the radiation appears folded forward. All the radiation emitted from the forward direction ($\theta' = 0$) to right angles from the direction of motion ($\theta' = 90°$) is contained in a cone with $tan\theta = c/\gamma v$ or roughly for $\beta \sim 1$ inside a cone with half angle $\theta = 1/\gamma = \sqrt{1-\beta^2}$. Thus as the object reaches an angle higher than $\alpha - \sqrt{1-\beta^2}$ any radiation from the front of the object goes over the observer's head or camera.

In fact due to the relativistic aberration only a very small part of the light emitted backward in the rest frame of the object will go backward in the laboratory frame. What will be observe? When an object such as a cube (radiating white light in its rest frame) approaches from very far away ($\alpha < \sqrt{1-\beta^2}$), then the observer sees its front side and shortened by perspective its bottom side both radiating in the the ultraviolet. The as the cube gets closer and the observation angle (α) grows, the cube seems to turn and if $\alpha > 1/\beta\gamma$, then we see only the bottom still violet. As the observation angle becomes greater, the one not only no longer sees the front but also can see the backside and the color is less violet. When the object passes over head ($\alpha = 90°$), one observes practically only the back side of the cube, radiating in the infrared. The picture remains nearly unchanged until the cube disappears in the distance.

Now let us consider this a little more quantitatively. Consider the cube at the moment it is at right angles to the observer. (The moment that the light it emits to the observer leaves at right angles from the cube in the observer's frame.) The observer will take a picture of the cube with light arriving in a wave front where the light arrives to the eye or camera simultaneously. If the cube is small compared to the distance to the camera, then to first order all the light from the bottom surface leaves for the camera at essentially the same instant but the light from the back face of the cube must leave earlier, the higher the point on the back face of the cube. The light leaving the top of the back face of the cube must leave a time $\Delta t = \ell_o/c$ and at a position of the cube that is $d = -v\Delta t = \beta\ell_o$ earlier (further back).



The image from below shows the cube with width ℓ_o transverse to the direction of motion and bottom length in direction of motion the Lorentz contracted $\ell_o\sqrt{1-\beta^2}$ and back edge with same width and length $\ell_o\beta$. This is exactly the perspective view one would get, if the cube were rotated through and angle β.



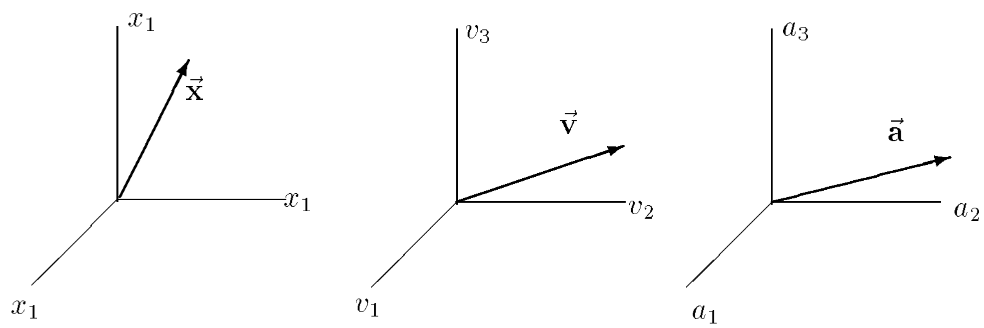
One can do these same calculations from any selected observation angle and finds similar results. The image (eye or photographic) appears to be a cube rotated by the aberration angle.

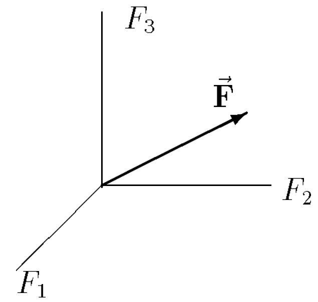
The key issue is that one is observing with light emitted from the object (cube in our example). In relativity light propagates with constant speed c independent of the observer's or source speed and the key point here is that the wave front always remains perpendicular to the direction of propagation. The only thing that changes is the direction of propagation (and thus wavefront angle) which is what we call relativistic aberration. Thus an image in one frame remains an image in the other and only the angle of observation changes.

This statement is true for the case of a small object which subtends a small solid angle. As one goes to larger angles, the aberration changes and a larger solid angle object would be rotated and distorted by the variation in aberration angle across the object being viewed.

10 Momentum Space

For any vector we may construct a vector space:





etc.

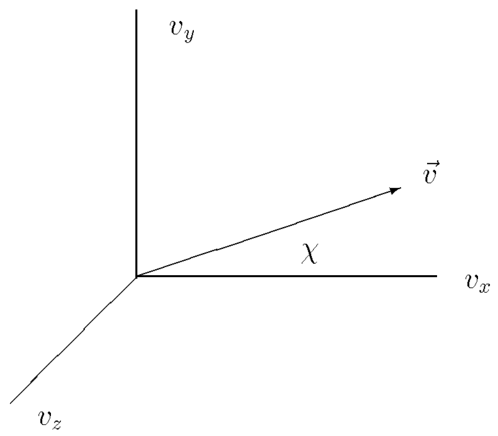
In such a space we may use any convenient coordinates: polar, rectangular, etc. Choice of coordinates is usually made to emphasize the symmetries present in the situation of interest.

We know how all Lorentz vectors transform under the set of special Lorentz transformations ($t||t'$, $x||x'$, $y||y'$, $z||z'$) where origins in space-time coincide and the relative velocity along the x and x' axes.

Therefore we can see how any property they have transforms.

Example:

$$tan\chi' = \frac{sin\chi \sqrt{1 - v^2/c^2}}{cos\chi - v/c} \tag{303}$$



$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \frac{\left[(v')^2 + u^2 + 2uv'cos\chi' - (uv'sin\chi/c)^2\right]^{1/2}}{1 + uv'cos\chi/c^2}$$

Any Lorentz vector transforms as

$$\begin{array}{rcl} x'_{\|} & = & \gamma(x_{\|} - \beta x_0) \\ x'_0 & = & \gamma(x_0 - \beta x_{\|} \\ x'_{\perp} & = & x_{\perp} \end{array} \tag{304}$$

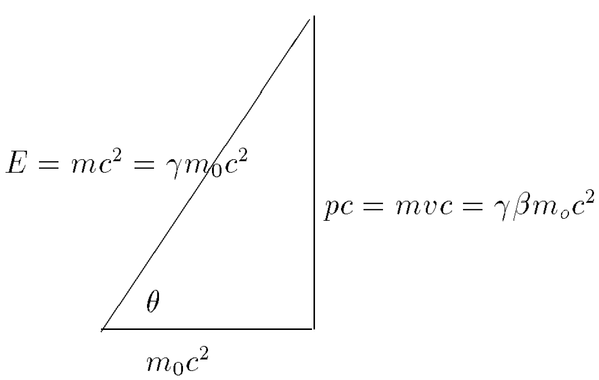
where $\|$ and $\perp$ refer to $\vec{v}$ frame.

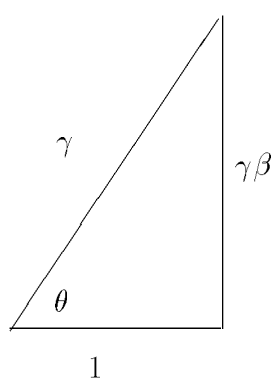
In momentum space we are interested in

$$\tilde{p} = (E/c, p_x, p_y, p_z) \tag{305}$$

$$\tilde{p} \cdot \tilde{p} = E^2/c^2 - |\vec{p}|^2 = (m_0 c)^2 \tag{306}$$

Pythagorean Theorem:





$sin\theta = \frac{pc}{E} = \frac{v}{c} = \beta$
$cos\theta = \frac{m_0 c^2}{E} = \frac{1}{\gamma}$

$$1 + (\beta\gamma)^2 = \gamma^2 \tag{307}$$

$$\begin{aligned} E^2 &= (m_0c^2)^2 + (pc)^2 \\ EdE &= c^2pdp \end{aligned} \tag{308}$$

10.1 Distribution Functions

We can define a distribution function $f(x)$ by how one determines the number (of something) in the interval (x_1, x_2) or

$$\text{Number} \equiv \int_{x_1}^{x_2} f(x)dx \tag{309}$$

Normalize to a single event and $f(x)$ becomes a probability with

$$\int_{\text{all } x} f(x)dx = 1$$

Change variables: $y = y(x)$, then

$$g(y)dy = f(x)dx \tag{310}$$

since number is invariant

$$g(y) = f(x)\left|\frac{dx}{dy}\right| \tag{311}$$

the absolute value is because $f \geq 0$ and $g \geq 0$.

For many (n) variables, one replaces $\left|\frac{dx}{dy}\right|$ by

$$\begin{vmatrix} \frac{\partial x_1}{dy_1}, & \frac{\partial x_1}{dy_2}, & \cdots & \frac{\partial x_1}{dy_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{dy_1}, & \frac{\partial x_n}{dy_2}, & \cdots & \frac{\partial x_n}{dy_n} \end{vmatrix} = \left|\frac{\partial x_i}{dy_i}\right|$$

Where the $|\ |$ is the functional determinant or Jacobian determinant of the transformation of the variables.

Examples: From (x, y, z) to (r, θ, ϕ), $|\ | = r^2 sin\theta$.

From (x, y) to (r, θ), $|\ | = r$.

Solid angle $d\Omega = sin\theta d\phi d\theta$ with $\int_{\text{sphere}} d\Omega = 4\pi$

$$\text{Volume Element} = dxdydz = r^2 sin\theta drd\theta d\phi = r^2 drd\Omega = r^2 dr|d(cos\theta)|d\phi \tag{312}$$

For a Lorentz transformation without reflection, $J = +1$, for any Lorentz transform $|J| = 1$. This is another indication that Lorentz transformations are rotations (neglecting any translation - i.e. keeping origins aligned at zero time).

$$d^4\tilde{p} = dp_1 dp_2 dp_3 dp_4 = |J| dp_1' dp_2' dp_3' dp_4' \tag{313}$$

Because the same events populate each volume element.

But the 4 components of $\tilde{p}$ are not independent. We focus interest on the 3-D volume element $d^2\vec{p}$. The result of this relationship is that

$$\frac{d^3\vec{p}}{E} = \frac{d^3\vec{p}'}{E'} \tag{314}$$

Further

$$p dE d\Omega = p' dE' d\Omega' \tag{315}$$

Now we can easily prove these results using the Lorentz transformation of momentum:

$$\begin{array}{ll} E' = \gamma\left(E - \beta c p_x\right) & p_y' = p_y \\ p_x' = \gamma\left(p_x - \beta E/c\right) & p_z' = p_z \end{array} \tag{316}$$

and

$$E = \sqrt{(pc)^2 + (m_0 c^2)^2} \tag{317}$$

Which we can insert into the equation for p_x' to obtain

$$p_x' = \gamma\left(p_x - (\beta/c)\sqrt{(p_x^2 + p_y^2 + p_z^2)c^2 + (m_0 c^2)^2}\right) \tag{318}$$

Now we can evaluate the Jacobian

$$\begin{aligned} \left|\frac{\partial(p_x', p_y', p_z')}{\partial(p_x, p_y, p_z)}\right| &= \begin{vmatrix} \frac{\partial p_x'}{\partial p_x} & \frac{\partial p_x'}{\partial p_y} & \frac{\partial p_x'}{\partial p_z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left|\frac{\partial p_x'}{\partial p_x}\right| \\ &= \gamma\left(1 - \frac{\beta}{c}\frac{p_x c^2}{\sqrt{(p_x^2 + p_y^2 + p_z^2)c^2 + (m_0 c^2)^2}}\right) \\ &= \frac{\gamma E - \beta\gamma p_x c}{E} = \gamma\left(E - \beta c p_x\right) = \frac{E'}{E} \end{aligned} \tag{319}$$

So that

$$f(p_x, p_y, p_z) = f'(p_x', p_y', p_z')\frac{E'}{E} \tag{320}$$

and

$$f(p_x, p_y, p_z) dp_x dp_y dp_z = f'(p_x', p_y', p_z') dp_x' dp_y' dp_z' \tag{321}$$

so

$$\frac{f}{f'} = \frac{E'}{E} = \frac{dp_x' dp_y' dp_z'}{dp_x dp_y dp_z} = \frac{d^3\vec{p}'}{d^3\vec{p}} \tag{322}$$

so

$$\frac{d^3\vec{p}}{E} = \frac{d^3\vec{p}'}{E'} = \textbf{invariant} \tag{323}$$

$$\frac{d^3\vec{p}}{E} = \frac{p^2 dp d\Omega}{E} = \textbf{invariant} = \frac{p\left(p dp d\Omega\right)}{E} \tag{324}$$

$c^2 p dp = E dE$ so that $p dp = E dE/c^2 = \textbf{invariant}$ so $p dp d\Omega$ **is invariant.**

10.1.1 Further Discussion

Suppose one wants a quantity describing a distribution of events characterized by $|\vec{p}| = p$: $g(p,\Omega)$ such that $\int g(p,\Omega)dpd\Omega$ = number of events (or 1, if normalized to be a probability).

$$\begin{aligned} g(p,\Omega) &= f(p,\Omega)\ p^2 \\ g'(p',\Omega')p^2 &= f'(p,\Omega)\ (p')^2 \end{aligned} \tag{325}$$

These formulae have made use of the relations of $p^2dpd\Omega$ in the first and $f(p,\Omega)p^2dpd\Omega = g(p,\Omega)dpd\Omega$ $d\Omega = d(cos\theta)d\phi$. Taking the quotients of the two equations one finds

$$\frac{g(p,\Omega)}{g'(p',\Omega')} = \frac{f}{f'}\frac{p^2}{(p')^2} = \frac{E'}{(p')^2}\frac{p^2}{E} \tag{326}$$

But $f/f' = E'/E$ so that

$$\frac{g(p,\Omega)E}{p^2} = \frac{g'(p',\Omega')E'}{(p')^2} = \textbf{invariant} \tag{327}$$

A distribution function may be expressed with respect to $(|\vec{p}|,\Omega) = (p\Omega)$ or (E,Ω) since $p = p(E)$ and $E = E(p)$.

Now let us require a distribution $h(E,\Omega)$ such that

$$\begin{aligned} \int h(E,\Omega)dEd\Omega &= \text{number of events} \\ &= \int h(E,\Omega)\frac{c^2pdp}{E}d\Omega \end{aligned} \tag{328}$$

because $EdE = c^2pdp$. Then

$$\begin{aligned} h(E,\Omega)c^2p/p &= g(p,\Omega) \\ h(E,\Omega) &= g(p,\Omega)\frac{E}{c^2p} \\ \frac{h(E,\Omega)}{h'(E',\Omega\prime)} &= = \frac{p'E}{pE'}\frac{g}{g'} = \frac{p'E}{pE'}\frac{p^2E'}{(p')^2E} = \frac{p}{p'} \end{aligned} \tag{329}$$

So

$$\frac{h(E,\Omega)}{p} = \frac{h'(E',\Omega')}{p'} = \textbf{invariant} \tag{330}$$

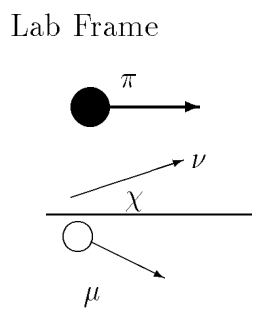
10.1.2 Example of a Distribution Function Problem

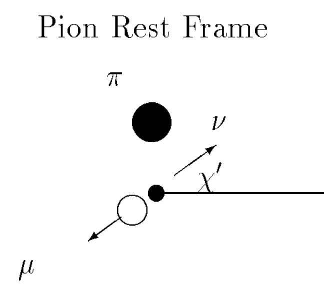
A narrow beam of pions with momentum 10.0 GeV/c decays ($\pi \rightarrow \mu + \nu$) in vacuum, providing a line source of neutrinos. In the pion rest frame the neutrino momentum distribution is

$$g'(p',\Omega') = g'(p',cos\chi',\psi') = \frac{1}{4\pi}\delta(p'-p_0).$$

The distribution is **isotropic** with $p_0 = 29.3 \times 10^{-3}$ GeV/c. Find the momentum distribution of the neutrinos in the laboratory.

$m_{\pi^\pm} = 0.1396$ GeV/c^2, $m_\mu = 0.1057$ GeV/c^2, $m_\nu \simeq 0$ GeV/c^2





$$\beta\gamma = \frac{pc}{m_0 c^2} = \frac{10.0}{0.1396} = 71.6$$

$$\beta = \frac{1}{\sqrt{1+(\beta\gamma)^2}} = 1 - 0.98 \times 10^{-3}$$

$$g(p,\Omega) = \frac{p^2 E'}{(p')^2 E} g'(p',\Omega')$$

$E_\nu = pc$, $\quad E' = p'c$, so that

$$g(p,\Omega) = \frac{p}{p'}\,\frac{1}{4\pi}\,\delta(p'-p_0)$$

$$\begin{aligned} E' &= \gamma(E - \beta pc\, cos\chi) \\ p'_{\chi'} &= \gamma(p\, cos\chi - \beta E/c) \end{aligned} \tag{331}$$

From the first expression with $E = pc$, $E' = p'c$:

$$p' = \gamma p(1-\beta cos\chi)$$

$$\frac{p}{p'} = \frac{1}{\gamma(1-\beta cos\chi)}$$

$$g(p,\Omega) = \frac{1}{4\pi}\,\frac{\delta(\gamma p(1-\beta cos\chi)-p_0)}{\gamma(1-\beta cos\chi)}$$

To find the distribution $G(p)$, integrate over Ω for fixed p

$$G(p) = \frac{1}{4\pi}\int\int d\psi\, d(cos\chi) g(p,\Omega)\frac{\delta(\gamma p(1-\beta cos\chi)-p_0)}{\gamma(1-\beta cos\chi)} \tag{332}$$

Set

$$\gamma p(1-\beta cos\chi) \quad = \quad w$$

$$
\begin{aligned}
-\beta\gamma p d(cos\chi) &= dw; \\
d(cos\chi) &= -\frac{dw}{\beta\gamma p}.
\end{aligned} \tag{333}
$$

$$
\begin{aligned}
G(p) &= \frac{1}{2}\int_{w=\gamma p(1-\beta)}^{w=\gamma p(1+\beta)} \frac{dw}{\beta\gamma p}\,\frac{p}{w}\delta(w-p_0) \\
&= \frac{1}{2}\,\frac{1}{\beta\gamma}\int \frac{dw}{w}\delta(w-p_0) \\
G(p) &= \frac{1}{2\beta\gamma p_0}
\end{aligned} \tag{334}
$$

Independent of p so the momentum distribution is **flat**. The distribution extends from a maximum of $\gamma p_0(1+\beta)$ to a minimum $\gamma p_0(1-\beta)$.

To check the calculation, evaluate

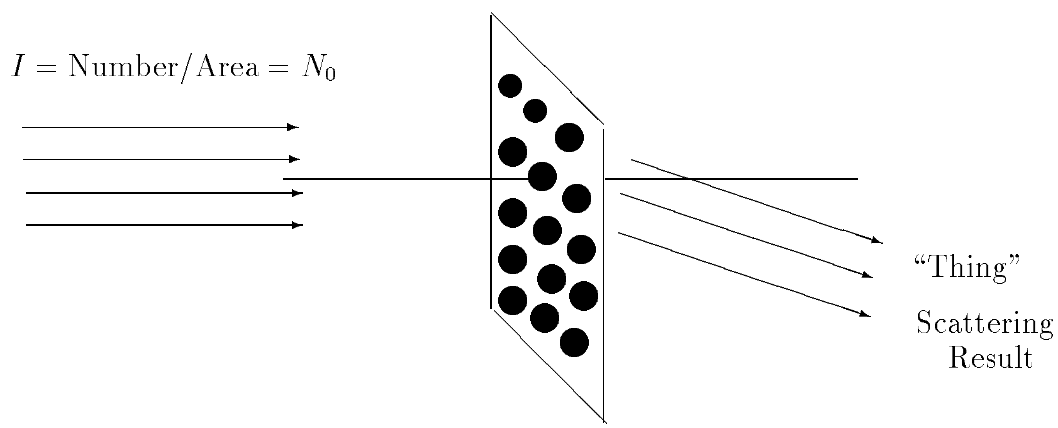
$$
\int_{p_{min}}^{p_{max}} G(p)dp = \frac{1}{2\beta\gamma p_0}\left[\gamma p_0(1+\beta) - \gamma p_0(1-\beta)\right] = 1 \tag{335}
$$

The result comes out to unity as it should, because in s'

$$
\int g(p',\Omega')dp'd\Omega' = \frac{1}{4\pi}\int d\Omega' \int dp'\delta(p'-p_0) = 1 \tag{336}
$$

10.2 Cross-Sections

Cross sections are useful for calculating interactions for a beam of incoming particles.



$$
dn = \text{number doing a certain thing} \tag{337}
$$

$$
\text{Probability} = \frac{dn}{N_0} = \frac{\text{Area for doing it}}{\text{Total Area}} \tag{338}
$$

Area for doing it is that of of the circles $= d\sigma$.

$$\sigma = \int \frac{d\sigma}{d\lambda} d\lambda \tag{339}$$

where λ is a "thing", e.g., within dE or $d\Omega$.

The multicomponent result is

$$\sigma = \int \frac{\partial^n \sigma}{\partial\lambda_1 \partial\lambda_2 \cdots \partial\lambda_n} d\lambda_1 d\lambda_2 \cdots d\lambda_n \tag{340}$$

10.2.1 Example

$$\sigma_{Total} = \int \frac{\partial^2 \sigma}{\partial p \partial\Omega} dp d\Omega \tag{341}$$

For a process mcose final state is defined by p, Ω (a binary process) but also

$$\sigma_{Total} = \int \frac{\partial^2 \sigma}{\partial E \partial\Omega} dE d\Omega \tag{342}$$

so we need to transform such differential cross-sections. Let

$$\frac{\partial^2 \sigma}{\partial p \partial\Omega} = g(p, \Omega) \tag{343}$$

Then

$$\frac{\partial^2 \sigma}{\partial p \partial\Omega} \frac{E}{p^2} = \frac{\partial^2 \sigma'}{\partial p' \partial\Omega'} \frac{E'}{(p')^2} \tag{344}$$

If

$$\frac{\partial^2 \sigma}{\partial E \partial\Omega} = h(E, \Omega) \tag{345}$$

Then

$$\frac{\partial^2 \sigma}{\partial E \partial\Omega} \frac{1}{p} = \frac{\partial^2 \sigma'}{\partial E' \partial\Omega'} \frac{1}{p'} \tag{346}$$

10.2.2 Invariants

$$\begin{array}{ccc} & & pdEd\Omega \\ Ef(p_x, p_y, p_z) \overset{\frac{d^3\vec{p}}{E}}{=} Ef(p^2 dp d\Omega) & & \\ g(p, \Omega)\frac{E}{p^2} & \frac{h(E,\Omega)}{p} & \\ \frac{E}{p^2}\frac{\partial^2\sigma}{\partial p \partial\Omega} & & \frac{1}{p}\frac{\partial^2\sigma}{\partial E \partial\Omega} \end{array}$$

10.2.3 Scattering Processes

The particles of Newton's physics were "eternal", never losing identity, changing mass, being created and destroyed. In modern atomic, nuclear, and particle physics particles alter their states and are created and destroyed.

Examples:

(1) Radiation by an atom in an excited state, $[E_i - E_f = h\nu]$ creating a photon.
(2) Nuclear transmutations, e.g.

$$\alpha + {}_7N^{14} \rightarrow p + {}_8O^{17}$$

where the sum of masses changes.
(3) Particle Production, e.g.

$$\pi^- + p \rightarrow \Lambda^0 + K^0$$

Two annihilations, two creations and the sum of masses changes. (This reaction is called associate production since the "strange" particles Λ^0 and K^0 are produced in association. The strong interaction respects and conserves "strangeness".)

All of these effects are well described by relativistic mechanics. However, it says little or nothing about what happens during the reaction; but before and after when we have independent free particles (which are well described) and **conservation laws** relating initial and final states.

We call all these reactions generalized scattering processes. If the final set is identical to the initial set (same masses, charges, spins, etc.), it is elastic scattering; otherwise, inelastic scattering.

Because of creation and destruction of particles, the older classifications, (simple vs. complex, 2-body vs. many-body, dynamics of particles vs. dynamics of systems of particles) are meaningless.

A system is characterized by
Energy
Momentum
Angular Momentum
Electric Charge
...

Its permanence is in its conserved quantities.

If at one time we describe a system as having particles A, B, C, $\cdots$, we must remember that each may be a complex structure. E.g. an atom is a nucleus plus electrons, a nucleus is neutrons and protons.

By calling something a particle we mean that its particle parameters (charge, proper mass, spin, magnetic moment, ...,) are constant in time before and/or after the reaction occurs. Because of **Quantization** none of these parameters change gradually, but only abruptly (destruction, creation). In the weak field of a mass spectrometer a molecular ion moves as a single particle, but in an intense laser beam, it might disassociate. Even particles now thought to be simple (such as the electron) are allowed the possibility of structure by our formalism.

10.3 Dynamics of Systems of Particles

$$\vec{P} = \sum \vec{p}_i, \qquad \tilde{P} = \sum \tilde{p}_i, \qquad E = \sum E_i \tag{347}$$

Definition of C.M. System: $\vec{P} = 0$. To find this Lorentz frame, start in an arbitrary frame, with $\vec{V}_{frame}$ parallel to $\vec{P}$ in that frame.

$$\begin{aligned} E' &= \gamma_F \left(E = \beta_f P c\right) \\ P'_{\|} &= \gamma_F \left(P = \beta_f E/c\right) \\ P'_{\perp} &= P_{\perp} \end{aligned} \tag{348}$$

In the desired C.M. frame, $P'_{\|} = 0$ so that

$$\beta_f = \frac{Pc}{E}, \qquad \vec{\beta}_f = \frac{\vec{P}c}{E} = \frac{M\vec{V}c}{E} \tag{349}$$

which is a restatement of $E = Mc^2$.

The invariant of $\tilde{P}$ is $\tilde{P}\cdot\tilde{P} = (m_0 c)^2$. This invariant for a system is proportional to the square of the total energy in the C.M. frame divide by c^2.

One may calculate the components of $\tilde{P}$ in any frame as if the system were a particle of rest mass M. The 4-velocity of C.M. $\equiv \tilde{W}$;

$$\tilde{P} = M_0 \tilde{W} \tag{350}$$

To measure actually the components of $\tilde{P}$ in any system, one must deal with the individual particles, and their energies and momenta in that system. But if they interact, one must also take into account the energy of interaction and its equivalent mass. This is universally true for every kind of interaction energy: thermal, electromagnetic, gravitational, atomic, nuclear, etc.

In classical mechanics of collisions we distinguish between conservation of momentum and conservation of energy (elastic versus inelastic). In Special Relativity they are components of a 4-vector and must be considered together. No exception is known. But is was necessary for Pauli (1930) to postulate the existence of the neutrino to save it.

However: we think that conservation of total energy is not the same as conservation of the total rest-mass (proper mass). This contradicts Newtonian assumptions.

There are only two types of entities.

(1) Rest Mass which can be "weighed" by bringing it to rest in some system

(2) massless particles moving with speed c, e.g. photons, neutrinos

A hot body weights more than an identical cold one. The relative kinetic energy of thermal motion has rest-energy and rest-mass. Also: A closed cavity of reflecting walls containing photons "weighs" more than the same cavity empty.

So: Composite Systems must be considered carefully. Not all of the parts can be brought to rest in the same frame. Referring to the rest mass of a composite system, we mean

$$M_{C.M.} = \sum \frac{E_i}{c^2} \tag{351}$$

of all parts including trapped photons and other "massless" particles.

A bound $p^+ + e^-$ is a hydrogen atom. It is lighter than the sum of separate masses of p and e, both at rest, inspite of the e^- motion having kinetic energy which has mass. This effect is due to the potential energy < 0, which is the difference between E-M field energies in the two configurations. This E-M energy is associated with photons having $m_0 = 0$.

10.3.1 Radiative Transitions & Decay

This leads us to radiative transitions, which are defined as emission or absorption of real photons which can be identified separately from other parts of the system, either at every early or very late times.

$$\text{photon momentum} \equiv |\vec{p}| = \frac{h\nu}{c}, \qquad E = h\nu \qquad |\tilde{p}|^2 = 0 \tag{352}$$

Now we consider the generic case of radiative decay of an initial state "mother" particle with a rest mass M_0 into a final state "daughter" particles with rest mass m_0 and a photon.

S: "Mother" Rest Frame $\qquad$ S': "Daughter Rest Frame

In reference frame S (initial "mother" particle at rest) one has

$$\begin{aligned} \tilde{P}_m &= \tilde{P}_d + \tilde{\Pi} \\ \tilde{P}_m = \left(M_0c^2, 0, 0, 0\right) &= \left(mc^2, -p, 0, 0\right) + (E_\gamma, p_\gamma, 0, 0) \\ &= \left(\gamma m_0 c^2, -E_\gamma/c, 0, 0\right) + (E_\gamma, E_\gamma/c, 0, 0) \end{aligned} \tag{353}$$

$$\tilde{P}'_m = \left(Mc^2, -E'_\gamma/c, 0, 0\right) \quad = \quad \left(m_0c^2, 0, 0, 0\right) + \left(E'_\gamma, E'_\gamma/c, 0, 0\right) \tag{354}$$

Conservation of four-momentum gives

$$\begin{aligned} S: \qquad M_0c^2 &= mc^2 + E_\gamma \\ S': \qquad Mc^2 &= m_0c^2 + E'_\gamma \end{aligned} \tag{355}$$

The Lorentz transformation gives us a way to handle this:

Lorentz Transform (L.T.)

$$E' = \gamma\left(E - \beta pc\right) \qquad p' = \gamma\left(p - \beta E/c\right)$$

Apply L.T.

$$Mc^2 = \gamma\left(M_0c^2 - \beta 0\right) \qquad m_o c^2 = \gamma\left(mc^2 - \beta(-E_\gamma)\right) \qquad E'_\gamma = \gamma\left(E_\gamma - \beta E_\gamma\right)$$

$$E'_\gamma = \gamma\left(0 - \beta m_0 c^2\right) \qquad 0 = \gamma\left(-E'_\gamma/c - \beta mc\right) \qquad E'_\gamma/c = \gamma\left(E_\gamma/c - \beta E_\gamma/c\right)$$

These yield

$$M = \gamma M_0 \qquad \gamma\frac{m}{m_0}\left(1 - (\frac{E_\gamma}{mc^2})^2\right) = 1 \qquad E'_\gamma = \gamma(1-\beta)E_\gamma = \sqrt{\frac{1-\beta}{1+\beta}}E_\gamma$$

$$E'_\gamma = -\beta\gamma M_0 c^2 \qquad \beta = -\frac{E_\gamma}{mc^2}$$

Now inverse L.T.

$$E = \gamma\left(E' + \beta pc\right) \qquad p = \gamma\left(p' + \beta E'/c\right)$$

$$M_0c^2 = \gamma\left(Mc^2 + \beta E'_\gamma\right) \qquad mc^2 = \gamma\left(m_0c^2 + \beta\ 0\right) \qquad E'_\gamma = \gamma\left(0 - \beta M_0 c\right)$$

$$0 = \gamma\left(E'_\gamma/c + \beta Mc\right) \qquad -E'_\gamma/c = \gamma\left(0 + \beta m_0 c\right) \qquad E_\gamma/c = \gamma\left(E'_\gamma/c + \beta E'_\gamma/c\right)$$

These give

$$\gamma\frac{M}{M_0}\left(1 - (\frac{E'_\gamma}{Mc^2})^2\right) = 1 \qquad m = \gamma m_0$$

$$\beta = -\frac{E'_\gamma}{Mc^2} \qquad E_\gamma = -\beta\gamma m_0 c^2 \qquad E_\gamma = \gamma\left(1+\beta\right)E'_\gamma = \sqrt{\frac{1-\beta}{1+\beta}}E_\gamma$$

There are 12 results from the forward and backward Lorentz transformations. All four on the photon give the same result:

$$\frac{E'_\gamma}{E_\gamma} = \sqrt{\frac{1-\beta}{1+\beta}} \tag{356}$$

Two give the "definitions"

$$\frac{M}{M_0} = \frac{m}{m_0} = \gamma \tag{357}$$

Two give

$$\beta = -\frac{E_\gamma}{mc^2} = -\frac{E'_\gamma}{Mc^2} \tag{358}$$

so that

$$ME_\gamma = mE'_\gamma \tag{359}$$

But since $M/M_0 = m/m_0 = \gamma$, This is not an independent result.

The last four relations can be combined with the above to give

$$\gamma^2\left(1 - \beta^2\right) = 1$$

which is an identity. There remain the conservation relations:

$$M_0c^2 = mc^2 + E_\gamma; \qquad Mc^2 = m_0c^2 + E'_\gamma$$

One may define

$$\Delta E \equiv \left(M_0 - m_0\right)c^2 \tag{360}$$

The system is fully defined by M_0c^2 and ΔE.

10.3.2 Radiative Decay Continued

In Mother particle rest frame S_M

$$\begin{aligned}
\tilde{P}_{\text{Mother}} &= \left(M_0c^2,\ 0,\ 0,\ 0\right)/c \\
\tilde{P}_{\text{Daughter}} &= \left(M_0c^2 - E_\gamma,\ -E_\gamma,\ 0,\ 0\right)/c \\
\tilde{P}_{\text{Photon}} &= \left(E_\gamma,\ E_\gamma,\ 0,\ 0\right)/c \qquad (361)
\end{aligned}$$

In Daughter rest frame S_d

$$\begin{aligned}
\tilde{P}_{\text{Mother}} &= \left(\frac{(M_0c^2)^2 - m_0E_\gamma}{\sqrt{(M_0c)^2 - 2M_0E_\gamma}},\ \frac{M_0E_\gamma}{\sqrt{(M_0c)^2 - 2M_0E_\gamma}},\ 0,\ 0\right)/c \\
\tilde{P}_{\text{Daughter}} &= \left([(M_0c)^2 - 2M_0E_\gamma]^{1/2},\ 0,\ 0,\ 0\right)/c \\
\tilde{P}_{\text{Photon}} &= \left(\frac{M_0E_\gamma}{\sqrt{(M_0c)^2 - 2M_0E_\gamma}}, \frac{M_0E_\gamma}{\sqrt{(M_0c)^2 - 2M_0E_\gamma}},\ 0,\ 0\right)/c \\
\sqrt{(M_0c)^2 - 2M_0E_\gamma}/c &= \text{rest mass of daughter in its own frame} \qquad (362)
\end{aligned}$$

A convenient path to solving for the various quantities is in terms of M_0 and ΔE is:

$$\beta = -\frac{E_\gamma}{mc^2} \quad \text{so} \quad \gamma = \frac{1}{\sqrt{1 - (E_\gamma/mc^2)^2}} \qquad (363)$$

and

$$M_0 - \Delta E/c^2 = m_0 = m\sqrt{1 - (E_\gamma/mc^2)^2} \qquad (364)$$

Square this equation

$$M_0^2 - 2M_0\Delta E/c^2 + \left(\Delta E/c^2\right)^2 = m^2 - \left(E_\gamma/mc^2\right)^2 \qquad (365)$$

Insert $m = M_0 - E_\gamma/c^2$

$$M_0^2 - 2M_0\Delta E/c^2 + \left(\Delta E/c^2\right)^2 = M_0^2 - 2M_0E_\gamma/c^2 + \left(E_\gamma/mc^2\right)^2 - \left(E_\gamma/mc^2\right)^2 \qquad (366)$$

and solve for E_γ:

$$\begin{aligned}
2M_0E_\gamma &= 2M_0\Delta E - (\Delta E/c)^2 \\
E_\gamma &= \Delta E\left(1 - \frac{\Delta E}{2M_0c^2}\right) \qquad (367)
\end{aligned}$$

From this, one finds by substitution the values of β, γ, E'_γ, M, and m. If a solution in terms of M_0 and m_0 is wanted, merely replace ΔE by $(M_0 - m_0)\,c^2$, obtaining

$$E_\gamma = \left(1 - \frac{m_0}{M_0}\right)^2 \frac{M_0c^2}{2} \qquad (368)$$

Simplifying Features for Radiative Decay
(1) One may choose all momenta parallel to x and x'. Only a single direction occurs in one frame.
(2) Conservation of $\tilde{P}$ is in general four scalar equations but only two for simple radiative decay.
(3) Problem is completely determined: only one possible outcome.
(4) Only two useful frames; no photon rest frame

Although there are very many quantities M_0, m_0, E_γ, v_{rel} between Mother and Daughter particles, ΔE, total energy of m in M's frame, Kinetic Energy of each in other's frame, β, γ. There are really only four that define the rest: m_0, m_0, E_γ, & V.
A particle emits a photon. One may be interested in the Mother's or the Daughter's frame:

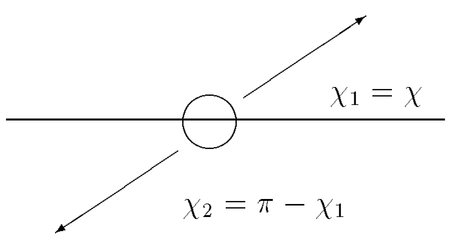
$$\begin{array}{cc} S_M & S_D \\ \tilde{P}_{\text{intial}} = (m_{in}c,\ 0,\ 0,\ 0) & \tilde{P}_{\text{in}} = \left(m_f c + E'_\gamma/c,\ -E'_\gamma/c,\ 0,\ 0\right) \\ \tilde{P}_{\text{final}} = (m_{in}c - E_\gamma/c,\ -E_\gamma/c,\ 0,\ 0) & \tilde{P}_{\text{f}} = (m_f c,\ 0,\ 0,\ 0) \\ \tilde{\Pi}_\gamma = (E_\gamma/c,\ E_\gamma/c,\ 0,\ 0) & \tilde{\Pi} = \left(E'_\gamma/c,\ E'_\gamma/c,\ 0,\ 0\right) \end{array}$$

Double Massless Consider decay with both daughters massless (e.g. photon, neutrinos)

Although similar, there is an important difference: There is only one special Lorentz frame - the rest frame of the original particle. Photons do not have rest frames. In the special Lorentz frame, we have

$$\tilde{P}_{\text{intial}} = \left(m_0 c^2,\ 0,\ 0,\ 0,\ 0\right) \tag{369}$$

$$\tilde{P}_{\text{final}} = (E_\gamma, E_\gamma cos\chi/c,\ E_\gamma sin\chi/c,\ 0) \rightarrow (E_\gamma, -E_\gamma cos\chi/c,\ -E_\gamma sin\chi/c,\ 0) \tag{370}$$



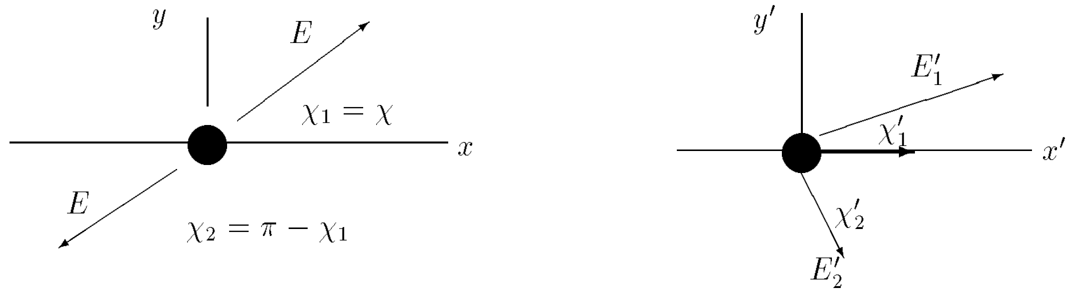
This diagram is constructed so that $\tilde{P}_{\text{initial}} = \tilde{P}_{\text{final}}$: $\quad 2E_\gamma = m_0 c^2$

$$cos\chi_1 = cos\chi, \qquad cos\chi_2 = cos(\pi - \chi_1) = -cos\chi_1 \tag{371}$$

Similarly

$$sin\chi_2 = -sin\chi_1 = -sin\chi \tag{372}$$

We can choose arbitrarily a second coordinate system in which the initial particle moves in the direction defined by χ'_1 with a velocity defined by β



It is evident that (1) and (2) lie in the $x - y$ and $x' - y'$ planes ($\psi = \psi' = 0$).

$$\begin{aligned} |\tilde{P}\mathrm{initial}|^2 &= |\tilde{P}\mathrm{final}|^2 \\ (m_0c^2)^2 &= \left(\tilde{P}_{1F} + \tilde{P}_{2F}\right) \cdot \left(\tilde{P}_{1F} + \tilde{P}_{2F}\right) \\ &= |\tilde{P}_{1F}|^2 + |\tilde{P}_{2F}|^2 + 2\tilde{P}_{1F} \cdot \tilde{P}_{2F} \\ &= 0 + 0 + 2\left(E_1'E_2' - \vec{p}_{1f} \cdot \vec{p}_{2f}\right) \end{aligned} \tag{373}$$

Now $\vec{p}_{if} = E_i'/c$ and $\vec{p}_{1f} \cdot \vec{p}_{2f} = E_1'E_2'cos\theta/c^2$ with θ the angle between the $\vec{p}'$'s.

Thus

$$(m_0c^2)^2 = 2E_1'E_2'\,(1 - cos\theta) \tag{374}$$

An important use of this relation is to recognize photons from π^0 or η^0 decay against a background of uncorrelated photons.

In S, $\theta = \pi$, $E_1' = E_2' = E$ and $m_0c^2 = 2E$.

For the individual photons the Lorentz transformation with $pc = E$, $p_{1||} = Ecos\chi$, $p'c = E'$, $p_{1||} = E'cos\chi'$ gives

$$\begin{aligned} E' &= \gamma E\,(1 - \beta cos\chi) \\ E'cos\chi' &= \gamma E\,(cos\chi - \beta) \\ E'sin\chi' &= Esin\chi \end{aligned} \tag{375}$$

10.4 General Case of Decay into Two Bodies of Any Masses

Consider a system 0 of rest mass m_0 goes to pieces 1 and 2 having non-zero rest masses m_1 and m_2.

$$\tilde{P}_0 = \tilde{P}_1 + \tilde{P}_2 \tag{376}$$

The rest frame of 0 is S and we label everything with left subscript 0

$$({}_0E_0,\ 0,\ 0,\ 0) = ({}_0E_1, {}_0\vec{p}_1c) + ({}_0E_2, {}_0\vec{p}_2c) \tag{377}$$

with ${}_0E_0 = m_0c^2$, ${}_0E_1^2 = ({}_0p_1c)^2 + (m_1c^2)^2$ and ${}_0E_2^2 = ({}_0p_2c)^2 + (m_2c^2)^2$ Thus

$${}_0E_0 =_0 E_1 +_0 E_2 \tag{378}$$

and

$$ {}_0p_1 +_0 p_2 = 0 \tag{379} $$

Denote $|{}_0p_1| = |{}_0p_2| = q_0$, then

$$ \begin{aligned} {}_0E_1 &= \sqrt{(q_0c)^2 + (m_1c^2)^2} \\ {}_0E_2 &= \sqrt{(q_0c)^2 + (m_2c^2)^2} \end{aligned} \tag{380} $$

Now express conservation of 4-momentum as

$$ \tilde{P}_2 = \tilde{P}_0 - \tilde{P}_1 \tag{381} $$

and form $|\tilde{P}_2|^2$:

$$ \begin{aligned} |\tilde{P}_2|^2 &= |\tilde{P}_0|^2 + |\tilde{P}_1|^2 - 2\tilde{P}_0 \cdot \tilde{P}_1 \\ (m_2c^2)^2 &= (m_0c^2)^2 + (m_1c^2)^2 - \left[E_0E_1/c^2 - \vec{p}_0 \cdot \vec{p}_1\right] \end{aligned} \tag{382} $$

which holds in any frame.

Choose to evaluate $\tilde{P}_0 \cdot \tilde{P}_1$ in frame S, where ${}_0\vec{p}_0 = 0$, ${}_0E_0 = m_0c^2$ and solve for E_1

$$ {}_0E_1 = \left(\frac{m_0^2 + m_1^2 - m_2^2}{2m_0}\right) c^2 \tag{383} $$

If instead we start with

$$ \tilde{P}_1 = \tilde{P}_0 - \tilde{P}_2 \tag{384} $$

we find

$$ {}_0E_2 = \left(\frac{m_0^2 + m_2^2 - m_1^2}{2m_0}\right) c^2 \tag{385} $$

We find q_0 from

$$ \begin{aligned} q_0c &= \sqrt{({}_0E_1)^2 - (m_1c^2)^2} = \left[\left(\frac{m_0^2 + m_1^2 - m_2^2}{2m_0}\right)^2 c^4 - \frac{4m_0^2m_1^2c^4}{(2m_0)^2}\right]^{1/2} \\ &= \frac{1}{2m_0}\left[\left(m_0^2 + m_1^2 - m_2^2\right)^2 - 4m_0^2m_1^2\right]^{1/2} c^2 \\ q_0 &= \frac{1}{2m_0}\left[m_0^2 + m_1^4 + m_2^4 - 2m_0^2m_1^2 - 2m_0^2m_2^2 - 2m_1^2m_2^2\right]^{1/2} c \\ &= \frac{1}{2m_0}\left[\left(m_0^2 - m_1^2 - 2m_1m_2 - m_2^2\right)\left(m_0^2 - m_1^2 + 2m_1m_2 - m_2^2\right)\right]^{1/2} c \\ &= \frac{1}{2m_0}\left[\left(m_0^2 - (m_1^2 + m_2^2)\right)\left(m_0^2 - (m_1^2 - m_2^2)\right)\right]^{1/2} c \end{aligned} \tag{386} $$

The speeds of the fragments in S are

$$ \beta_1 = \frac{v_1}{c} = \frac{{}_0p_1}{{}_0E_1/c} = \frac{q_0c}{{}_0E_1} = \frac{\sqrt{({}_0E_1)^2 - (m_1c^2)^2}}{{}_0E_1} = \sqrt{1 - \left(\frac{m_1c^2}{{}_0E_1}\right)^2} \tag{387} $$

These velocities are completely determined by $\tilde{P}_{\text{Total}} = \text{constant}$; except for the direction.

We may also evaluate the dynamical variables in the rest frame of particle 1, S_1. All quantities with left subscript 1.

$$ {}_1\tilde{P}_0 = {}_1\tilde{P}_1 - {}_1\tilde{P}_2 \tag{388} $$

Square this to find the modulus gives

$$ \begin{aligned} |{}_1\tilde{P}_0|^2 &= |{}_1\tilde{P}_1|^2 + |{}_1\tilde{P}_2|^2 + 2{}_1\tilde{P}_1 \cdot {}_1\tilde{P}_2 \\ (m_0c^2)^2 &= (m_1c^2)^2 + (m_2c^2)^2 - \left[{}_1E_{11}E_2/c^2 - \vec{p}_1 \cdot \vec{p}_2\right] \\ m_0^2 &= m_1^2 + m_2^2 + 2m_{11}E_2/c^2 \\ {}_1E_2 &= \frac{1}{2m_1}\left(m_0^2 - m_1^2 - m_2^2\right)c^2 \end{aligned} \tag{389} $$

Similarly,

$$ {}_1E_0 = \frac{1}{2m_1}\left(m_0^2 + m_1^2 - m_2^2\right)c^2 \tag{390} $$

and of course

$$ {}_1E_1 = m_1c^2 $$

10.4.1 Examples:

Consider the nuclear decay

$$ {}_{92}\mathrm{U}^{238} \rightarrow {}_{90}\mathrm{Th}^{234} \; + \; {}_2\mathrm{He}^4 \tag{391} $$

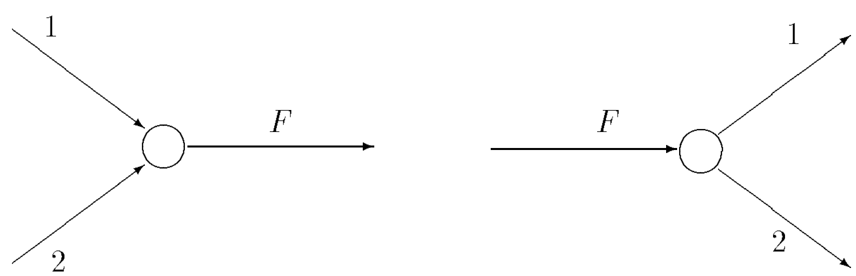
and electron scattering

$$ \begin{aligned} e^- \; + \; \mathrm{Hg} \;&\rightarrow\; e^-\mathrm{Hg} \qquad \text{Elastic} \\ e^- \; + \; \mathrm{Hg} \;&\rightarrow\; e^-\mathrm{Hg}^* \qquad \text{Inelastic} \end{aligned} \tag{392} $$

The changes in total proper mass contradicts Newtonian mechanics directly.

10.5 Two-Body Initial State

The two body initial state is a binary collision. Initially, two independent systems, 1 and 2. The final state F may have 1, 2, or many "particles". We start with the 4-momentum $\tilde{P}_0$ of F.



The second schematic is just the first with the arrows reversed.

$$\tilde{P}_1 + \tilde{P}_2 = \tilde{P}_0 \tag{393}$$

The frame S_f is the center-of-momentum frame of the colliding particles. m^*c^2 is their total energy in S_f. We are using an * (asterisk) for quantities in this frame. $(m^*c^2 = E^*)$

$$E_j^* = \frac{1}{2E^*}\left((E^*)^2 + (m_jc^2)^2 - (m_kc^2)^2\right) \tag{394}$$

for $j = 1,\ k = 2$ or $j = 2,\ k = 1$.

$$E_j^* = \frac{(m^*)^2 + m_j^2 - m_k^2}{2m^*}c^2 \tag{395}$$

$$\begin{aligned} q_0 &= \frac{1}{2m^*}\left((m*)^4 + m_1^4 + m_2^4 - 2(m^*)^2m_1^2 - 2(m^*)^2m_2^2 - 2(m_1)^2m_2^2\right)^{1/2} \\ &= \frac{1}{2m^*}\left(\left[(m^*)^2 - (m_1+m_2)^2\right]\left[(m^*)^2 - (m_1-m_2)^2\right]\right)^{1/2} \end{aligned} \tag{396}$$

In many experiments one of the two initial bodies is at rest in the laboratory frame; it is useful to find the components of $\tilde{P}$ in its frame, say that of 2. Squaring $\tilde{P}_0 = \tilde{P}_1 + \tilde{P}_2$ yields

$$(m^*)^2 = m_0^2 + m_2^2 + 2m_2({}_2E_1)/c^2, \tag{397}$$

or

$${}_2E_1 = \frac{(m^*)^2 - m_1^2 - m_2^2}{2m_2} \tag{398}$$

By squaring $\tilde{P}_1 = \tilde{P}_0 - \tilde{P}_2$ yields

$$m_1^2 = m_0^2 + m_2^2 - 2m_2({}_2E_0)/c^2, \tag{399}$$

From these the momentum magnitude $|{}_2\vec{p}_1| = |{}_2\vec{p}_0|$ can be found by

$$(pc)^2 = E^2 - (mc^2)^2 \tag{400}$$

We can evaluate m^* in terms of the quantities in frame S_2:

$$(m^*)^2 = ({}_2E_1 + {}_2E_2)^2/c^4 - ({}_2\vec{p}_1 + {}_2\vec{p}_2)^2/c^2 = \left({}_2E_1 + m_2c^2\right)^2/c^4 - {}_2{p_1}^2/c^2; \tag{401}$$

or

$$(m^*)^2 = m_1^2 + m_2^2 + 2m_{22}E_1/c^2 \tag{402}$$

Which is identical the equation for ${}_1E_0$.

The beam particle kinetic energy is

$${}_2\mathrm{KE}_1 = {}_2E_1 - m_1c^2 \tag{403}$$

so that equation for m^* is

$$(m^*)^2 = m_1^2 + m_2^2 + 2m_{22}\mathrm{KE}_2/c^2 \tag{404}$$

To transform between S_0 and S_2 we need the relative velocity

$$\beta = {}_0v_2 = \frac{|{}_0\vec{p}_2|c}{{}_0E_2} = \frac{({}_1p_2 + 0)c}{{}_1E_2 + m_2c^2} \tag{405}$$

This can be found in terms of either the beam momentum or the beam energy by using

$${}_1{E_2}^2 = {}_1{p_2}^2c^2 + (m_1c^2)^2. \tag{406}$$

10.5.1 Two Body Initial States: Summary

General Features:
(1) A unique direction is defined in the intial state momenta. If another direction enters in describing the final state, an element of simplicity is lost.
(2) The conservation of $\tilde{P}$ involves four scale equations of constraint. If only two particles come out, this second defined direction and the intial one define a plane which may be taken as the $x-y$ plane equal to the $x'-y'$ plane and the four equations reduce to three.
(3) If only one particle comes out the problem is completely determined. Retudes to the previous problem of one in and two out, reversed. If two come out, three more quantities are unfixed. These may be take as

(1) mass ratio of daughters

(2,3) their directions (opposite: θ, ϕ) in the c.m. system of the incoming pair.

These last two are on a different footing: Totally undetermined by the general conservation of 4-momentum. so, if the final rest-mass ratio is given, the problem is fully defined and all directions in the c.m. system are equally valid as to conservation of $\tilde{P}$. There may be additional physics that relates to the relative directions.

10.5.2 Special Cases of two-body initial states

(1) One body final state
(2) Elastic Scattering (same two bodies in final state)
(3) Inelastic Scattering - two body final state
(4) General Case: many final bodies
(1) One body final state Completely inelastic collision where the two bodies stick together. This can happen only, if there exists an excited bound state of 1 and 2 with just the right rest energy, to allow conservation of both energy and momentum.

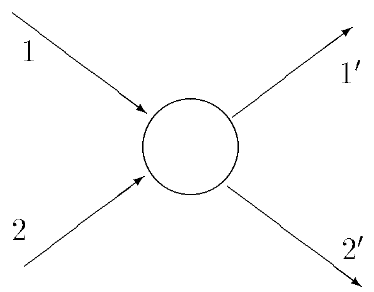
Consider the head-on collision of two putty balls. Conservation of three momentum $\vec{p}$ yields

$$0 = \gamma_1 m_1 v_1 + \gamma_2 m_2 v_2 = \gamma_0 m_0 v_0 \tag{407}$$

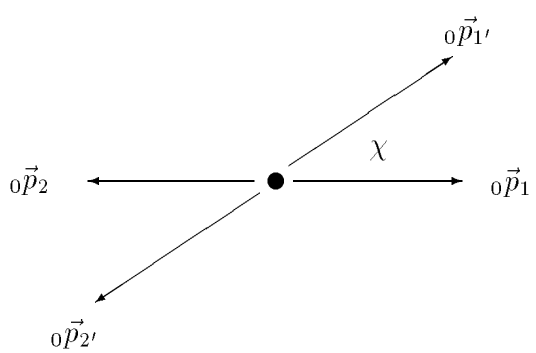
which works only if $v_0 = 0$ and thus $\gamma_0 = 1$. The time-like part of the 4-vector $\tilde{P}$ conservation is energy conservation

$$\begin{aligned} \gamma_1 m_1 + \gamma_2 m_2 &= \gamma_0 m_0 = m_0 \\ m_1 + \mathrm{KE}_1/c^2 + m_2 + \mathrm{KE}_2/c^2 &= m_0; \\ m_0 &= m_1 + m + 2 + (\mathrm{KE}_1 + \mathrm{KE}_2)/c^2 \end{aligned} \tag{408}$$

Macroscopic kinetic energy is converted to chaotic internal energy (heat), contributing to m_0 and thus to inertia.
<u>(2) Elastic Scattering</u> Two bodies come in and the same two come out of the scattering process. $1 + 2 \rightarrow 1' + 2'$; $m_{1'} = m_1$, & $m_{2'} = m_2$.



Four-momentum conservation in the C.M. frame S gives the sum of the two input 3-momenta and the sum of the two output momenta are zero, $\vec{p}_1 + \vec{p}_2 = 0$, and $\vec{p}_{1'} + \vec{p}_{2'} = 0$. And thus $|\vec{p}| = q_0$. So all four momenta vectors can be drawn as:



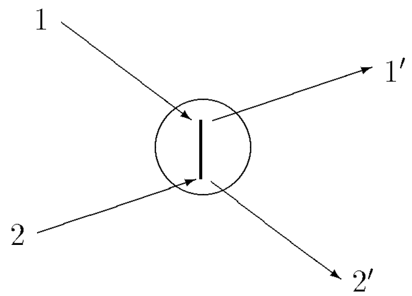
Quantum field theories gives ways of calculation the distribution function for χ and also ψ; but $\tilde{P}_{\mathrm{total}}+$ constant. Only reduces the number of independent variables by 4.

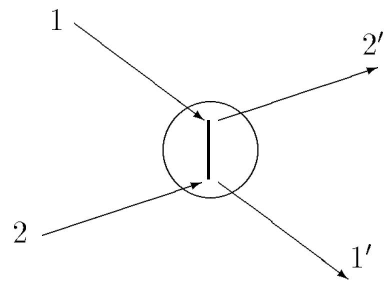
10.5.3 4-Momentum Transfer

Define the 4-momentum transfer $\tilde{\Delta}$ or sometimes $\tilde{Q}$ as

$$\tilde{\Delta} \equiv \tilde{P}_{1'} - \tilde{P}_1 = \tilde{P}_2 - \tilde{P}_{2'} \tag{409}$$

The 4-momentum transfer is useful, if we regard the interaction as in two steps as shown in the following figures were are generic Feymann diagrams:





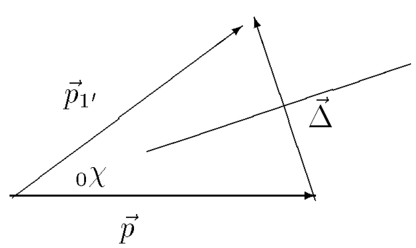
Let us evaluate $\tilde{\Delta} \cdot \tilde{\Delta}$, the four-momentum squared

$$\begin{aligned}
\tilde{\Delta} \cdot \tilde{\Delta} &= |\tilde{P}_{1'}|^2 + |\tilde{P}_1|^2 - 2P_{1'} \cdot \tilde{P}_1 \\
&= (m_1 c)^2 + (m_1 c)^2 - 2\left(E_1 E_{1'}/c^2 + \vec{p}_{1'} \cdot \vec{p}_1\right) \\
&= 2\left((m_1 c)^2 - ({}_1E_0/c)^2 + q_0^2 cos_0\chi\right) \\
&= -2\left(q_0^2 - q_0^2 cos_0 \chi\right) \\
&= -2q_0^2\left(1 - cos_0\chi\right) \qquad (410)
\end{aligned}$$

The four-momentum transfer $\tilde{\Delta}$ is a space-like vector ($|\tilde{\Delta}|^2 < 0$. It is usually defined with the opposite sign to the convention that I use here but is space-like.

For a grazing collision ${}_0\chi \rightarrow 0$, so $|\tilde{\Delta}|^2 \rightarrow 0$.
For a head-on collision, ${}_0\chi \rightarrow \pi$ so $|\tilde{\Delta}|^2 \rightarrow 4q_0^2$.

A simpler derivation is:



$$\frac{\Delta}{2} = psin\left(\frac{{}_0\chi}{2}\right), \qquad (411)$$

$$\Delta^2 = 4p^2 sin^2\left(\frac{{}_0\chi}{2}\right) = 2p^2\left(1 - cos_0\chi\right) \qquad (412)$$

Since $|\tilde{\Delta}|^2$ is linear in $cos_0\chi$, equal intervals of $|\tilde{\Delta}|^2$ correspond to equal intervals of solid angle in the C.M..

Because $|\tilde{\Delta}|^2$ is a Lorentz invariant, we may calculate its value in any frame. Do so in S_2, the rest frame of particle 2.

$$\begin{aligned}
|\tilde{\Delta}|^2 &= |\tilde{P}_2|^2 + |\tilde{P}_{2'}|^2 + 2\tilde{P}_2 \cdot \tilde{P}_{2'} \\
&= (m_2 c)^2 + (m_2 c)^2 + 2\left(E_e E_{2'}/c^2 - \vec{p}_2 \cdot \vec{p}_{2'}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2(m_2c)^2 + 2\left(m_2E_{2'} + 0\right) \\
&= 2m_2\left(E_{\mathrm{restenergy}} - E_{\mathrm{total}}\right) \\
&= 2m_{22}\mathrm{KE}_2 \qquad (413)
\end{aligned}
$$

The last, ${}_2\mathrm{KE}_2$ is the laboratory kinetic energy acquired by the struck particle – the particle that was previously at rest in the lab.

If the final velocity is $\ll c$, then ${}_2\mathrm{KE}_2 \simeq |{}_2\vec{p}_2|^2/(2m_2)$, so that $|\tilde{\Delta}|^2 \simeq |{}_2p_2|^2$; square of the four-momentum transfer is approximately the square of the 3-momentum transfer.

10.5.4 Cross-Momentum Transfer

The 4-momentum transfer was defined as

$$
\begin{aligned}
\tilde{\Delta} &\equiv \tilde{P}_{1'} - \tilde{P}_1 = \tilde{P}_2 - \tilde{P}_{2'} \\
&= \tilde{\Delta}(1,1') = -\tilde{\Delta}(2,2') \qquad (414)
\end{aligned}
$$

These had an arbitrary sign choice, since one particle gains and one loses.

We can also define

$$
\begin{aligned}
\tilde{\Delta}(1,2') &\equiv -\tilde{\Delta}(2,1') \\
&= \tilde{P}_{2'} - \tilde{P}_1 = \tilde{P}_{1'} - \tilde{P}_2 \qquad (415)
\end{aligned}
$$

The calculations are similar, resulting in

$$
|\tilde{\Delta}(1,2')|^2 = (m_1c)^2 + (m_2c)^2 - 2_0E_{20}E_1/c^2 + 2q_0^2cos(\pi_0\chi) \qquad (416)
$$

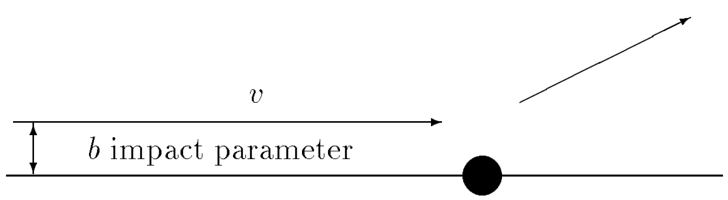
This has a maximum when the first $\tilde{\Delta}|^2$ is minimum and vice versa; since it varies linearly with the opposite sign of $cos_0\chi$.

In frame S_2,

$$
|\tilde{\Delta}(1,2')|^2 = (m_1 - m_2)^2c^2 - 2m_{22}\mathrm{KE}_{1'} \qquad (417)
$$

Where the last term ${}_2\mathrm{KE}_{1'}$ is the lab kinetic energy of the "incident" particle after the collision.

The basic variables in elastic scattering are: m_0 and $|\tilde{\Delta}|^2$. They are analogous to E, l instead of b, v in Coulomb scattering.



The 4-momentum transfer $\tilde{\Delta}$ is useful for:

(1) For kinematics in general

(2) Corresponds to model of process for exchange of a virtual particle or when the initial particle carries "identity", e.g. electric charge, spin direction, etc.

10.5.5 Inelastic Scattering: 2 in, 2 out

Two particles in and two particles out but with $m_1 \neq m_{1'}$ and $m_2 \neq m_{2'}$.

In C.M. system the incoming particles have equal and opposite momenta with magnitude q_0. In C.M. system the outgoing particles have equal and opposite momenta with different magnitude q_0'. q_0' can be less than (exothermic) or greater (endothermic) than q_0. The results change from the earlier two to two elastic case:

$$ {}_0E_{1'} = \frac{1}{2m_0}\left(m_0^2 + m_{1'}^2 - m_{2'}^2\right)c^2 \tag{418} $$

which is now not equal to

$$ {}_0E_1 = \frac{1}{2m_0}\left(m_0^2 + m_1^2 - m_2^2\right)c^2 \tag{419} $$

also same with $1 \rightarrow 2$ and $1' \rightarrow 2'$. Also

$$ \begin{aligned} q_0' &= \frac{1}{2m_0}\left[m_0^4 + m_{1'}^4 + m_{2'}^4 - 2\left(m_0^2 m_{1'}^2 + m_{1'}^2 m_{2'}^2 + m_{2'}^2 m_0^2\right)\right]^{1/2} \\ \neq q_0 &= \frac{1}{2m_0}\left[m_0^4 + m_1^4 + m_2^4 - 2\left(m_0^2 m_1^2 + m_1^2 m_2^2 + m_2^2 m_0^2\right)\right]^{1/2} \end{aligned} \tag{420} $$

As m_0 decreases, q_0 and q_0' decrease. If $q_0' < q_0$, it will arrive at zero first. Then for further decease of m_0 (smaller initial total energy), the reaction cannot occur. (One finds an imaginary q_0'.)

We may factor q_0' differently:

$$ q_0' = \frac{1}{2m_0}\left[\left(m_0^2 - [m_{1'} + m_{2'}]^2\right)\left(m_0^2 - [m_{1'} - m_{2'}]^2\right)\right]^{1/2} c \tag{421} $$

So the threshold is when the first ( ) reaches zero for

$$ m_{0\ \text{threshold}} = m_{1'} + m_{2'} \tag{422} $$

At that point there is just enough total energy to make the two final rest energies, but none to give them any kinetic energy.

There are many expressions for a threshold: e.g., if particles 2 is fixed in the laboratory reference system S_2:

$$ {}_2\mathrm{KE}_{1\ \text{threshold}} = \frac{1}{2m_0}\left[(m_{1'} + m_{2'})^2 - (m_{1'} - m_{2'})^2\right]c^2 \tag{423} $$

Which can be derived from the equation

$$ m_0^2 = (m_{1'} + m_{2'})^2 + 2m_2\ {}_2\mathrm{KE}_1 \tag{424} $$

Then there is the momentum and cross-momentum transfer. There are many more relations which can be derived. An example is

$$ |\tilde{\Delta}(1,1')|^2 = (m_{1'}c)^2 + (m_1 c)^2 - 2({}_0E_{1'})({}_0E_1)/c^2 + 2q_0 q_0' cos({}_0\chi) = (m_{2'} - m_2)^2 c^2 + 2m_2\ {}_2\mathrm{KE}_{2'} \tag{425} $$

10.5.6 The General Case with ≥ 3 Final Particles

One may proceed by grouping into two system, then breaking each system down. Consider $N = 3$.

$$\begin{array}{ccccccccc} 1 & + & 2 & \rightarrow & 1' & + & 2' & + & 3' \\ 1 & + & 2 & \rightarrow & (1' & + 2') & + & 3' & \end{array} \tag{426}$$

In this case we group particles $1'$ and $2'$ into a composite particle in some other frame moving with momentum p, where p then decays to $1'$ and $2'$. This is a lot of work. What is simple? Nothing, except thresholds.

The parsimonious (energy efficient) was to make anything (in terms of energy expended) is to make it at rest.

$$m_{0\ \text{threshold}} = m_{1'} + m_{2'} + \cdots + m_{N'} \tag{427}$$

The threshold energy is the sum of the rest masses in the final state and $m_{0\ \textbf{threshold}} = (\sum E_i/c^2)_{\text{initial state}}$

10.5.7 Colliding Beams

Consider two equal energy beam colliding colliding in the laboratory frame L.

$$\text{C.M. Energy} = m_{0\ \textbf{threshold}} = {}_LE_1 + {}_LE_2 \tag{428}$$

Suppose each beam particle has 30 GeV: $m_)c^2 = 60$ GeV.

What is the equivalent accelerator laboratory energy (p, p) for a target at rest?

$$60 = \sqrt{2(1 + E_{Lab})}; \qquad E_{Lab} = 1800 \text{ GeV} \tag{429}$$

The colliding p-p beams at CERN 20-30 GeV each. Several labs collide e on e:

11 Radiation From Accelerated Charge

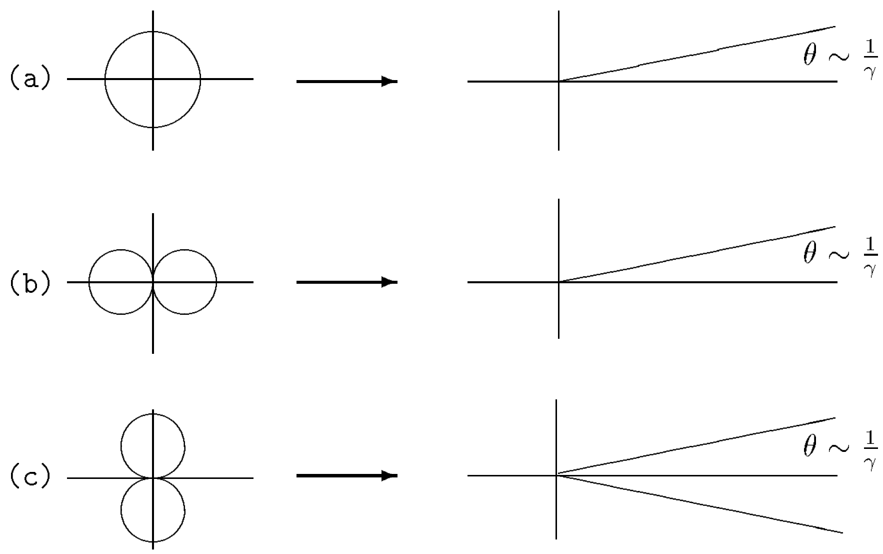
11.1 Introduction

You have learned about radiation from an accelerated charge in your classical electromagnetism course. We review this and treat it according to the prescriptions of Special Relativity to find the relativistically correct treatment.

Radiation from a relativistic accelerated charge is important in:
(1) particle and accelerator physics – at very high energies ($\gamma \gg 1$) radiation losses, e.g. synchrotron radiation, are a dominant factor in accelerator design and operation and radiative processes are a significant factor in particle interactions.
(2) astrophysics – the brightest sources from the greatest distances are usually relativistically beamed.
(3) Condensed matter physics and biophysics use relativistically beamed radiation as a significant tool. An example we will consider is the Advanced Light Source (ALS) at the Lawrence Berkeley Laboratory. Now free electron lasers are now a regular tool.

We will need to use relativistic transformations to determine the radiation and power emitted by a particle moving at relativistic speeds.

Lets look at the concept of relativistic beaming to get an idea before we go into the details which require a fair amount of mathematics.



Radiation from an accelerated relativistic particle can be greatly enhanced. Part of this effect is due to the aberration of angles and part due to the Doppler effect.

11.2 Plane Wave Electromagnetic Field

At long distances from accelerating charges the radiation field components of $F^{\mu\nu}$ dominate. They are related to the acceleration and satisfy all the properties of the plane wave electromagnetic field:

$$\vec{H}_{PW} = \hat{n} \times \vec{E}_{PW}, \quad \hat{n} \cdot \vec{E}_{PW} = \hat{n} \cdot \vec{H}_{PW} = \vec{E}_{PW} \cdot \vec{H}_{PW} = 0 \tag{430}$$

where$\hat{n}$ is the direction of propagation. At shorter distances the components of $F^{\mu\nu}$, which do not depend on acceleration, dominate. Specifically, $\hat{n} \cdot \vec{E} \neq 0$.

11.3 Doppler Effect

From time dilation we are used to the notion that a moving clock or system operating at frequency ν' in its rest frame will appear to be slower to a reference system.

$$\Delta t = \gamma \Delta t' \tag{431}$$

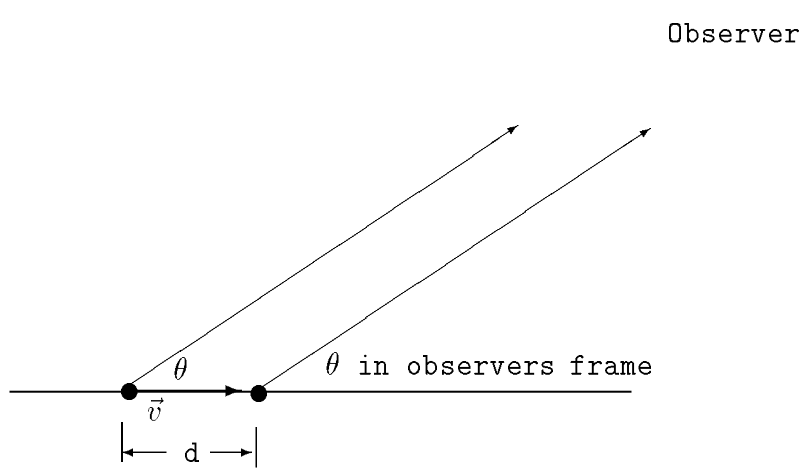
so that if the period in the rest frame is $\delta t' = 1/\nu'$, then

$$\nu = \nu'/\gamma \tag{432}$$

The factor leads to the relativistic transverse Doppler shift. The frequency shift one would observe for a clock or system moving transversely to the line of sight.

Thus the time between wave peaks (crests) or pulses is

$$\Delta t = \gamma \Delta t' = \frac{\gamma}{\nu'} \tag{433}$$



If the sources is moving at an angle θ to the observer's line of sight, then the difference in arrival times, Δt_A, of successive pulses or crests is

$$\Delta t_A \;=\; \Delta t - \frac{d}{c} = \Delta t(1 - \frac{v}{c} cos\theta)$$

$$\frac{1}{\nu_{obs}} = \frac{\gamma}{\nu'}(1 - \frac{v}{c}cos\theta_{obs}) \tag{434}$$

which when inverted or multiplied by c yields:

$$\nu_{obs} = \frac{\nu'}{\gamma(1 - \frac{v}{c}cos\theta_{obs})} \qquad \lambda_{obs} = \gamma\lambda'(1 - \frac{v}{c}cos\theta) \tag{435}$$

11.4 Radiation by an Accelerated Charge Near Rest

In 1897 Larmor derived the formula for the radiation by an accelerated charged particle. He found for the power and angular distribution:

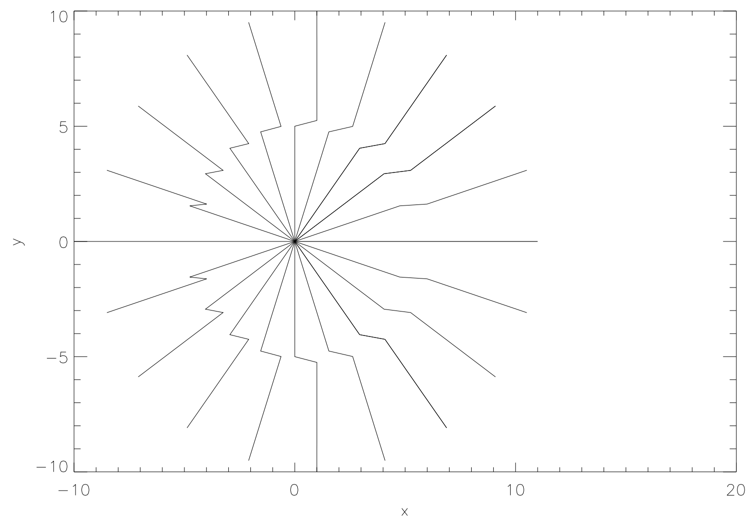
$$P = \frac{2q^2}{3c^3}\vec{a}\cdot\vec{a} \qquad \frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|a|^2 sin^2\Theta \tag{436}$$

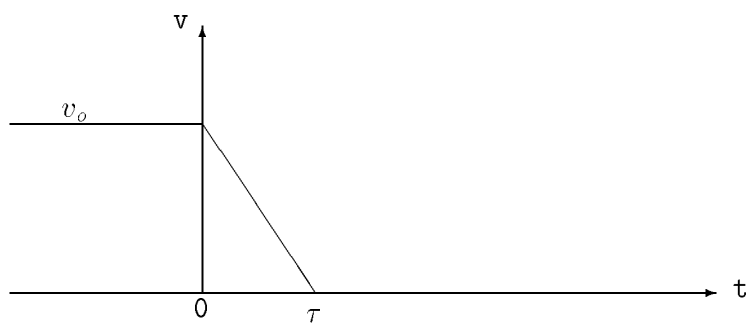
where Θ is the angle to the direction of acceleration. It is our task to find the relativistically consistent and correct version of these formulae.

We can rederive the Larmor formula for your education. We consider the electric field to be a real physical entity that points radially back to a charge at rest. If we go into a moving frame, the electric field lines will continue to point radially back to the instanteous position of the charge. The transformation of the electric field works out precisely that way. The Lorentz-Fitzgerald contraction along the direction of motion causes an increase by the factor γ of the transverse component of the field. Gauss's law continues to hold in that an integral over a closed surface, such as a sphere, gives the net charge within.

Now if a charge is diverted from uniform motion, then by our earlier arguments about causality, the electric field lines out at radius R can not be effected by that change from uniform motion until a time $t = R/c$ later. (In fact we expect that the electic field lines will change at the speed of light since light is an electromagnetic phenomenon.) Thus at a time t after a brief $\delta t = \tau$ disturbance (change from one state of uniform motion to another - also called acceleration) there is a critical radius $R = ct$. Inside of radius $R - c\tau$ the electric field lines point radially to the new instanteous position of the charge and outside of radius $R + c\tau$ the electric field lines point radially to the virtual instanteous position of the undisturbed charge. The virtual instanteous position is where the charge would have been had it not been disturbed. There is a near discontinuity in the field lines where they must make a jaunt nearly perpendicular to radial. Nearly means that the angle between the field line and perpendicular to radial is of order $c\tau/vt$ where v is the velocity change due to the disturbance.

Consider: a charge moving with velocity $v \ll c$ abruptly, at time $t = 0$, is decelerated at a constant rate a until it comes to rest.





At $t = 0$, $x = 0$ and at $t = \tau$, $x = v_o\tau/2$.

Now consider fields at a time $t_f \gg \tau$. At a distance $r > ct_f$, the field will be that of a uniformly moving charge, emanating from the "virtual present position" (the point where the particle would have been, $x = v_0 t_f$, if it had continued unaccelerated. At a distance $r < c(t_f - \tau)$, the field will be that of a charge at rest with $x = v_o\tau/2$.

There is a transition region which is nearly a spherical shell ($v_o \ll c$) A particular field line L defines a cone of angle, θ, inside, which contains a certain flux. Its continuation L' defines another cone which must contain the same flux by reason of Gauss's law relating the field flux and the enclosed charge. Thus $\theta' = \theta$ and L' is parallel to L.

Consider the portion connecting these two regimes.

The radial component of the electric field, E_r must be the same in the shell as just outside of it on either side (Gauss's law).

$$E_r = \frac{q}{r^2} = \frac{q}{ct_f r} = \frac{q}{(ct_f)^2} \tag{437}$$

By the geometry of the situation

$$\frac{E_\theta}{E_r} = \frac{v_o t_f sin\theta}{c\tau} \tag{438}$$

$$E_\theta = \frac{v_o t_f sin\theta}{c\tau} E_r = \frac{v_o t_f sin\theta}{c\tau} \frac{q}{(ct_f)^2} = \frac{q v_o sin\theta}{c^3 t_f \tau}. \tag{439}$$

Now $ct_f = r$, and $a = v_o/t_f$, so that

$$E_\theta = \frac{qasin\theta}{c^2 r} \tag{440}$$

The significance of this result is that $E_\theta \propto 1/r$ while $E_r \propto 1/r^2$. At a large distance the tangential electric field E_θ will dominate.

From our general knowledge of varying vacuum fields we know that there will be a component of $\vec{B}$ of strength equal to $\vec{E}$ and perpendicular both to $\vec{E}$ and $\vec{r}$.

The energy density (energy per unit volume) in the transition layer is

$$u = \frac{\text{Energy}}{\text{Volume}} = \frac{E_\theta^2}{8\pi} + \frac{B_\phi}{8\pi} = \frac{E_\theta^2}{4\pi} = \frac{q^2 a^2 sin^2\theta}{4\pi c^4 r^2} \tag{441}$$

The volume of the shell is it area $(4\pi r^2)$ times its thickness $(c\tau)$ and the average of $sin^2\theta = 2/3$,

$$\begin{aligned} < sin^2\theta > &= \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} sin^2\theta d(cos\theta)d\phi = \frac{1}{2}\int_{-1}^{1} sin^2\theta d(cos\theta) = \frac{1}{2}\int_{-1}^{1}(1-x^2)dx \\ &= \frac{1}{2}(x - \frac{1}{3}x^3)|_{-1}^{1} = \frac{2}{3} \end{aligned} \tag{442}$$

so that the energy in the transition layer is

$$E = \frac{2}{3}\frac{q^2 a^2 \tau}{c^3}.$$

The radiated power is then the energy per unit time:

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}. \tag{443}$$

which is precisely the formula derived by Larmor in 1897.

11.5 Radiation from Circular Orbit

Suppose that $\vec{a} \perp \vec{B}$, giving motion in a circle and that $\gamma \gg 1$:

insert figure: Reference frame S Laboratory and reference frame S$'$ which is the electron instantaneous rest frame

two column

The laboratory reference frame S has $\vec{B}$ perpendicular to the plane of the circular orbit ($B_x = B_y = 0$, and $B_z = B$ and $\vec{E} = 0$

$$F = e|\vec{v} \times \vec{B}| = \frac{\gamma m_o v^2}{r} \tag{444}$$

By transformation law $F' = \gamma^{-2,3} F$

column 2 Transform fields:

$$E' = \gamma\,(E - \beta B) \tag{445}$$

or more precisely

$$\begin{aligned} \vec{E}' &= \gamma\left(\vec{E} - \vec{\beta} \times \vec{B}\right) \\ &= \gamma\,(0 + \beta_x B_z(-\hat{e}_y)) \\ &= -\beta\gamma B\hat{e}_y \end{aligned} \tag{446}$$

Thus

$$\vec{F} = (-e)\vec{E}' = +\beta\gamma q B\hat{e}_y = m_o\vec{a} \tag{447}$$

Thus

$$a = \frac{\beta\gamma q B}{m_o} \tag{448}$$

The power emitted is

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3} = \frac{2}{3}\frac{\beta^2\gamma^2 q^4 B^2}{m_o^2 c^3} \tag{449}$$

This is the correct relativistic form! because P is energy per unit time and each is the 0-component of a 4-vector!!

To make a relativistic generalization we employ the concept of covariance.

The Poynting vector is the 0-0 component of the electromagnetic stress tensor and the radiated electromagnetic energy is the 0-component of a Lorentz 4-vector. Time is the 0-component of a Lorentz 4-vector. So the ratio energy per time is an invariant.

Can one find a Lorentz invariant that reduces to Larmor's formula as $\beta \to 0$? If so, it will be the correct relativistic formula! Is it unique? Yes, if we require it to involve only $\vec{\beta}$ and $d\vec{\beta}/dt$ and not higher powers.

How is one to construct it? Non-relativistically,

$$\frac{dv}{dt} = \frac{1}{m_o}\frac{dp}{dt} \tag{450}$$

So the Larmor power is

$$P = \frac{2}{3}\frac{e^2}{m_o^2 c^3}\left(\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}\right) \tag{451}$$

To get an invariant, experience tells us to substitute $d\tau$ for dt, i.e. $d\vec{p}/dt \to d\vec{p}/d\tau$, and add the fourth component:

$$\frac{d\tilde{p}}{d\tau}\cdot\frac{d\tilde{p}}{d\tau} = \left(\frac{dE}{d\tau}\right)^2 - c^2\left(\frac{d\vec{p}}{d\tau}\right)^2 \tag{452}$$

Notice that

$$EdE = c^2 pdp \tag{453}$$

So

$$\left(\frac{dE}{c}\right)^2 = \frac{(pc)^2}{E^2}(dp)^2 = \left(\frac{mvc}{mc^2}\right)(dp)^2 = \beta^2 (dp)^2 . \tag{454}$$

So that

$$\frac{1}{c^2}|\frac{d\tilde{p}}{d\tau}|^2 = |\frac{d\vec{p}}{d\tau}|^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2 \tag{455}$$

Giving

$$P = \frac{2}{3}\frac{e^2}{m_o^2 c^3}\left[|\frac{d\vec{p}}{d\tau}|^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2\right] \tag{456}$$

It is possible to write this in many ways. One way is

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[|\dot{\vec{\beta}}|^2 - |\vec{\beta}\times\dot{\vec{\beta}}|^2\right] \tag{457}$$

11.6 Power and Angular Distribution Summary

We can calculate these in a consistent way by using these formula as correct in the rest (primed) frame of the electron and transform the accelerations (forces), angles, frequencies, etc. into the laboratory frame. What we need is to show that powe is a Lorentz invariant $P = P'$ for any emitter that emits with front-back symmetry (zero net momentum) in its instantaneous rest frame. To do this we make use of the invariance of $\tilde{a}\cdot\tilde{u}$ which is zero for all systems.

$$\tilde{a}\cdot\tilde{u} = \frac{d\tilde{u}}{d\tau}\cdot\tilde{u} = \frac{1}{2}\frac{d}{d\tau}(u^\mu u_\mu) = \frac{1}{2}\frac{d}{d\tau}(c^2) = 0$$

This is a consequence invariance of the speed of light and four-vector velocity.

In the zero net radiation momentum (in instanteous rest frame) case $\tilde{a}\cdot\tilde{a} = \vec{a}\cdot\vec{a}$ since in the rest frame $a_0 = 0$. Thus the power can be evaluated in any frame can be found by computing the acceleration in that frame and squaring it.

$$P = \frac{2q^2}{3c^3}\vec{a}' \cdot \vec{a}' = \frac{2q^2}{3c^3}\left(a'^2_\perp + a'^2_{||}\right) = \frac{2q^2}{3c^3}\gamma^4\left(a^2_\perp + \gamma^2 a^2_{||}\right) \tag{458}$$

where $a_\perp$ is the acceleration perpendicular to the motion of the charged particle and $a_{||}$ is the acceleration component parallel to the charge particle motion. In the last line we have made use of the transformation of accelerations $a'_{||} = \gamma^3 a_{||}$ and $a'_\perp = \gamma^2 a_\perp$ evaluated in the instantaneous rest frame (primed) of the electron. Note that there is a factor of γ difference in the transformation of accelerations perpendicular and parallel to the direction of motion. This translates into a difference between γ^4 and γ^6 in the perpendicular and parallel cases.

We get a similar expression for the angular distribution:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{a^2_\perp + \gamma^2 a^2_{||}}{(1-\beta cos\theta)^4} sin^2\Theta' \tag{459}$$

We are making use of the conversion

$$\frac{dP}{d\Omega} = \frac{1}{\gamma^4(1-\beta cos\theta)^4}\frac{dP'}{d\Omega'}$$

Evaluation for perpendicular and parallel cases yields:

$$\frac{dP_\perp}{d\Omega} = \frac{q^2 a^2_\perp}{4\pi c^3}\frac{1}{(1-\beta cos\theta)^4}\left[1 - \frac{sin^2\theta cos^2\phi}{\gamma^2(1-\beta cos\theta)^2}\right] \rightarrow_{\gamma\gg 1} \approx \frac{4q^2a^2_\perp}{\pi c^3}\gamma^{\mathbf{8}}\frac{1-2\gamma^2\theta^2 cos2\phi + \gamma^4\theta^4}{(1+\gamma^2\theta^2)^6} \tag{460}$$

$$\frac{dP_{||}}{d\Omega} = \frac{q^2 a^2_\perp}{4\pi c^3}\frac{sin^2\theta}{(1-\beta cos\theta)^6} \rightarrow_{\gamma\gg 1} \approx \frac{4q^2 a^2_{||}}{\pi c^3}\gamma^{\mathbf{10}}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^6} \tag{461}$$

Note the large powers of γ 8 and 10 which shows the seriousness of the relativistic effects. Before we follow this up in detail, we review radiation near the rest frame of the emitting particle.

11.6.1 Case I: acceleration parallel to motion

Consider $\vec{\beta} \parallel \dot{\vec{\beta}}$; acceleration parallel to motion $\vec{\beta} \times \dot{\vec{\beta}} = 0$. Recalling that $\beta^2 = 1 - 1/\gamma^2$, then

$$\beta\dot{\beta} = \frac{\dot{\gamma}}{\gamma^3};$$

$$\dot{\beta} = \frac{\dot{\gamma}}{\beta\gamma^3}; \qquad \dot{\beta}^2 = \frac{\dot{\gamma}^2}{\beta^2\gamma^6}$$

$$P = \frac{2}{3}\frac{e^2}{c}\left(\frac{\dot{\gamma}}{\beta}\right)^2 \tag{462}$$

$$\begin{aligned}
P &= \frac{2}{3}\frac{e^2}{m_o^2 c^3}\left(1-\beta^2\right)\left(\frac{dp}{d\tau}\right)^2 \\
&= \frac{2}{3}\frac{e^2}{m_o^2 c^3}\gamma^2\frac{1}{\gamma^2}\left(\frac{dp}{dt}\right)^2 \qquad d\tau = \frac{dt}{\gamma} \\
&= \frac{2}{3}\frac{e^2}{m_o^2 c^3}(m_o c^2)^2\left[\frac{d(\beta\gamma)}{dt}\right] \qquad p = \beta\gamma m_o c \\
&= \frac{2}{3}\frac{e^2}{c}\left(\frac{\dot{\gamma}}{\beta}\right)^2
\end{aligned} \tag{463}$$

where the conversion makes use of the relations

$$\begin{aligned}
(\beta\gamma) &= \sqrt{\gamma^2-1} \\
\frac{d(\beta\gamma)}{dt} &= \frac{1}{2}\frac{2\gamma\dot{\gamma}}{\sqrt{\gamma^2-1}} \\
&= \frac{\gamma\dot{\gamma}}{\beta\gamma} = \frac{\dot{\gamma}}{\beta}
\end{aligned} \tag{464}$$

$$\begin{aligned}
\frac{\dot{\gamma}}{\beta} &= \frac{c}{v}\frac{d}{dt}\left(\frac{E}{m_o c^2}\right) \\
&= \frac{1}{m_o c}\frac{dE}{v\,dt} = \frac{1}{m_o c}\frac{dE}{dx}
\end{aligned} \tag{465}$$

$$\frac{P}{dE/dt} \sim \frac{2}{3}\frac{e^2/m_o c^2}{m_o c^2}\frac{dE}{dx} \tag{466}$$

So that the power radiated compared to the energy change per unit distance is

$$P = \frac{2}{3}\frac{e^2}{m_o^2 c^3}\left(\frac{dE}{dx}\right)^2 \tag{467}$$

Now we can compare the radiated power with the acceleration power

$$\frac{P}{dE/dt} = \frac{2}{3}\frac{e^2/m_o c^2}{m_o c^2}\frac{dE/dx}{dE/dt}\frac{dE}{dx} \tag{468}$$

Note $(dE/dx)/(dE/dt) \sim dt/(cdt)$ when $\beta \sim 1$. The ratio of powers, radiated to acceleration, is negligible unless energy gain in 2.8×10^{-13} cm is of order of the rest mass - i.e. for an electron ~ 0.511 MeV.

11.6.2 Case II: acceleration perpendicular to motion

Centripetal acceleration: $\dot{\vec{\beta}} \perp \vec{\beta}$. $\vec{a} = c\dot{\vec{\beta}}$ and $\vec{v} = c\vec{\beta}$.

insert figure / diagram to show vector directions

Then in the relation to find the rate of change of the energy-momentum four-vector

$$\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2 - |\frac{d\vec{p}}{dt}|^2$$

$\dot{E}/dt \cong 0$; since $\vec{F} \perp \vec{v}$, so that no work is being done on the particle. Then

$$P = -\frac{2}{3}\frac{e^2}{m_o c^3}|\frac{d\vec{p}}{d\tau}|^2 \tag{469}$$

and

$$|\frac{d\vec{p}}{d\tau}| = \gamma\omega|\vec{p}| \tag{470}$$

where $\omega = \beta c/\rho$ is the orbital angular frequency of an orbit with radius ρ. One can derive this relationship

$$\frac{dp}{p} = d\theta = \frac{ds}{\rho} = \frac{vdt}{\rho} = \omega dt$$

$$\frac{dp}{dt} = \omega p$$

Thus

$$\omega = \frac{\beta c}{\rho}$$

Now we can move on to the power loss rate

$$d\tau = \frac{dt}{\gamma} \qquad p = \beta\gamma m_o c$$

$$\begin{aligned} P &= \frac{2}{3}\frac{e^2}{m_o^2 c^3}\gamma^2\omega^2|\vec{p}|^2 \\ &= \frac{2}{3}\frac{e^2}{m_o^2 c^3}\left(\frac{\beta c}{\rho}\right)\gamma^2\left(\beta\gamma mc\right)^2 \end{aligned} \tag{471}$$

$$P = \frac{2}{3}\frac{e^2 c}{\rho^2}\beta^4\gamma^4 \tag{472}$$

The energy gain for a particle per turn in an accelerator is

$$\delta E = 2\pi\rho\frac{P}{v} \tag{473}$$

The radiation loss is

$$\delta E = \frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4 \tag{474}$$

In practical units

$$\delta E/(1\ \mathrm{MeV}) = 8.85 \times 10^{-2}\frac{(E/1\ \mathrm{GeV})^4}{\rho/(1\ \mathrm{meter})} \tag{475}$$

The power radiated by a bunch of electrons

$$\mathrm{Power}/(1\ \mathrm{watt}) = 10^6[\delta E/(1\ \mathrm{MeV\ turn})][J/(1\ \mathrm{amp})] \tag{476}$$

provided the radiation is incoherent.

Aside: How to get these practical unit relations:

$$\delta E = \frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4$$

Start by putting $\beta = 1$. If β is not very near to 1, then one gets negligible radiation power.

$$\gamma = \frac{E}{m_o c^2} = \frac{E(\mathrm{in\ GeV})}{0.511\ \mathrm{MeV}}$$

$$\frac{e^2}{\rho} = \frac{e^2}{m_o c^2}\frac{m_o c^2}{\rho} = r_o\frac{m_o c^2}{\rho} = (2.8 \times 10^{-13}\ \mathrm{cm})\frac{0.511\ \mathrm{MeV}}{\rho\ (\mathrm{in\ cm})}$$

and the conversion from ρ in cm to m is $\rho_{cm} = 100\rho_m$. So that

$$\begin{aligned}
\delta E &= \left[\frac{4\pi}{3}\frac{2.8\times 10^{-13}}{100}\frac{0.511}{(5.11\times 10^{-4})^4}\right]\frac{[E(\mathrm{in\ GeV})]^4}{\rho\ (\mathrm{in\ m})}\ \mathrm{MeV} \\
&= \left[8.85\times 10^{-2}\frac{[E(\mathrm{in\ GeV})]^4}{\rho\ (\mathrm{in\ m})}\right]\ \mathrm{MeV} \\
&= 88.5\frac{[E(\mathrm{in\ GeV})]^4}{\rho\ (\mathrm{in\ m})}\ \mathrm{keV}
\end{aligned} \tag{477}$$

Giving the conversion used above.

$$\begin{aligned}
\mathrm{Power} &= \frac{\delta E}{\mathrm{Electron\ turn}} \times \frac{\mathrm{turn}}{\mathrm{sec}} \times \mathrm{Number\ of\ electrons} \\
&= \delta E\frac{V}{2\pi\rho}\frac{2\pi\rho J}{eV} = \frac{\delta E \times J}{e} \\
\mathrm{Power\ (in\ kW)} &= 88.5[E(\mathrm{in\ GeV})]^4 J\ (\mathrm{in\ amps})/\mathrm{R}\ (\mathrm{in\ m}) \\
&= 26.5[E(\mathrm{in\ GeV})]^3 B\ (\mathrm{in\ teslas}) J\ (\mathrm{in\ amps}) \\
\frac{\delta E\ (\mathrm{in\ MeV})}{e} &= \delta V\ (\mathrm{in\ MV})
\end{aligned} \tag{478}$$

Now Some Numbers and History E.O. Lawrence invented the cyclotron and the first was built here at Berkeley. Later his colleague Edwin McMillen (and

Table 3: Parameters for Sample Accelerators

Accelerator	LBL	Cornell	LHC	ALS	FermiLab	SSC	Elo
Max. Energy (GeV)	0.3	10	45	1-2	1000	20,000	10^4
Particle	e	e	e	e	p-$\overline{p}$	p-p	p
B (Tesla)		0.33	0.135	1.248	4.4	6.6	7.7
Radius (m)	1	100	4249		1000	11.7 km	50 km
Bending R (m)				4.01		10.1 km	
Beam Current (ma)				400		73	100
Single Bunch (ma)				1.6			0.00167
E-gain/turn (MeV)	0.05	10.5	350	1		5.26	
E-loss/turn (MeV)	0.001	8.8		0.112	0.001		18
Synchrotron Power				45 kW		9.1 kW	1.8 MW
RF Power (kW)			16000	300	1600		61000
RF (MHz)		713.94	352	500	53.1	374.74	412
Harmonic		1800	31324	328	1113	103,680	146500
Beam lifetime (hrs)				14	4	~24	48
Fill time			30 min	2.1 min		40 min	4 hrs

independently in the Soviet Union by V.I. Veksler) invented the idea of phase stability which made the synchrotron possible.

Synchrotron radiation was first observed in a laboratory in 1947. That laboratory was in Berkeley.

Early Synchrotrons:

First synchrotron was operated with 8 MeV electrons in 1946 by Goward and Barnes in Woolwich Arsenal, UK. In 1947 GE labs operated an electron synchrotron at 70 MeV. Soon after there were many operating.

An early synchrotron at Berkeley had a radius of about 1 meter and a maximum energy of about 0.3 GeV. The synchrotron radiation $\delta E_{max} \sim 1$ keV/turn could be noticed. The acceleration voltage was only a few keV/turn.

At big electron synchrotron was built at Cornell and operated at 10 GeV . The radius was about 100 meters. It encloses a football field. The magnetic field was $B = 3.3$ kG (0.33 Tesla). The accelerator voltage was about 10.5 MeV/turn and the synchrotron losses were $\delta E_{rad} \sim 8.8$ MeV/turn.

LBL Advanced Light Source is designed to provide synchrotron radiation as a tool for research.

11.7 Synchrotron Radiation Basics

Consider the non-relativistic case of a charged particle in a circular orbit caused by a magnetic field. That particle will radiate electromagnetic waves at a frequency given by the orbit frequency (or the Lamor frequency)

$$\omega_L = \frac{qB}{mc} \qquad \nu_L = \frac{qB}{2\pi mc}$$

due to the acceleration of bending in the magnetic field. As the particle's energy is increased relativistic effects will become important. For the same orbit the particle will both begin to radiate more energy and at more frequencies - which are at the orbit frequency and its harmonics. The peak power will be emitted at a frequency which is at $\approx \gamma^3$ times the orbit frequency. In the next sections we will understand this.

11.7.1 Synchrotron Emitted Power

To find the total emitted power we can use the Lamor (1897) formula

$$P_{emitted} = \frac{2}{3}\frac{q^2}{c^3}|\vec{a}_o|^2 = \frac{2}{3}\frac{q^2}{c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$$

where $\vec{a}_o$ is the particle acceleration in its instantaneous rest frame and the right hand side of the equation uses the acceleration transform law from the particle rest frame.

$$\frac{dE}{dt} = q\vec{v}\cdot\vec{E}$$

and since $E = 0$ we have $\gamma =$ constant.

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(\gamma m_o \vec{v}) = q\vec{v}\times\vec{B}$$

With $\gamma =$ constant,

$$\gamma m_o \frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}$$

Thus

$$\frac{dv_\parallel}{dt} = 0, \qquad \frac{dv_\perp}{dt} = \frac{q}{\gamma m_o}\vec{v}_\perp\times\vec{B}$$

We can conclude $|v_\parallel| =$ constant and $|v_\perp| =$ constant. We have uniform circular motion of the projected motion on the normal plane. That is a simple helical motion around the uniform magnetic field.

The frequency of rotation or gyration is

$$\omega_B = \frac{qB}{\gamma m_o c}, \quad \rightarrow \quad a_\perp = \omega_B v_\perp$$

Note that the gyration (orbit) frequency is the Lamor frequency divided by γ.

We can now evaluate the transformation of the Lamor formula for the power radiated since we know $a_\perp = \omega_B v_\perp$ and $a_{||} = 0$

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^4\omega_B^2 v_\perp^2 = \frac{2}{3}\frac{q^2}{c^3}\gamma^4\left(\frac{qB}{\gamma m_o c}\right)^2 v_\perp^2 = \frac{2}{3}\frac{q^4B^2}{m_o^2 c}\beta_\perp^2\gamma^2$$
$$= \frac{2}{3}r_o^2 c\beta_\perp^2\gamma^2 B^2$$
$$= 2\beta_\perp^2\gamma^2 c\sigma_T U_B = 2\beta^2\gamma^2 c\sigma_T U_B sin^2\alpha$$

where $r_o = e^2/m_e c^2$ is the classical radius of the electron, $\sigma_T = 8\pi r_o^2/3$ is the Thomson crossection, $U_B = B^2/8\pi$ is the energy density of the magnetic field, and α is the helix pitch angle (angle of the gyrating particle with respect the magnetic field lines). This is the relativistically correct form that we saw previously.

11.7.2 Synchrotron Radiation Frequency Spectrum

First we consider the frequency distribution of a monoenergetic distribution, i.e. we consider the radiation from a particle at an energy E corresponding to γ. When the particle's energy increases (as γ grows larger) the aberration of angles moves most of the radiated power into a cone of half angle $\Delta\theta \sim 1/\gamma$ in the instantaneous direction of motion of the particle. Thus an observer will see a pulse of radiation whenever the particle's instantaneous velocity sweeps past his direction. This will happen once per orbit. This pulse will be narrow both because the aberration of angles and because of the time dilation and Doppler effect. Since the relativistic particle is moving towards the receiver (observer), the received pulse is sharpened (compressed in time) by a factor of order γ^{-2}. The time compression goes at

$$\frac{dt}{d\tau} = 1 - \beta cos\theta \sim 1 - \beta + \Delta\theta^2/2 \rightarrow \gamma^{-2}$$

where the limit comes for $\gamma \gg 1$ since $\beta = \sqrt{1 - 1/\gamma^2} \rightarrow 1 - \gamma^{-2}/2$ and $\Delta\theta^2/2 \sim \gamma^{-2}/2$.

Thus the observer will see a pulse every orbit with width γ^{-3} of the pulse separation. Fourier theory tells us that the signal will appear at the orbit frequency and its harmonics and that the power will peak at a frequency which is near $\gamma^3\nu_L$ (where $\nu_B = \omega_L/2\pi = qB/m_e c$).

For a magnetic field $B = 10^{-5}$ Gauss, which is a typical value in the Galaxy and many powerful radio galaxies, $\nu_L = 28$ Hz. The electrons that produce emission at radio frequencies of a few GHz therefore have Lorentz factors $\gamma \sim 10^3 - 10^4$. The spacing between successive harmonics is $\nu_B = \nu_L/\gamma$, for very high γ this spacing is so narrow as to negligible for all but the highest frequency resolution observations. In astrophysical sources, this is often blurred and smoothed by variations in the electron energy (a power law spectrum) and by variations in the magnetic field intensity and direction.

11.8 Astrophysical Synchrotron Radiation

11.8.1 Historical Note

Although nonthermal radiation had been observed from the Galaxy from the opening of radio astronomy in the pioneering work by Karl Jansky in 1933, there was no clear evidence of its origin. In 1950 Kiepenheuer suggested that Galactic nonthermal radio emission was synchrotron radiation and Alfvén and Herlofson proposed that non-thermal discrete sources were emitting synchrotron radiation. Kiepenheuer showed that the intensity of the nonthermal Galactic radio emission can be understood as the radiation from relativistic cosmic ray electrons that move in the general interstellar magnetic field. He found that a field of 10^{-6} Gauss (10^{-10} Tesla) and relativistic electrons of energy 10^9 eV would give about the observed intensity. The early 1950s saw the development of these ideas (e.g. Ginzburg et al. 1951 and following papers, see Ginzburg 1969) that synchrotron emission was the source of non-thermal "cosmic" radiation. This model was later supported by maps which showed that the sources of the non-thermal components were extended nebulae and external galaxies and by the discovery that the radiation was polarized as predicted by theory. The synchrotron theory is widely accepted and is the basis of interpretation of all data relating to nonthermal radio emission.

11.8.2 Context

Synchrotron radiation is a common phenonmen in astrophysics as there are almost always plasma and magnetic fields present and energetic electrons. Because of stochastic scattering processes, the energetic electrons tend to be isotropically distributed.

For an isotropic distribution of velocities one needs to average over all angles for a given speed β. If α is the pitch angle, the angle between the magnetic field direction and the particle velocity, then

$$< \beta_\perp^2 >= \frac{\beta}{4\pi} \int sin^2 \alpha d\Omega$$

Thus

$$P = \left(\frac{2}{3}\right)^2 r_o^2 c \beta^2 \gamma^2 B^2 = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 U_B$$

where $\sigma_T = 8\pi/3\ r_o^2$ is the Thomson cross section and $U_B = B^2/8\pi$ is the energy density in magnetic field.

Electrons of a given energy ($E = \gamma m_e c^2$) radiate over a wide spectral band, with the distribution peaking roughly at $\nu_c \approx 16.08(B_{\text{eff}}/\mu\text{G})(E/\text{GeV})^2$ MHz, with a long low-power tail at higher frequencies, and most of the radiation in a 2:1 band from peak. The peak intensity is at $\nu_{max} = 0.29\nu_c = 4.6(B_{\text{eff}}/\mu\text{G})(E/\text{GeV})^2$ MHz.

The radiation from a single electron is elliptically polarized with the electric vector maximum in the direction perpendicular to the projection of the magnetic field on the plane of the sky. Explicitly the total emissivity of a single electron via synchrotron radiation is the sum of parallel and perpendicular polarization

$$j(\nu) = \frac{\sqrt{3}e^3 B sin\alpha}{16\pi^2\epsilon_0 c m_e} F(x) \tag{479}$$

where α is the electron direction pitch angle to the magnetic field B and $F(x) \equiv x\int_x^\infty K_{5/2}(\eta)d\eta$ is shown graphically in Figure **??**.

The quantity x is the dimensionless frequency defined as $x \equiv \omega/\omega_c = \nu/\nu_c$ where ω_c and ν_c are the critical synchrotron frequencies. An electron accelerated by a magnetic field B will radiate. For nonrelativistic electrons the radiation is simple and called cyclotron radiation and its emission frequency is simply the frequency of gyration of the electron in the magnetic field.

However, for extreme relativistic ($\gamma \gg 1$) electrons the frequency spectrum is much more complex and extends to many times the gyration frequency. This is given the name synchrotron radiation. The cyclotron (or gyration) frequency ω_B is

$$\omega_B = \frac{qB}{\gamma m c} \tag{480}$$

For the extreme relativistic case, aberration of angles cause the radiation from the electron to be bunched and appear as a narrow pulse confined to a time period much shorter than the gyration time. The net result is an emission spectrum characterized by a critical frequency

$$\omega_c \equiv \frac{3}{2}\gamma^2\omega_B sin\alpha = \frac{3\gamma^2 qB}{2mc} sin\alpha \tag{481}$$

To understand the astrophysical radiation, one must consider that cosmic ray electrons are an ensemble of particles of different pitch angles α and energies E. It can generally be assumed that the directions are fairly isotropic so that integration over pitch angles is straightforward.

The next step is integration over electron energy spectrum to determine the total synchrotron radiation spectrum.

If the electrons' direction of motion is random with respect to the magnetic field, and the electrons' energy spectrum can be approximated as a power law: $dN/dE = N_0 E^{-p}$, then the luminosity is given by

$$I(\nu) = \frac{\sqrt{3}e^3}{8\pi m c^2}\left(\frac{3e}{4\pi m^3 c^5}\right)^{(p-1)/2} L N_0 B_{\text{eff}}^{(p+1)/2} \nu^{-(p-1)/2} a(p), \tag{482}$$

where $a(p)$ is a weak function of the electron energy spectrum (see Longair, 1994, vol. 2, page 262 for a tabulation of $a(p)$), L is the length along the line of sight through the emitting volume, B is the magnetic field strength, and ν is the frequency.

At very low frequencies synchrotron self-absorption is very important as according to the principle of detailed balance, to every emission process there is a corresponding absorption process. At the lowest frequencies synchrotron self-absorption predicts an intensity that increases as $\propto \nu^{5/2}$.

The local energy spectrum of the electrons has been measured to be a power law to good approximation, for the energy intervals describing the peak of radio synchrotron emission (at GeV energies). The index of the power law appears to increase from about 2.7 to 3.3 over this energy range (Webber 1983, Nishimura et al 1991). Such an increase of the electron energy spectrum slope is expected, as the energy loss mechanisms for electrons increases with the square of the electron energy.

The synchrotron emission at frequency ν is dominated by cosmic ray electrons of energy $E \approx 3(\nu/\mathrm{GHz})^{1/2}$ GeV. The range of energies contributing to the radiation intensity at a given frequency depends on the electron energy spectrum: the steeper the electron distribution, the narrower the energy range (Longair 1994). For the case of most of the Galaxy, this range is of order 15 to 50. The observed steepening of the electrons' spectrum at GeV energies is used to model the radio emission spectrum at GHz frequencies (e.g. Banday & Wolfendale, 1990, Platania et al. 1998).

11.9 Free electron Lasers

The Free Electron laser (FEL) is a classical device that converts the kinetic energy of an electron beam into electromagnetic radiation by passing it through a transverse periodic magnetic field (called the "wiggler"). In contrast with conventional lasers, the radiation of the FEL is not constrained by the discrete energy levels that fix the wavelength of emission. The wavelength of FEL radiation depends mainly on the wavelength of the periodic magnetic field and the energy of the electron beam. High peak powers and its large range of operational wavelengths make it a laser of the future. A simple schematic representation of the FEL is given in the following figure.

A key feature is that the FEL is a true laser producing coherent radiation. Coherent radiation happens when the FEL is biased in the resonant condition. This leads to an effect where the electrons bunch more tightly so that they radiate as a single coherent bunch. For N electrons acting independently, the radiation is proportional to Ne^2. If the N electrons act coherently, as if a single particle, then the radiation is proportional to N^2e^2.

One could seed the laser with an electromagnetic wave for specific applications but to have a completely tunable laser, generally the FEL operates on the principle of a single-pass free electron laser operating the self-amplified spontaneous emission (SASE) mode. Electron motion through the undulator with alternating magnetic fields forces the electrons into a sinusoidal trajectory leading to electromagnetic radiation which recouples to the electron bunch causing laser action through SASE. The radiated power increases along the electron beam path leading to exponential increase in intensity. With high enough electron current and long enough undulator the power is saturated and energy oscillates between the electron and photon beam. If the resonant condition is met the energy exchange between the electron and photon beam leads to microbunching and coherent emission.

It should be noted that the FEL does not require any mirrors or resonating laser cavity structure. This is a great advantage at short wavelengths where, for example, mirrors and optics are technicaly difficult.

One can think about the FEL in steps: (1) What is the wavelength of light emitted by an electron traveling down the FEL magnet structure? Once can find this by using the synchrotron radiation formula or by transforming to the rest frame of the electron to find the frequency of oscillation by the magnets and then transforming the radiation to the lab by the Doppler formula. The approximate answer is

$$\lambda_\gamma = \frac{\lambda_{\text{magnetic structure}}}{2\gamma^2}$$

(2) What is the resonant condition? The undulator gives a resonance condition between the electron bunch and the electromagnetic wave, when one undulator period (travel length) λ_u gives a time difference between the electron bunch and electromagnetic wave corresponding to one period of the electromagnetic wave. In that situation the electrons are always going uphill against the electric field and thus adding power to the electromagnetic wave. That condition for very small transverse

movement of the electrons is that

$$\Delta t = \lambda_u/v - \lambda_u/c = \lambda_\gamma/c = \lambda_u/(2\gamma^2 c)$$

11.10 High Gain Free Electron Lasers

Motivation for high-gain FELs are as microwave sources for advanced accelerators and efficient sources of short wavelength radiation. The basic physics is that a beam of electrons is injected along the axis of an undulator (a transverse, periodic (λ_0), magnetostatic field $B_o(z)$, N_o periods). The electrons are periodically deflected and as a result radiate synchrotron radiation. The primary features of synchrotron radiation are spontaneous emission which is incoherent: $I \sim N_e$, in a narrow cone: $\theta \sim 1/\gamma$, and narrow bandwidth:

$$\frac{dI}{d\omega d\Omega} \sim sinc^2\left(\pi N_0 \frac{\omega - \omega_s}{\omega_s}\right) \tag{483}$$

which peaks at $\omega = \omega_s = 2\pi/\lambda_s$ (and we will see that the resonant condition is at $\lambda_s = (1-\beta_{||})\lambda_o/\beta_{||}$.)

In the electron rest frame the wiggler field looks like N_0 period radiation field with wavelength

$$\lambda_s' = \lambda_0' = \lambda_0/\gamma_{||}$$

where $\gamma_{||}^2 = 1/(1-\beta_{||}^2)$. Thus the electron oscillaes N_0 times. It produces a wavepacket of length $N_0\lambda_0'$ peaked at wavelength λ_0'. The spectrum of the radiation is the Fourier transform of a plane wave truncated after N_0 oscillations:

$$I(\omega) = sinc^2\left(\pi N_0 \frac{\Delta\omega}{\omega_s}\right) \tag{484}$$

In the laboratory frame, λ_s is the Doppler upshifted wavelength

$$\lambda_s = \frac{\lambda_s'}{2\gamma_{||}} \simeq \frac{\lambda_0}{2\gamma_{||}^2} \tag{485}$$

The exact solution is

$$\lambda_s = \frac{1-\beta_{||}}{\beta_{||}}\lambda_0$$

A free electron laser has tunability via change in electron energy or the undulator.

$$\frac{1}{\gamma^2} = 1 - \beta_{||}^2 - \beta_\perp^2 \simeq \frac{1}{\gamma_{||}^2} - \frac{a_0^2}{\gamma^2} \tag{486}$$

where a_0 is he dimensionaless vector potential of the undulator

$$a_0 = \frac{e\lambda_0 B_0}{2\pi m_0 c^2} \qquad \text{helical}$$

$$
\begin{aligned}
&= 0.934 B_0 \lambda_0 \qquad \text{per Tesla cm} \\
&= \frac{e\lambda_0 B_0}{2\sqrt{2}\pi m_0 c^2} \qquad \text{planar} \\
&= 0.66 B_0 \lambda_0 \qquad \text{per Tesla cm}
\end{aligned} \tag{487}
$$

and

$$\gamma_{\|}^2 = \frac{\gamma^2}{1+a_0^2} \tag{488}$$

or

$$\lambda_s = \frac{\lambda_0}{2\gamma^2}\left(1+a_0^2\right) \tag{489}$$

11.10.1 Stimulated Emission

Inject a laser beam with $\lambda \simeq \lambda_s$ along the axis of the undulator. The electrons move along curved path at $v_e<c$. Therefore $v_{\|}<c$. Light moves down the4 axis at $v_z = c$. If an electron has the resonant energy $E_R = \gamma_R m_0 c^2$

$$\gamma_R^2 = \frac{\lambda_0}{2\lambda}\left(1+a_0^2\right) \tag{490}$$

then the relative phase between transverse electron and radiation oscillations remains constant. Depending upon the phase the electron can give energy to the field and decellerate, $\dot{\gamma}<0$ (stimulated emission) or take energy from the field and accelerate, $\dot{\gamma}>0$.

An issue is that at the entrance of the undulator the electron phases are randomly distributed. For low gain, half of electrons will accelerate and half will decellerate. For low gain $<\gamma_0>>\gamma_R$. This is what is observed for the first FEL, the Madey laser in 1976 operated at 10.6 μm.

But if undulator is long enough and the current is high enough, then energy modulation will result in space modulation. There will be "self-bunching" and it will be around a "right" phase for gain. Most electrons will have the same phase and the intensity will be proportional to the number of electrons squared. $I \propto N_e^2$. This is collective instability of self-bunching and exponential gain.

11.10.2 Self-Consistent Theory

To fully describe FELs, we need a many particle, self-consistent theory that combines relativity for the electron mechanics and trajectories including the transverse current $J_\perp$, Maxwell's equations (or the special relativistic version), and an expression for the radiation field.

Wiggler Field

$$\vec{B}_0 = \vec{\nabla} \times \vec{A}_0$$

Radiation Field

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

Trajectory Equation

$$\frac{dp}{dt} = \frac{d}{dt}(\gamma m_0 v) = e\left(\vec{E} + \frac{1}{c}\vec{v} \times (\vec{B}_0 + \vec{B})\right)$$

Energy Equation

$$\frac{dE}{dt} == \frac{d}{dt}(\gamma m_0 c^2) = e\vec{E} \cdot \vec{v} = eEv_\perp$$

The total field on electrons from the vector potential $\vec{A}_{tot}$

$$\vec{A}_{tot} = \vec{A}_0 + \vec{A}$$

which is the total from the wiggler and radiation. $\vec{A}_0$ is periodic (spatially) either planar or helical

$$\vec{A}_0 = \frac{1}{\sqrt{2}}(\hat{e}e^{-ik_0 z} + c.c.)$$

for the helical field which leads to circularly polarized radiation:

$$\vec{A} = -\frac{i}{\sqrt{2}}\left[A\hat{e}e^{I(k_\parallel - \omega t)} - c.c.\right]$$

where $\omega = ck = c\sqrt{k_\parallel + k_\perp^2}$ where $k_\perp$ allows for waveguides. Let $k_\perp = 0$. Then

$$\begin{aligned} \frac{d}{dt}(\gamma m_0 v_\perp) &= e\left[E + \frac{1}{c}(v \times B)_\perp\right] \\ &= -\frac{e}{c}\left[\frac{\partial A_{tot}}{\partial t} - (v \times \nabla \times A_{tot})_\perp\right] \\ &= -\frac{e}{c}\frac{d}{dt}A_{tot} \end{aligned} \tag{491}$$

$$\frac{d}{dt}(\gamma\beta_\perp) = -\frac{e}{mc^2}\frac{dA_{tot}}{dt}dt a_{tot} \tag{492}$$

For perfect on-axis injection $\beta_\perp(0) = 0$ and

$$\beta_\perp = -\frac{a_{tot}}{\gamma} \simeq -\frac{a_0}{\gamma}$$

12 Uniform Acceleration

This material is to prepare a transition towards General Relativity via the *Equivalence Principle* by first understanding uniform acceleration.

The *Equivalence Principle* stated in a simple form: *Equivalence Principle:* A uniform gravitational field is equivalent to a uniform acceleration.

This is not very precise statement and one lesson we have learned in Special Relativity is the need to be precise in our statements, definitions, and use of coordinates. We will come to a more precise statement of the *Equivalence Principle* in terms like at a space-time point with gravitational acceleration $\vec{g}$ there is a tangent reference frame undergoing uniform acceleration that is equivalent. This is similar to the instantaneous rest frame of Special Relativity in the case of an object undergoing acceleration.

We first need to understand carefully what is a uniform acceleration reference frame, which we will do in steps.

First imagine a reference frame – a rigid framework of rulers and clocks, our standard reference frame – undergoing uniform acceleration. In classical nonrelativistic physics we can imagine a rigid framework to which we can apply a force which will cause it to move with constant acceleration.

However, in Special Relativity no causal impulse can travel faster than the speed of light, thus the frame work cannot be infinitely rigid. When the force causing the acceleration is first applied, the point where the force is first applied begins to accelerate first and as the casual impulse moves out, the other portions join in the acceleration.

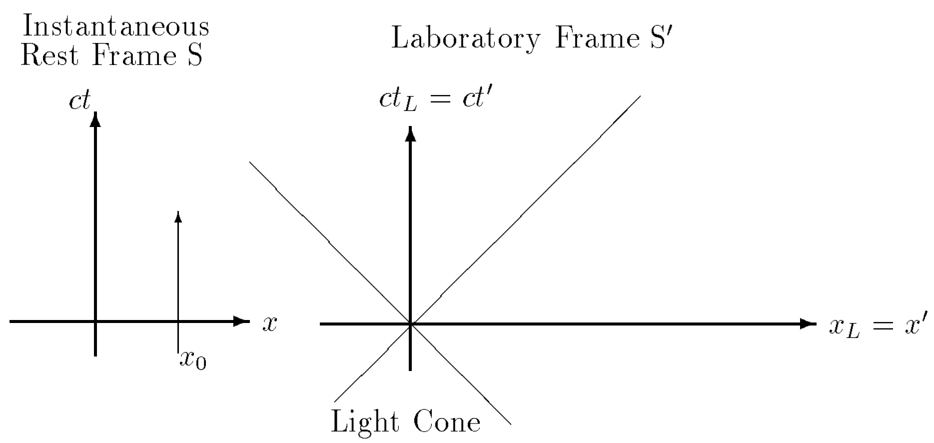
Consider a simple long rod as an example: If one pulls on a long rod, it will lengthen at first as the end being pulled starts moving before the other end even knows it is. Then as it gains speed, Lorentz-FitzGerald contraction will cause it to shorten. If one pushes on the long rod from behind, it will first shorten as the end with the force moves toward the other end which sits there unaware of the some to arrive acceleration. All objects, however rigid, evidently display some degree of elasticity during acceleration. It is clear that in Special Relativity no rod can be infinitely rigid but must be elastic at some level. (Home work problem: prove that since the speed of sound is less than or equal to the speed of light, that the rigidity of any material is less than xxx?)

As a body accelerates, it moves in a continuous fashion from one inertial system to another. If it is to retain its same rest length in its instantaneous rest system, then it length relative to its original inertial system will have to decrease continuously because of Lorentz-FitzGerald length contraction. If, on the other hand, it retained the same length relative to the original inertial system, then the Lorentz-FitzGerald contraction would require its rest length to increase as its gains speed. This is not very satisfactory.

Either way, the metric will depend upon time. If we want a direct comparison to gravity, we need to require an accelerated coordinate system to have a time independent form.

12.1 Accelerating a Point Mass

A uniformly accelerating point mass is one that is subject to the same force in each and every one of its instantaneous rest systems. I.e. a uniformly accelerating point mass is subject to a constant force $\vec{F} = m_o\vec{g}$ along the $+x$-axis in a coordinate system which is the inertial frame where its velocity is zero (instantaneous rest frame).



Acceleration transforms as

$$a_x = \frac{d^2x}{dt^2} = \frac{a'_x}{\gamma^3\left(1+\frac{vu'_x}{c^2}\right)^3} \tag{493}$$

so that in the instantaneous rest frame $a = \gamma^{-3}a'$. In the instantaneous rest frame $F_x = F'_x$. Now we can solve the equation of motion in either of two ways: from the acceleration or from the force. In Problem Set 2 we solved the problem for a uniformly accelerating rocket using the acceleration transformation. [3]

Here we use force transformation.

$$\begin{aligned} F'_x = \frac{dp'_x}{dt'} &= m_o g = F_x \\ \frac{d\gamma m_o \beta c^2}{dct'} &= m_o g \\ d(\gamma\beta) &= \frac{g}{c^2}d(ct') \\ \gamma\beta &= \frac{g}{c^2}(ct') = \frac{gt'}{c} \end{aligned} \tag{494}$$

where the constant of integration is set equal to zero because we define the time zero to be when $\beta = 0$. This can be turned into an equation for β alone:

$$\begin{aligned} \gamma\beta &= \frac{\beta}{\sqrt{1-\beta^2}} = \frac{g}{c^2}(ct') = \frac{gt'}{c} \\ \frac{\beta^2}{1-\beta^2} &= \left(\frac{gt'}{c}\right)^2 \end{aligned}$$

[3] In rocket frame the acceleration was $a'_x = g$ (Note reversal of S′ and S compared to discussion in this section.) Thus the acceleration in the Earth frame was $a_x = \left(1 - v^2/c^2\right)^{3/2} g = dv_x/dt$. Regrouping we had $gdt = du_x/\left(1-v^2/c^2\right)^{3/2}$ and integrating gives $gt = v/\sqrt{1-v^2/c^2}$ or $v/c = gt/c/\sqrt{1+(gt/c)^2}$.

$$\beta = \frac{(gt'/c)}{\sqrt{1+(gt'/c)^2}} \tag{495}$$

So that, from the laboratory, observing the test particle start from rest we first see its velocity increasing linearly with time as we classically expect for a particle under uniform acceleration. Then as the velocity begins to be a significant fraction of the speed of light, the term in the denominator becomes increasing important and the velocity increases ever more slowly in time and only approaches the speed of light asymptotically. The shape of the trajectory of a particle undergoing uniform acceleration is a hyperbola and not the classical parabola, but for low velocities they are indistinguishable conics.

Note also

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{1+\left(\frac{gt'}{c}\right)^2} \tag{496}$$

If we look at the Lorentz factor γ, we see that it is first very nearly unity and then as the velocity begins to saturate, γ increases linearly with time. This is simply conservation of energy, as the constant acceleration (force in the instantaneous rest frame) is constantly doing work $W = cF$.

Now take a side step and evaluate in terms of proper time.

$$d\tau = \frac{dt'}{\gamma} = \frac{dt'}{\sqrt{1+\left(\frac{gt'}{c}\right)^2}}$$

$$\tau = \int \frac{dt'}{\sqrt{1+\left(\frac{gt'}{c}\right)^2}} = \frac{c}{g} sinh^{-1}\left(\frac{gt'}{c}\right)$$

or inverting the equation to find t' in terms of proper time τ

$$t' = \frac{g}{c} sinh\left(\frac{g\tau}{c}\right) \tag{497}$$

end aside

Now we can solve for x' using the definition of $\beta = dx'/d(ct')$.

$$\begin{aligned} dx' &= \beta d(ct') \\ x' &= x_o + \int_{ct'=0}^{ct'=ct'} \beta d(ct') = x_o + \int_{ct'=0}^{ct'=ct'} \frac{gt'/c}{\sqrt{1+(gt'/c))^2}} d(ct') \\ &= \frac{c^2}{g}\sqrt{1+\left(\frac{g}{c^2}(ct')\right)^2}\Big|_{t'=0}^{ct'=ct'} + x'_o \\ &= \frac{c^2}{g}\sqrt{1+(gt'/c)^2} - \frac{c^2}{g} + x'_o \end{aligned} \tag{498}$$

Define

$$x'_P \equiv x'_o - \frac{c^2}{g} \tag{499}$$

Then our equation becomes

$$\begin{aligned}
x' - x'_P &= \frac{c^2}{g}\sqrt{1 + \left(\frac{g}{c^2}ct'\right)^2} \\
\left(\frac{g}{c^2}(x' - x'_P)\right)^2 &= 1 + \left(\frac{g}{c^2}ct'\right)^2 \\
\left(\frac{g}{c^2}(x' - x'_P)\right)^2 - \left(\frac{g}{c^2}ct'\right)^2 &= 1
\end{aligned} \tag{500}$$

This last equation describes a hyperbola. Note that we can find a formula for x in terms of τ given the last equation for x' in terms of t'.

$$(g/c^2)^2(x' - x'_P)^2 - (gt'/c)^2 = 1$$

$$(g/c^2)^2(x' - x'_P)^2 = 1 + (gt'/c)^2 = 1 + sinh^2(g\tau/c) = cosh^2(g\tau/c)$$

or

$$x' = x'_P + \frac{c^2}{g}cosh\left(\frac{g\tau}{c}\right) = x'_0 - \frac{c^2}{g} + \frac{c^2}{g}cosh\left(\frac{g\tau}{c}\right)$$

Note that x'_0 is directly related to the x_0 in the accelerating frame.

Because the world line is a hyperbola in Minkowski space, the world line of the point mass approaches the light line asymptotically. This means all events on the world line will have a space like relationship to all events to the left of the focal point $P \equiv (0, x'_P)$.

$$x'_P = x'_o - \frac{c^2}{g} \tag{501}$$

So that the distance between the rest point and focal point is proportional to the inverse of the acceleration.

insert figure here showing frames with small acceleration and with large accelerations.

$$\beta = \frac{gt'/c}{\sqrt{1 + \left(\frac{gt'}{c}\right)^2}} \qquad \frac{g}{c^2}\left(x' - x'_p\right) = \sqrt{1 + \left(\frac{gt'}{c}\right)^2} \tag{502}$$

Therefore

$$\beta = \frac{ct'}{x' - x'_P} = tan\theta \tag{503}$$

where θ is the horizontal angle.

insert figure here showing θ etc. The line from point P, $(0, x'_P)$ to point (ct', x') is the x axis in the instantaneous rest frame. Defines simultaneity in instantaneous rest frame is changing constantly since the instantaneous rest frame is continuously changing.

insert figure here showing world lines etc. and that P is a pivot point.

The observer A no matter where along his world lines never knows the future of the observer passing through the pivot point and objects to the left are never in casual contact but if they did would appear to move backward through time. ....

Now calculate the distance from event P = $(0, x'_P)$ to event (ct', x')

$$\left(\frac{c^2}{g}\right)^2 = \left(x' - x'_P\right)^2 - c^2 t'^2 \tag{504}$$

combining that with the equation for β yields

$$\begin{aligned} \beta &= \frac{ct'}{x' - x'_P} \\ \beta^2 \left(x' - x'_P\right)^2 &= c^2 t'^2 \end{aligned} \tag{505}$$

Evaluate this for $t' = 0$ to get the distance, x_{PA}, between event P and where A crosses the x' axis.

$$(x_{PA})^2 = \left(\frac{c^2}{g}\right)^2 = \left(x' - x'_P\right)^2 - \beta^2 \left(x' - x'_P\right)^2 = \left(1 - \beta^2\right)\left(x' - x'_P\right)^2 \tag{506}$$

$$(x' - x'_P) = \gamma x'_{PA} \tag{507}$$

Lorentz contraction $x'_{PA} = x_{PA}/\gamma$.

It is easy to show that the distance from the pivot point to any point on the hyperbolic trajectory is the same. The accelerating system moves in such a way that the distance to the pivot point is increasing in inertial space by precisely its instanteous gamma so that the Lorentz length contraction makes the distance to the pivot point in its rest frame constant. I.e. if the line of simultaneity intersects A's trajectory at point B then from the hyperbola formula above for all B we have

$$(x'_B - x'_P) = \gamma x'_{PA} \tag{508}$$

Thus $x_B - x_P = x_{PB} = \gamma x'_{PA}$ The distance from the pivot point event $(0, x_P)$ to the mass point at B as measured in the accelerated coordinate system is the same as the distance from the pivot point event $(0, x_P)$ to the mass point when it was at rest or any other point on its trajectory. Therefore to an observer in the accelerated system the point mass maintains a fixed distance to the pivot point event $(0, x_P)$ throughout its motion. Thus despite accelerating away continously the eternal moment remains a fixed distance away.

12.2 Uniformly Accelerated Reference Frame

We are now in a position to discuss a uniformly accelerated reference frame.

insert figure of two uniformly accelerating masses with same focal point.

Consider two observers (1) and (2) both with the same focus point x'_p and both cross the x'-axis at the same $t' = 0$. Then there is always the same distance from x'_p and thus each other. As a result they will have to have different accelerations because they have the same focus

$$a_1 = g_1 = c^2/x'_1 \qquad a_2 = g_2 = c^2/x'_2 \tag{509}$$

This is what one sees in the figure with the curves further away from the focal point being flatter. A straight line is a the world line for a non-accelerating particle.

One can make a uniformly accelerated frame, if the acceleration of each point is inversely proportional to its distance from the focus point x'_p. Actually $(ct', x') = (0, x'_p)$.

An observer riding with a meter stick in this accelerated frame would say it maintained a constant length. An observer in an inertial frame (e.g. our Lab frame) claims the rod is shrinking in time as it accelerates away. However, as it approaches the origin, it lengthens and slows down.

A rod on the other side of the origin accelerates to the left rather than the right.

This situation is called a Rindler Space.

Note that the coordinate choices are different from our usual every day conventions. Usually we chose the vertical axis to be the z-axis and have the effective acceleration "downward" toward negative z. What we would observe conventionally from our inertial frame would be an elevator rushing doward towards us at high speed and decellerating at a rate g coming to a stop at a distance and then accelerating upwards away retracing its path.

Note that the acceleration depends on distance away from pivot point. That is the only system in which the clocks in the instanteous rest frame do not have to continuously be reset relative to each other.

12.2.1 Coordinate System for a Uniformly Accelerating System

There are two natural frames in which to describe a uniformly accelerating system. They are the accelerating coordinates τ the proper (rest frame) time and related coordinates and the coordinates of one inertial frame that matches the accelerating frame at one instant.

The transformation between the coordinate systems is given by

$$x' = -\frac{c^2}{g} + \left(x + \frac{c^2}{g}\right) cosh\left(\frac{g\tau}{c}\right)$$

$$ct' = \left(x + \frac{c^2}{g}\right) sinh\left(\frac{g\tau}{c}\right) \tag{510}$$

where g is standing in for the uniform 3-acceleration magnitude a. Note that this is what one must expect for the form of the time transformation. At the pivot point $x = x_P = -c^2/g$ time must be frozen (thus the name pivot point). If we put that value in, then we find $ct' = 0$ independent of τ As one moves away from the pivot point, the conversion to t' must increase linearly with the distance from the pivot point. Thus it must be proportional to $x + c^2/g$.

12.2.2 The Metric for a Uniformly Accelerating System

We want to find $g_{\mu\nu}$

$$(cd\tau)^2 = ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

For an inertial frame we have the Minkowski metric $\eta_{\mu\nu}$ which in Cartesian coordinates is diagonal and constant

$$\eta_{\mu\nu} = [1, -1, -1, -1]$$

How about expressing this in terms of the proper time τ and the instanteous rest frame coordinates x, y, and z?

$$dx^0 = cdt = sinh(g\tau/c)dx^1 + (1 + gx/c^2)cosh(g\tau/c)d\tau$$

$$dx^1 = dx' = cosh(g\tau)dx^1 dx^1 + (1 + gx/c^2)sinh(g\tau/c)d\tau$$

$$(cd\tau)^2 = ds^2 = (1 + gx/c^2)^2(cd\tau)^2 - (d\vec{x})^2$$

$$g_{\mu\nu} = [(1 + gx/c^2)^2, -1, -1, -1] \qquad g^{\mu\nu} = [(1 + gx/c^2)^{-2}, -1, -1, -1]$$

Note the difference between covariant and contravariant.

12.3 Alternate Discussion to be integrated

We revisited the Lorentz transformation in the case of circular motion, that is, motion with a uniform speed but continously changing direction, in the case that results in Thomas precession. Now we consider the velocity transformation.

12.3.1 Instanteous Velocity Transformation

The Lorentz transformations of space-time coordinates

$$\begin{aligned} t' &= \gamma\,(t-\beta x/c) \\ x' &= \gamma\,(x-\beta ct) \\ y' &= y \\ z' &= z \end{aligned} \tag{511}$$

and their converse (primes exchanged with unprimes and $\beta = v/c$ with $-\beta$ are differentiated with respect to t' and used to find the velocity

$$\begin{aligned} \vec{u} &= (u_1,u_2,u_3) = (\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}) \\ \vec{u}' &= (u'_1,u'_2,u'_3) = (\frac{dx'}{dt'},\frac{dy'}{dt'},\frac{dz'}{dt'}) \end{aligned} \tag{512}$$

$$u'_1 = \frac{u_1 - v}{1-u_1v/c^2}, \quad u'_2 = \frac{u_2}{\gamma(1-u_1v/c^2)}, \quad u'_3 = \frac{u_3}{\gamma(1-u_1v/c^2)} \tag{513}$$

$$u_1 = \frac{u'_1 - v}{1-u_1v/c^2}, \quad u_2 = \frac{u'_2}{\gamma(1-u'_1v/c^2)}, \quad u_3 = \frac{u'_3}{\gamma(1-u'_1v/c^2)} \tag{514}$$

No assumption as the uniformity of $\vec{u}$ (or $\vec{u}'$) has been made. These equations apply equally to the instantaneous velocity in non-uniform (or circular) motion.

Now consider the magnitudes u and u' defined as

$$u^2 = u_1^2 + u_2^2 + u_3^2, \qquad u'^2 = u_1'^2 + u_2'^2 + u_3'^2 \tag{515}$$

Now we can readily calculate the $\gamma(u)$ transformation laws by factoring out the $(dt)^2$ and $(dt')^2$ from $(cd\tau)^2 = (cdt)^2 - (d\vec{r})^2 = (cdt')^2 - (d\vec{r}')^2$ and substituting in for u'_i

$$dt^2(c^2-u^2) = (dt')^2(c^2-u'^2) = dt^2\gamma^2(v)(1-u_1v/c^2)^2(c^2-u'^2). \tag{516}$$

$$c^2 - u'^2 = \frac{c^2(c^2-u^2)(c^2-v^2)}{(c^2-u_1v)^2} \tag{517}$$

$$\frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left(1-\frac{u_1v}{c^2}\right) \qquad \frac{\gamma(u)}{\gamma(u')} = \gamma(v)\left(1+\frac{u'_1v}{c^2}\right) \tag{518}$$

Now note how simple this is in the instanteous rest frame:

$$\gamma(u') = \gamma(v)\gamma(u)$$

This should remind you of the rapidity formulation given in the homework. where the rapidity is defined as the rotation angle

$$\phi(u) = tanh^{-1}\left(\frac{u}{c}\right), \qquad tanh(\phi(u) = \frac{u}{c} \tag{519}$$

$$\phi(u) = \phi(u') + \phi(v) \tag{520}$$

Differentiating this with respect to time gives us a simple way to work out the acceleration transformation.

12.3.2 Acceleration Transformation

$$\frac{d}{dt}\phi(u) = \frac{d}{dt'}\phi(u')\frac{dt'}{dt} \tag{521}$$

Since the derivative of the hyperbolic tangent is the hyperbolic secant

$$\frac{d}{dt}\phi(u) = \frac{1}{c}\gamma^2(u)\frac{du}{dt} \tag{522}$$

Since

$$\frac{dt'}{dt} = \frac{\gamma(u')}{\gamma(u)} \tag{523}$$

Substituting we obtain the acceleration transformation formula

$$\gamma^3(u')\frac{du'}{dt'} = \gamma^3(u)\frac{du}{dt} \tag{524}$$

Under the Galilean transformation, the acceleration is invariant; but, acceleration is not in Special Relativity.

We need to define the proper acceleration

$$|\tilde{a}| \equiv \alpha \equiv= \gamma^3(u)\frac{du}{dt} = \frac{d}{dt}\left[\gamma(u)u\right] \tag{525}$$

where α is measured in the instantaneous rest frame.

Now constant instanteous acceleration (constant proper acceleration) is a particularly simple case. Integrating and chosing $u = 0$ at $t = 0$ (or vice versa) one finds

$$\alpha t = \gamma(u)u \tag{526}$$

Thus at low velocity u increases linearly with t and as $u \to c$ $\gamma(u)$ grows linearly with time. Squaring, solving for u, and integrating again, chosing zero as the constant of integration yields

$$x^2 - (ct)^2 = c^4/\alpha \equiv X^2 \tag{527}$$

Thus, for obvious reasons, rectilinear motion with constant proper acceleration is called hyperbolic motion.

12.4 Rindler Space, Symmetry and GR

The equivlance principle implies a new symmetry and thus associated invariance. With a realization and the uniqueness of solutions give a formulation to the theory of gravity.

The strong and weak Equivalence Principle: The weak equivalence principle is that gravitational and inertial masses are precisely equal (also includes Lorentz invariance). The strong equivalence principle applies to all laws of nature that no experiment can distinguish between an accelerating frame of reference and a uniform gravitational field.

We can also use this symmetry approach to find the Rindler space. Consider an "generalized elevator" as a kind of rocket ship in outer space far from the strong influence of Earth or any other body. Now give the "elevator" a constant acceleration g upwards. All inhabitants of the "elevator" will feel the pressure from the floor, just as if they were living in the gravitational field at the surface of the Earth (or equivalent). This is a method of constructing "artificial" gravitational field. We now consider this artificial gravitational field more carefully.

Suppose we want this artificial gravitational field to be constant in space and time. We will find that we can make the artificial gravitational field uniform in time and two spatial directions but it must decrease in the direction of the field itself. The inhabitants will feel a constant acceleration.

Consider a coordinate grid for an elevator free to accelerate uniformly or be in a uniform gravitational field, which we take to be ξ^μ inside the elevator, such that points on the elevator wall and floor are given by ξ^i and are constant. The zeroth component $\xi^0 = c\tau$, where τ is the proper time (elapsed instanteous rest time in the elevator). An observer in outer space uses a standard Cartesian grid x^μ in an inertial frame there. The motion of the elevator is described by the function $x^\mu(\tilde{\xi})$.

That is the elevator is free to move only along one axis (the "vertical" axis). We designate the "vertical" direction to be the z-axis. The origin of the $\tilde{\xi}$ coordinates is a point in the middle of the floor of the elevator, which for convenience coincides with the origin of the $\tilde{x}$ coordinates at $t = \tau = \xi^0(\tau) = 0$. Thus the coordinates of the origin (center point of elevator floor) will be

$$\tilde{\xi} = (c\tau, 0, 0, 0) \qquad \tilde{x}_c = (ct(\tau), 0, 0, z(\tau)) \tag{528}$$

Time (τ) run at a constant rate for the observer inside the elevator.

$$\left(\frac{\partial x^\mu}{\partial \tau}\right)^2 = \left(\frac{\partial ct}{\partial \tau}\right)^2 - \left(\frac{\partial z}{\partial \tau}\right)^2 = c^2. \tag{529}$$

The acceleration is set to be $\vec{g}$, which is the spatial portion of the four-acceleration:

$$\tilde{a} = \frac{\partial^2 x^\mu}{\partial \tau^2} = g^\mu. \tag{530}$$

At $\tau = 0$ we can specify that the velocity of the elevator is zero:

$$\frac{\partial x^\mu}{\partial \tau} = (c, \vec{0}) \quad (\text{at } \tau = 0). \tag{531}$$

We can make use of the differential proper time along any world line $d\tau = dt/\gamma$. Using the relation

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{1 + \left(\frac{gt}{c}\right)^2} \tag{532}$$

we find

$$\tau = \int \frac{dt}{\sqrt{1 + \left(\frac{gt}{c}\right)^2}} = \frac{g}{c} sinh^{-1}\left(\frac{gt}{c}\right) \tag{533}$$

Inverting this equation we find a relationship for t in terms of τ

$$\frac{gt}{c} = sinh\left(\frac{g\tau}{c}\right) \tag{534}$$

This equation works for the origin. The acceleration depends upon location so that the more general formula becomes

$$ct = \left(\xi^3 + \frac{c^2}{g}\right) sinh\left(\frac{g\tau}{c}\right) \tag{535}$$

$$z = x^3 = \left(\xi^3 + \frac{c^2}{g}\right) cosh\left(\frac{g\tau}{c}\right) - \frac{c^2}{g} \tag{536}$$

At that moment t and τ coincide, and if the acceleration $\vec{g}$ is to be be constant, then at $\tau = 0$, $\partial\vec{g}/\partial\tau = 0$, so that

$$\frac{\partial}{\partial \tau} g^\mu = (F, \vec{0}) = \frac{F}{c}\frac{\partial}{\partial \tau} x^\mu \qquad \tau = 0, \tag{537}$$

where F is an unknown constant.

Now this equation is Lorentz covariant. So not only at $\tau = 0$, but also at all times we should have

$$\frac{\partial}{\partial \tau} g^\mu = \frac{F}{c}\frac{\partial}{\partial \tau} x^\mu \tag{538}$$

Combining equations x and y gives

$$\begin{aligned} g^\mu &= \frac{F}{c}(x^\mu + A^\mu) = \frac{g^2}{c^2}(x^\mu + A^\mu) = \frac{g^2}{c^2}(x^\mu + \delta_3^\mu \frac{c^2}{g}), \\ x^\mu(\tau) &= B^\mu cosh(g\tau/c) + C^\mu sinh(g\tau/c) - A^\mu, \end{aligned} \tag{539}$$

F^μ, A^μ, B^μ, and C^μ are constants. $F = g^2/c$ can be found from the derivative of four acceleration evaluated at $\beta = 0$. Then from equations 16, 17, and the boundary conditions:

$$(g^\mu)^2 = cF = g^2, \quad B^\mu = \frac{c^2}{g}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad C^\mu = \frac{c^2}{g}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A^\mu = B^\mu, \tag{540}$$

and since at $\tau = 0$, the acceleration is purely spacelike. We find that the parameter g is the absolute value of the acceleration.

We notice that the position of the elevator floor at "inhabitant time" τ is obtained from the position at $\tau = 0$ by a Lorentz boost around the point $x^\mu = -A^\mu$. This must imply that the entire elevator is Lorentz-boosted. The boost is given by the rotation matrix with angle $\chi = g\tau/c$. This observation immediately gives the coordinates of all other points in the elevator. Suppose at $\tau = 0$,

$$x^\mu(0, \vec{\xi}) = (0, \vec{\xi}) \tag{541}$$

Then at other τ values

$$x^\mu(c\tau, \vec{\xi}) = \begin{pmatrix} sinh(g\tau/c)\left(\xi^3 + \frac{c^2}{g}\right) \\ \xi^1 \\ \xi^2 \\ cosh(g\tau/c)\left(\xi^3 + \frac{c^2}{g}\right) - \frac{c^2}{g} \end{pmatrix} \tag{542}$$

The 0 and 3 (height) components of the ξ coordinates, imbedded in the x coordinates, are pictured in the next figure. The light cone defines the boundary of the space at $\tau = 0$ the coordinates lie on the positive x^3 axis in a very ordinary way. Each x^3 coordinate follows a hyperbola in x^3 and $c\tau$ that keep it in the right quadrant (in x^3- $c\tau$ plane. The description of the quadrant of space time in terms of the ξ coordiates is called "Rindler space".

It should be clear that an observer inside the elevator feels no effects that depend explicity on his time coordinate τ, since a transition for τ to t' is nothing but a Lorentz transformation.

We also notice some important effects:
(i) Equal τ lines (lines of simultaneity) converge at the left (at the $x^3 - c\tau$ origin). It follows that the local clock speed, which is given by $\eta = \sqrt{(\partial x^\mu/\partial c\tau)^2}$. varies with height $\equiv x^3$:

$$\eta = 1 + g\xi^3/c^2, \tag{543}$$

(ii) The acceleration or gravitational field strength felt locally is $\eta^{-2}\vec{g}(\xi)$, which is proportional to the distance to the point $x^\mu = -A^\mu$. So even though the field is constant in the transverse direction and with time, it decreases with height (x^3).

(iii) The region of space-time described by the observer in the elevator is only part of all of space-time, where $x^3 + c^2/g > |x^0|$. The boundary lines are called (past and future) horizons.

All of these are trypically relativistic effects. In the non-relativistic limit $(g \to 0)$ the coordinates simplfy to

$$x^3 = \xi^3 + \frac{1}{2} g\tau^2; \quad x^0 = c\tau. \tag{544}$$

According to the equivalence principle the relativistic effects discovered here should also be features of gravitational fields generated by matter (or energy). Let us inspect them individually.

Observation (i) suggest that clocks will run slower, if they are deep down in a gravitational field. Indeed as one suspects equation x will generalize to

$$\eta = 1 + \Phi(x)/c^2 \tag{545}$$

where $\Phi(x)$ is the gravitational potential. This will be true, provided that the gravitational field is stationary (not time varying). This effect is called the gravitational redshift.

Relativistic effect (ii) could have been predicted by the following argument. The energy density of a gravitational potential is negative. Since the energy of two masses M_1 and M_2 at a distance r apart is $E = -G_n M_1 M_2 / r$, we can calculate the energy density of a field $\vec{g}$ as $T_{00} = -(1/8\pi G_n)|\vec{g}|^2$. If we have normalized $c = 1$, this is also its mass density. But then this mass density in turn should generate a gravitational field! This would imply

$$\vec{\partial} \cdot \vec{g} = 4\pi G_n T_{00} = -\frac{1}{2}|\vec{g}|^2$$

so that the field strength should decrease with height. However, this reasoning is too simplistic, since the field obeys a differential equation but without the coefficient 1/2.

The possible emergence of horizons (iii) turns out to be a new feature of relativistic gravitational fields. Under normal circumstances the fields are so weak that no horizon will be seen, but gravitational collapse may produce horizons. If this happens, there will be regions of space-time from which no signals can be observed.

The most important conclusion to be drawn is that in order to describe a gravitational field, one may have to perform a transformation from the coordinates ξ^μ that were used inside the elevator where one feels the gravitational field, toward coordinates x^μ that describe empty space-time, in which freely falling objects move along straight lines. Now we know that in an empty space without gravitational fields the clock speeds and the lengths of the rulers are described by a distance fuction $c\tau$ or ℓ as

$$(cd\tau)^2 = -(d\ell)^2 = g_{\mu\nu} dx^\mu dx^\nu; \text{ where } g_{\mu\nu} = \eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1) \tag{546}$$

In tems of the coordinates ξ^μ appropriate for the elevator, we have for infinitesimal displacement $d\xi^\mu$,

$$\begin{aligned} dx^0 &= sinh(g\tau/c)d\xi^3 + (1+g\xi^3/c^2)cosh(g\tau/c)d\tau, \\ dx^3 &= cosh(g\tau/c)d\xi^3 + (1+g\xi^3/c^2)sinh(g\tau/c)d\tau. \end{aligned} \tag{547}$$

This implies

$$(cd\tau)^2 = -(d\ell)^2 = (1+g\xi^3/c^2)^2(dc\tau)^2 - (d\vec{\xi})^2. \tag{548}$$

If we write this in the form

$$(cd\tau)^2 = -(d\ell)^2 = g_{\mu\nu}(\xi)d\xi^\mu d\xi^\nu = (1+g\xi^3/c^2)^2(dc\tau)^2 - (d\vec{\xi})^2. \tag{549}$$

then we see that all effects that the gravitational field have on rulers and clocks can be described in terms of space and time dependent field $g_{\mu\nu}(\xi)$. Only in the gravitational field of a Rindler space can one find coordinates x^μ inerms of these the function $g_{\mu\nu}$ takes the simple form shown. We will see that $g_{\mu\nu}(\xi)$ is all that is ned to describe the gravitational field completely.

Spaces in which the infinitesimal distance $cd\tau$ or $d\ell$ is described by a space time dependent fuction $g_{\mu\nu}(\xi)$ are called curved or Riemann spaces. Space-time is apparently a Riemann space.

We can write the metric more explicitly as

$$g_{\mu\nu} = \begin{pmatrix} \eta(\xi)^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1/\eta(\xi)^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{550}$$

where $\eta(\xi) = 1 + g\xi^3/c^2$.

$$dx^\mu = (cd\tau, d\vec{r}) \qquad sx_\mu = \eta^2 cd\tau, d\vec{r}) \tag{551}$$

Note since the metric has the form

$$(cd\tau)^2 = -(d\ell)^2 = (1+g\xi^3/c^2)^2(dct^2 - (d\vec{r})^2. \tag{552}$$

then an object stationary at fixed $\vec{r}$ has proper time $d\tau = \eta dt$. A particle moving with velocity $\vec{v} = d\vec{r}/dt$ will have proper time

$$d\tau = \sqrt{(\eta dt)^2 - (d\vec{r}/c)^2} = dt\sqrt{\eta^2 - \beta^2} = dt/\gamma^* \tag{553}$$

where $\gamma^* = 1/\sqrt{\eta^2 - \beta^2}$.

12.4.1 Uniformly Accelerating Clocks - Gravitational Freqency Shift

We could have derived these results by simply considering two clocks in a spaceship ("rocket elevator") with constant acceleration or the Doppler effect on a photon emitted at one end of the spaceship and received at the other and comparing this to our thought experiment about a photon in a gravitational field. This approach is much more physical but does not show all the features of the Rindler space.

Consider that the inertial (and by inference gravitational) "mass" of a particle is given by the sum of its rest energy plus all other energies divided by c^2. Thus the inertial and gravitational mass of photon is $E/c^2 = h\nu/c^2$. If a photon changes its gravitational potential through simple propagation then it must change its energy by and amount $\Delta E = Egh/c^2$ where g is the acceleration of gravity and h is the height change. Thus

$$\frac{\Delta E}{E} = \frac{\Delta \nu}{\nu} = \frac{gh}{c^2} \tag{554}$$

The fractional frequency change is the change in gravitational potential divided by c^2.

For comparison consider a lab accelerating at rate g with the two clocks separated by a instanteous distance h along the acceleration direction. If the first clock sends a photon of frequency ν_{source}, then the second clock recieves a photon observed at frequency ν_{observed} we know that they are related by the Doppler formula by

$$\nu_{\mathrm{observed}} = \nu_{\mathrm{source}} \gamma \left(1 + \beta cos\theta\right) \tag{555}$$

Since the angle is either 0 or 180°,

$$\frac{\nu_{\mathrm{observed}}}{\nu_{\mathrm{source}}} = \gamma \left(1 \pm \beta\right) \tag{556}$$

The time it takes for the photon to get from the first to the second clock is approximately $\Delta t = h/c$ and the velocity change is $\Delta v = g\Delta t = gh/c$ or $\beta = gh/c^2$ Differentially one has $\nu_2/\nu_1 = 1 + gx/c^2$

The gravitational redshift was fist measured directly in the laboratory in 1960 by Pound and Rebka where they let a 14.4 keV γ-ray, emitted in the radioactive decay of ^{57}Fe, to fall 22.6 meters down an evacuated shaft where $gh/c^2 = 2.47 \times 10^{-15}$, and they measured a fractional change in frequency of $(2.57 \pm 0.26) \times 10^{-15}$, thus verifying to that level the equivalence principle.

One could anticipate that for a spherical mass (M) in an otherwise flat space-time that the rate of clocks would vary as

$$\frac{dt(r)}{dt(\infty)} = 1 - \frac{GM}{c^2 r} \tag{557}$$

12.5 Local Coordinates

It is sometimes better to use local standard clocks for the determination of velocity and acceleration at each point, rather than referring to a single coordinate clock located at the origin. The former run faster than the latter by the factor η, so that the local velocity β_L of an object moving over coordinate intervals dx and dt is given by

$$(\beta_L)^i = \frac{d}{\eta d\tau}(x^i) \quad \text{or} \quad (\beta_L)^i = \frac{1}{\eta}\beta^i \tag{558}$$

A second application of this time derivative operator to $(\beta_L)^i$ gives the connection between coordinate and local acceleration:

$$\left(\dot{\beta}_L\right)^i = \frac{d}{\eta d\tau}(\beta_L)^i = \frac{d}{\eta d\tau}\left(\frac{dx^i}{\eta d\tau}\right) = \frac{1}{\eta^2}\left[\dot{\beta}^i - \frac{\partial_x \eta}{\eta}\beta^i\beta_x\right] \tag{559}$$

since $\partial_\tau \eta = (\partial_x \eta)\beta_x$; hence,

$$(a_L)^i = \frac{1}{\eta^2}\left[a^i - \frac{g}{\eta}\beta^i\beta_x\right] \tag{560}$$

Thus the local acceleration of a free-falling body is

$$(a_L)_x = -\frac{g}{\eta}\left[1 - \frac{\beta_x^2}{\eta^2}\right], \quad (a_L)_y = \frac{g}{\eta}\frac{\beta_x\beta_y}{\eta^2}, \quad (a_L)_z = \frac{g}{\eta}\frac{\beta_x\beta_z}{\eta^2} \tag{561}$$

Thus the acceleration depends upon the local velocity and the local value of g at any point is found to be $g_L = g/\eta$ with the local velocity $(\beta_L)^i = \beta^i/\eta$ so that one can write

$$(a_L)_x = -g_L\left[1 - (\beta_L)_x^2\right], \quad (a_L)_y = -g_L(\beta_L)_x(\beta_L)_y, \quad (a_L)_z = -g_L(\beta_L)_x(\beta_L)_z \tag{562}$$

Freee-falling local acceleration appear here exclusively in terms of local velocities and the local acceleration constant g_L.

When an object falls vertically, its acceleration $(a_L)_x$ ranges between $-g_L$ and 0 depending n $(\beta_L)_x$, rather than between $-g$ and $+g$ as it does at the origin.

12.6 Dynamics

The 4-D momentum is defined in an accelerated system just as it is defined in an inertial frame.

$$\tilde{p} = m_0\tilde{u} \qquad p^\mu = m_0 u^\mu = (p^0, \vec{p}) = m_0\gamma^*(1, \beta_x, \beta_y, \beta_z) \tag{563}$$

where $\vec{p} = \gamma^* m_0 c^2 \beta$ and $p^0 = \gamma^* m_0 c^2$ where $E = m_0 c^2$ is the proper energy. For $m_0 = 0$ in these equations one replaces $\gamma^* m_0 c^2$ by E_0. Because this is a 4-D vector multiplied by an invariant, it can be found either by using he known values of β^μ in the accelerated system or by transforming the inertial 4-D momentum to the accelerated system.

In its covariant form, it is

$$p_\mu = g_{\mu\nu} p^\nu = (\eta^2 p^0, -\vec{p}) = (p_0, -\vec{p}) \tag{564}$$

so that

$$p_0 = \gamma^* \eta^2 m_0 c^2$$

There are evidently two entirely different energies of an object in the accelerated system; the covariant and he contravariant energies.

If momentum and energy are conserved in a local interaction in an inertial system, then the momentum and the covariant and contravariant energies are conserved in the accelerated system. ($\Delta p = 0$ is an invariant.)

However, if there a quantity is conserved over the path of a freely-falling particle? The answer as we will show later is that the covariant energy is.

More generally, the covariant energy of an object is constant for any time-independent metric.

12.7 Gravitational Redshift

The Equivalence principle leads directly to two interesting predictions about the behavior of light in the presence of gravity. The first effect is that as light climbs up a gravitational gradient, its frequency decreases. The second is that light is deflected by a gravitational field.

These effects are obvious, if one knows that light consists of photons where $E = h\nu$ is the relation between the photon's kinetic energy E and the photon's frequency ν. Einstein's formula relating inertial mass m_I to energy $E = m_I c^2$. The weak Equivalence Principle states $m_I = m_G$. For the work done by a gravitational field with potential Φ on a particle of gravitational mass m_G as it traverses a potential difference $d\Phi$ is $-m_G d\Phi$. This must equal DE, the gain in the particle's kinetic energy. For a photon, $dE = hd\nu$, and so

$$hd\nu = -m_G d\Phi = -m_I d\Phi = -\frac{E}{c^2} d\Phi = -\frac{h\nu}{c^2} d\Phi, \tag{565}$$

and thus

$$\frac{d\nu}{\nu} = -\frac{d\Phi}{c^2} \tag{566}$$

Integrating this equation over a finite path from A to B, one fines

$$\frac{\nu_A}{\nu_B} = e^{-(\Phi_B - \Phi_A)/c^2} = \frac{e^{-\Phi_B/c^2}}{e^{-\Phi_A/c^2}} \tag{567}$$

As for light bending in a gravitational field, imagine a ray of light as a stream of photons; since these photons have inertial and gravitational mass, we expect them to obey Galileo's principle and follow a curved path just like a Newtonian bullet traveling at velocity c. That wold make, for example, the downward curvature of a horizontal beam in the earth's gravitational field with x horizontal and z vertical equal to

$$\frac{d^2z}{dx^2} = \frac{d^2z}{c^2dt^2} = -\frac{g}{c^2}. \tag{568}$$

In units of years and light-yeas $c = 1$, and it so happens that $g \approx 1$.

12.8 Static and Stationary SpaceTimes

A stationary field is one that does not change in time and a static one is one where the sources do not move. The most important property of stationary spacetimes is that they admit a preferred time. The metric of every static field can be brougt to the canonical form

$$(cd\tau)^2 = ds^2 = e^{2\Phi/c^2}c^2dt^2 - d\vec{r}^2 = \eta^2(cdt)^2 - d\vec{r}^2 \tag{569}$$

where the last part uses our previous notation. We can calculate the elapsed proper time

$$d\tau^2 = dt^2\left(e^{2\Phi/c^2} - \beta^2\right) = dt^2\left(\eta^2 - \beta^2\right) \tag{570}$$

$$d\tau = dt\sqrt{e^{2\Phi/c^2} - \beta^2} = dt\sqrt{\eta^2 - \beta^2} = dt/\gamma^* \tag{571}$$

In the weak field limit$\Phi/c^2 << 1$, $e^{2\Phi/c^2} \simeq 1 + 2\Phi/c^2$.

$$ds^2 \simeq (1 + 2\Phi/c^2)c^2dt^2 - dl^2$$

For a particle-worldline between two events P_1 and P_2, we have

$$\int_{P_1}^{P_2} ds = \int_{t_1}^{t_2} \frac{ds}{dt}dt = c\int_{t_1}^{t_2}\left(1 + \frac{2\Phi}{c^2} - \frac{v^2}{c^2}\right)^{1/2} dt, \tag{572}$$

where $v = dl/dt$ is the coordinate velocity of the particle. The binomial approximation gives

$$\int_{P_1}^{P_2} ds = c\int_{t_1}^{t_2}\left(1 + \frac{2\Phi}{c^2} - \frac{v^2}{c^2}\right)^{1/2} dt = c\int_{t_1}^{t_2}\left(1 + \frac{\Phi}{c^2} - \frac{1}{2}\frac{v^2}{c^2}\right) dt = C(T_1 - T_2) - \frac{1}{c}\int_{t_1}^{t_2}\left(\frac{1}{2}v^2 - \Phi\right) dt. \tag{573}$$

The condition that $\int ds$ be maximal is therefore equivalent to the last integral being minmal. That is exacatly Hamilton's Principle.

One consequence which we can read off immediately is what is called the Shapiro time delay. A light-ray satisfies $ds^2 = 0$ and thus $e^{\Phi}cdt = \pm dl$, the two signs corresponding to the two possible directions of travel. Consequently a radar, or other light signal, reflected from a distant object will return to its emission point after a coordinate time

$$\Delta t = 2\int e^{-\Phi}dl \tag{574}$$

has elapsed there, where the integration is performed over the path of the signal.

13 Differential Geometry

There is a strong connection between the geometry (symmetry) of Minkowski space and dynamics. Further generalization of relativity requires incorporating a new symmetry - the Equivalence Principle. That will be manifested as the more universal Riemannian geometry of simple curved space-time. Riemannian invariance is not limited to orthogonal or even linear transformations but includes all real, single-valued transformations that are continuous with finite first and second derivatives. This will include curved space-times that do not admit a global Cartesian coordinate system.

Through the Equivalence Principle gravity is replaced by space-time curvature. Gravity is replaced by a local effect. Thus we do not need to know the full geometry and topology but only the local differential geometry.

13.1 Invariant Length and the Metric Tensor

We want to generalize the concept of invariant length of Minkowski space. We define the invariant length and the generalized Riemannian metric by the equation

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{575}$$

where the indices μ and ν are repeated and thus summed and in our case of a 3+1 dimensional geometry cycle from 0 to 3. It is easy to see that $g_{\mu\nu}$ is a tensor. If the transformation for $\vec{x}'$ to $\vec{x}$ is $x^\mu = x^\mu(\vec{x}')$, then

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu = a^\mu_{\nu\prime} dx'^\nu \tag{576}$$

If ds^2 is the invariant length, then

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\sigma\rho} dx'^\sigma dx'^\rho = g_{\mu\nu} a^\mu_{\sigma\prime} dx'^\sigma a^\nu_{\rho\prime} dx'^\rho \tag{577}$$

From this we can conclude

$$g'_{\sigma\rho} = g_{\mu\nu} a^\mu_{\sigma\prime} a^\nu_{\rho\prime} \tag{578}$$

These geometry techniques have been more generalized to other dimensions by mathematicians and we will work out some problems and examples in lower dimensions for illustration and ease. This general bilinear form (product of two differentials) of the metric is the class of Riemannian spaces. In Riemannian geometry one must nearly always pay attention to the difference between covariant and contravariant (co and contra variation with respect to the transformations).

For example, consider the metric for a two dimensional space

$$dx^i = (dx, dy) \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{579}$$

$$dx^i = (dr, d\theta) \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{580}$$

$$dx_i = (dr, r^2 d\theta) \qquad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \tag{581}$$

13.2 The Transformation

The general case is any primed coordinate system and any unprimed coordinate systems connected by a transform on $x^i = x^i(x^{i\prime})$. There are no restrictions on the transformation except that it is continuous, real, single-valued, and its derivatives exist over the region of interest. The indices run from 1 to n, the dimension of the space. In differential form the transformation equations are

$$dx^i = a^i_{j\prime} dx^{j\prime} \quad \text{and} \quad dx^{j\prime} = a^{j\prime}_i dx^i \tag{582}$$

where

$$a^i_{j\prime} = [\mathbf{a}] = \frac{\partial x^i}{\partial x^{j\prime}} \quad \text{and} \quad a^{j\prime}_i = \left[\mathbf{a^{-1}}\right] = \frac{\partial x^{j\prime}}{\partial x^i} \tag{583}$$

giving

$$a^{j\prime}_i a^i_{k\prime} = \delta^{j\prime}_{k\prime} \tag{584}$$

The coefficients $a^{j\prime}_i$ and $a^i_{j\prime}$ are called the transformation coefficients and the condition

$$a^{j\prime}_i a^i_{k\prime} = \delta^{j\prime}_{k\prime} \quad \text{or} \quad \left[\mathbf{a^{-1}}\right] \cdot [\mathbf{a}] = [\mathbf{1}] \tag{585}$$

reflects the general requirement that any transformation followed by its inverse is unity.

13.3 Review of Definition of Tensors

The concept of tensors is actually more generalized than simply transforming from one inertial frame to another according to the generalized Lorentz Trnasformation. We can make use of our understanding of the transformation from one coordinate system to another to define a more generalized concept of a tensor.

First let us generate a new vector space with exactly the same properties as the differential forms: xx^i and $dx^{\prime i}$. That is to say that the geometry of those vector spaces has the same properties and thus transformation between coordinate systems as the differential forms.

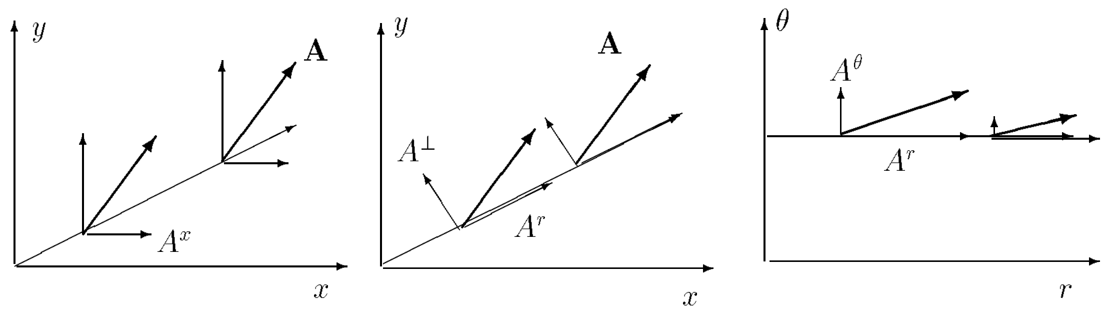
$$A^i = a^i_{j\prime} dA^{j\prime} \quad \text{and} \quad A^{j\prime} = a^{j\prime}_i A^i \tag{586}$$

The tensor notation is the same as before with the point transformation properties requiring as many $a^i_{j\prime}$ or $a^{j\prime}_i$ as there are indices. There is the same issue with upper and lower indicies.

13.4 Parallel Displacement/Transport

When a vector is carried from one point to another without changing its magnitude or direction, it undergoes parallel displacement. In a flat Euclidean space, the idea of of parallel displacement is perfectly clear. In terms of its Cartesian components, it means that the vector's components remain unchanged as it moves from one place to another.

Consider a (contravariant) vector $\vec{A} = (A^x, A^y)$ that is displaced along a line that radiates from the origin. Its components A^x and A^y remain constant under parallel displacement, so they have the same magnitude all along the path. See figure. The vector can also be specified in terms of components that are parallel and perpendicular to the radial line. $\vec{A} = (A^r, A^\perp)$ where the unit component direction vectors are $\hat{r}$ and $\hat{\theta}$ respectively.



(a) (b) (c)

In polar coordinates, the components of $\vec{A} = (A^r, A^\theta))$ in the radial direction also remains the same through out the displacement. That is the value of A^r are constant and the same as in the $(A^r, A^\perp)$ case. However, the A^θ are not the same as $A^\perp$. The relationship between the two is given by $A^\perp = rA^\theta$.

From a variation of this equation and recognizing that $A^\perp$ is constant, δA^θ can be found as a function of r and δr:

$$\delta A^\perp = \delta(rA^\theta) = A^\theta \delta r + r\delta A^\theta = 0 \tag{587}$$

or

$$\delta A^\theta = -\frac{A^\theta}{r}\delta r \tag{588}$$

As the vector moves outward in a $+r$-direction, its θ-component decreases inversely with r.

There is another way of obtaining this result in a more general way. Because $\vec{A}$ is a vector, we can its transformation to polar components from the transformation from polar to Cartesian coordinates

$$x = rcos\theta \qquad y = rsin\theta$$

The corresponding differential interval transformations are

$$dx = cos\theta dr - sin\theta d\theta \qquad dy = sin\theta dr + rcos\theta d\theta$$

or inverted

$$dr = cos\theta dx + sin\theta dy \qquad d\theta = -\frac{1}{r}sin\theta dx + \frac{1}{r}cos\theta dy$$

So the forward and backward transformations are

$$\begin{pmatrix} dr \\ d\theta \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

So vectors can be expected to transform the same way

$$\begin{pmatrix} A^r \\ A^\theta \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} A^x \\ A^y \end{pmatrix} \tag{589}$$

or written explicitly

$$\begin{aligned} A^r &= A^x cos\theta + A^y sin\theta \\ A^\theta &= -\frac{1}{r}A^x sin\theta + \frac{1}{r}A^y cos\theta \end{aligned} \tag{590}$$

Given that A^x, A^y, and θ are constant for a radial translation, a variation of the first equation is $\delta A^r = 0$ as already noted and a variation on the second equation gives

$$\begin{aligned} \delta A^\theta &= \delta\{-\frac{1}{r}A^x sin\theta + \frac{1}{r}A^y cos\theta\} \\ \delta A^\theta &= \frac{1}{r}\{\frac{1}{r}A_x sin\theta - \frac{1}{r}A^y cos\theta\}\delta r = -\frac{A^\theta}{r}\delta r \end{aligned} \tag{591}$$

A vector is parallel displaced, if it moves without changing its magnitude or direction. But it is clear from this example that the directional change is dependent on the coordinate system. The magnitude (i.e. the square root of the invariant length) is unchanged. under coordinate transformation, but when viewed in polar coordinates, its components do vary and hence changes apparent direction. This is not exactly what was defined as parallel transport though it is in some coordinate systems. The definition of parallel transport will have to be broadened.

Define the indexed (not tensor) symbol Γ^k_{ij} called a Christoffel symbol or affine connection. As a vector A^i is parallel displaced over a differential δx^j, its components A^k are changed by an amount determined by the Christoffel symbols in the defining equation

$$\delta A^k = -\Gamma^k_{ij} A^i \delta x^j \tag{592}$$

The values of these Christoffel symbols in Cartesian coordinates are zero, but they are not zero in polar coordinates.

13.5 Comparison to Curved Surface

We just did the transformation for flat space. Consider embedding a curved two dimensional surface in a three dimensional Euclidean space. Most specifically consider a spherical surface. We know the polar coordinate trnsformations:

$$x = rcos\phi sin\theta \qquad y = rsin\phi sin\theta \qquad z = rcos\theta$$

$$ds^2 = g_{ij}dx^i dx^j = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

Considering a fixed radius to give us a spherical surface, the metric on the surface is

$$ds^2 = r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) = (dx')^2 + (dy')^2$$

where $dx' = rd\theta$ and $dy' = rsin^2\theta d\phi$. In small regions where $|sin\theta|$ is not near zero and changing rapidly, on the surface this looks like a Euclidean formulation. It is only when one looks over larger regions or globally is it apparent that this is a curved surface.

13.6 Geodesic Path

In Riemannian geometry, the term geodesic path means: the path of shortest invariant distance. There is a strong connection between geodesic paths and the idea of parallel displacement. A vector that moves tangent to a geodesic is, at the same time, being parallel displaced as it moves along.

The geodesic path can be found using the variational principle to find the shortest distance between two points, which in Riemannian geometry requires that $\delta \int ds = 0 = \delta \int d\tau$ The calculus of variations gives

$$\frac{d^2 x^x}{ds^2} = -\frac{1}{2} g^{ks} \left[g_{is,j} + g_{js,i} - g_{ij,s}\right] \dot{x}^i \dot{x}^j \tag{593}$$

where the dot means derivative with respect to ds (or $cd\tau$) $\dot{x}^k \equiv dx^k/ds$ and $g_{ij,k} = \partial_k g_{ij}$.

Multiplying by

$$d\dot{x}^k = -\frac{1}{2} g^{ks} \left[g_{is,j} + g_{js,i} - g_{ij,s}\right] \dot{x}^i dx^j \tag{594}$$

This equation describes how a contravariant vector $\dot{x}^j$ changes as it moves over a path dx^j. It is the same as the definition of the Christoffel symbol. Clearly,

$$\Gamma^k_{ij} = \frac{1}{2} g^{ks} \left[g_{is,j} + g_{js,i} - g_{ij,s}\right] \tag{595}$$

This equation allows the Christoffel symbols to be calculated from the metric tensor alone. The geodesic equation becomes

$$\frac{d\dot{x}^k}{ds} = -\Gamma^k_{ij} \dot{x}^i \dot{x}^j \tag{596}$$

13.7 Parallel Displacement of Covariant Vectors

Covariant vectors also change direction under parallel displacement. Consider the inner invariant product of two vectors A^i and B_i. When the two are parallel displaced together in a Cartesian system. If δ represents the small variation of the product along the line of displacement, then because it is an invariant,

$$0 = \delta(A^i B_i) = A^k \delta B_k + B_i \delta A^i \tag{597}$$

Substituting $\delta A^i = -\Gamma^i_{jk} A^j \delta x^k$ yielding

$$A^i \delta B_i = -B_k \delta A^k = -B_k(-\Gamma^k_{ij} A^i \delta x^j) A^i \delta B_i = A^i \Gamma^k_{ij} B_k \delta x^j \tag{598}$$

Because this expression is true for any vector A^i, its coefficients on either side of the equation must be equal

$$\delta B_i = \Gamma^k_{ij} B_k \delta x^j \tag{599}$$

The Christoffel symbol gives the changes in a covariant vector as well as the contravariant one:

$$\delta A^k = \Gamma^k_{ij} A^i \delta x^j \qquad \delta A_i = \Gamma^k_{ij} A_k \delta x^j \tag{600}$$

δA^i and δA_i are not vectors. Note that δA^i identically zero in the Cartesian system but non-zero in the polar system. A zero vector cannot be transformed into a non-zero one.

It is also possible to determine how a mixed tensor changes under a parallel displacement. Consider the inner product between tensor T^i_j and two vectors A_i and B^j, while they are parallel displaced together: $\delta(T^i_j A_i B^j) = 0$ since the inner product is invariant.

$$\delta T^i_j = \Gamma^r_{js} T^i_r \delta x^s - \Gamma^i_{rs} T^r_j \delta x^s \tag{601}$$

$$\delta T^i_{jk} = \Gamma^r_{js} T^i_{rk} \delta x^s + \Gamma^r_{ks} T^i_{jr} \delta x^s - \Gamma^i_{rs} T^r_{jk} \delta x^s \tag{602}$$

13.8 Covariant Derivatives

Consider a contravariant vector A'^i in the Cartesian system that is transformed to a Cartesian system

$$A^i = a^i_{i\prime} A'^i \tag{603}$$

Taking the ordinary derivative of this gives

$$dA^i = d(a^i_{i\prime}) A'^i + a^i_{i\prime} dA'^i \tag{604}$$

If the Cartesian vector does not change its components in any way as it is parallel displaced, then $dA'^i = 0$ and the change in A^i will be given only by the term $d(a^i_{i\prime})A^{i\prime}$. This is precisely what we called δA^i previously in the discussion of parallel transport. Therefore

$$dA^i = \delta A^i + a^i_{i\prime} dA'^i \quad or \quad a^i_{i\prime} dA'^i = dA^i - \delta A^i$$

Multiplying through by the inverse transformation yields

$$dA'^i = a^{i\prime}_i [dA^i - \delta A^i]$$

We can generalize this to additional systems that are not necessarily Cartesian In general the quantity $dA^i - \delta A^i$ will transform as a vector.

$$D \equiv dA^i - \delta A^i$$

The first part represents the change in the vector A component i that is observed in the coordinate system. The second is the change δA^i that is due to the parallel displacement of the vector component A^i. This difference subtracts out the variation that has only to do with the curvilinear characteristics of the coordinate system.

Consider the vector $dx^i/ds = \dot{x}^i$ along a curve defined by the parameter s. Dividing by the invariant interval ds gives

$$\frac{D\dot{x}^i}{ds} = \frac{d\dot{x}^i}{ds} - \frac{\delta\dot{x}^i}{ds} = \frac{d\dot{x}^i}{ds} + \Gamma^i_{jk}\dot{x}^j\dot{x}^k$$

If the line is a straight line in a Cartesian system, then the vector $D\dot{x}^i/ds$ is zero in all coordinates so that

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{jk}\dot{x}^j\dot{x}^k$$

13.9 Review: Equations of Motions for Free Particle

If a particle is not being acted on by any forces, thus free of influence, then we can find its equation of motion by several methods: parallel transport, variational principle (geodesic path), and by setting the covariant derivative of the (four) velocity equal to zero becouse there is no force to cause acceleration.

13.9.1 Parallel Transport

We consider that the free particle has a given velocity (or momentum) and if there are no forces acting upon it, then it will simply move forward in parallel transport of its velocity (vector or momentum). This is the simplest way to see that the effect is strictly a local.

The parallel transport of the velocity gives

$$\delta u^k = -\Gamma^k_{ij}u^i\delta x^j$$

Dividing through by $d\tau$ or ds gives

$$\frac{du^k}{d\tau} = -\Gamma^k_{ij}u^iu^j$$

This is a simple equation that says that if the rate of change in the velocity is equal to the transport of that velocity times the effect of the changing coordinate metric.

13.9.2 Geodesic Equation

This comes from the requirement that the path taken is the minimum (extremum) of path between the end points.

$$\frac{du^k}{d\tau} = -\Gamma^k_{ij} u^i u^j$$

where ds is replaced by $d\tau$.

13.9.3 Covariant Derivative of Velocity

If no force is acting on the particle, then the covariant derivative of the velocity should be zero.

$$\frac{Dp^\mu}{d\tau} = F^\mu = 0 \quad \text{implies} \quad \frac{Du^\mu}{d\tau} = 0$$

Thus

$$\frac{Du^k}{d\tau} = \frac{du^k}{d\tau} + \Gamma^k_{ij} u^i u^j = 0$$

Once again we find the geodesic equation through a different conceptual approach.

13.10 Space-Time Differential Geometry

13.11 Spherical Surface as an Example

If we consider a two-dimensional spherical surface, then we can easily see that parallel transport of vector around a closed path does not return to point the same direction, E.g. consider a spherical triangle with one vertex at the north pole and two sides thus being lines of constant longitude (ϕ = constant) and the other connecting parallel to the equator (line of constant latitude). Take vectors parallel (or perpendicular) to the sides and start the parallel transport from the pole. The parallel vector will stay tangent to the constant longitude line (it is a geodesic and so that is clear). Across the constant latitude side the formerly parallel vector will stay perpendicular to that side and at the end will be parallel to the last side. When transported back to the pole it will stay parallel to the last side. Thus back at the same point, the pole, the parallel transported vector will point at a different angle - the angle of the opening of the triangle at the pole. This triangle was constructed so that angle at the pole is the excess angle. That is the angle by which the sum of the angles of the spherical triangle exceeds π radians (180°). Once can use the Christoffel symbols for a spherical surface to show that the result of the parallel transport of a vector acround any spherical triangle on the sphere is displaced by the excess angle. By Gauss's most beautiful theorem the excess (or deficit angle) is the ratio of the area of the triangle to the area of the sphere divided by 4π:

$$\alpha = \frac{A_{triangle}}{R^2}$$

Intuitively one knows that the curvature tensor for the space defined as on the surface of a sphere should not be equal to zero. Every point on a sphere's surface is intrinsically curved.

13.12 Curvature Measures

If a vector is parallel displaced around a closed loop in a flat space, the it will return to its original magnitude and direction. That should be true independent of the coordinate systems in which the displacement is observed. So if a parallel-displaced vector is changed by a trip around a closed loop, it follows that its region of travel is a "curved" space.

A surface observer can use this procedure to detect the existence of curvature, when visual recognition or embedding in a higher dimensional flat Cartesian space is not part of the mathematical procedure.

$$\delta V^{\alpha} = \delta x^{\mu} \delta x^{\nu} \left[\Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\nu} \Gamma^{\sigma}_{\beta\mu} - \Gamma^{\alpha}_{\sigma\mu} \Gamma^{\sigma}_{\beta\nu} \right] V^{\beta} \tag{605}$$

13.12.1 Christoffel Symbols

$$\Gamma^{m}_{ij} = \frac{1}{2} g^{mk} \left[g_{ik,j} + g_{jk,i} - g_{ij,k} \right] \tag{606}$$

13.12.2 Curvature Tensor

$$\text{Riemann} \qquad R^{k}_{ars} = \Gamma^{k}_{ar,s} - \Gamma^{k}_{as,r} + \Gamma^{b}_{ar} \Gamma^{k}_{sb} - \Gamma^{b}_{as} \Gamma^{k}_{rb}$$

The intrinsic curvature at any point can be found in terms of derivatives of the metric at that point.

One can find the curvature by transporting the same vector half way around a differentially small rectangle in both directions and comparing the results. The procedure is to first take the covariant derivative of a vector $\vec{v}$ with respect to x^i, then take the covariant derivative of this result with respect to x^j:

$$D_{x^j} \left(D_{x^i} \vec{v} \right)$$

Then reverse the order and subtract

$$D_{x^j} \left(D_{x^i} \vec{v} \right) - D_{x^i} \left(D_{x^j} \vec{v} \right)$$

The vector $\vec{v}$ can be taken out of the additive terms in this tensor equation, leaving the differential operators in the form of the tensor $[R]$. $[R]$ is called the Riemann-Christoffel curvature tensor.

$$D_{x^j} \left(D_{x^i} \vec{v} \right) - D_{x^i} \left(D_{x^j} \vec{v} \right) = \left[D_{x^i} \left(D_{x^j} \right) - D_{x^i} \left(D_{x^j} \right) \vec{v} = [R] \cdot \vec{v}$$

or

$$\Delta \vec{v} / \Delta \text{area} = [R] \ \cdot \vec{v}.$$

Remember that when a tensor is zero in one coordinate system, it is zero in all coordinate systems. So if the Riemann-Christoffel tensor is zero at any point in any system of coordinates, then the space at that point is flat. On the other hand, if $[R]$ is nonzero at a point, then the space has an intrinsic curvature at that point.

The Riemann-Christoffel tensor can be explicitly derived by carrying out the operations outlined above. This can be done using contravariant vector A^k. First take the covariant derivative of A^k

$$A^k_{;r} = A^k_{,r} + \Gamma^k_{ar}A^a$$

Now take the next covariant derivative

$$A^k_{;rs} = \frac{\partial}{\partial x^s}(A^k_{;r}) + \Gamma^k_{as}A^a_{;r} - \Gamma^a_{rs}A^k_{;a}$$

where the index s representing the second covariant derivative is placed directly behind the index r representing the first covariant derivative.

$$A^k_{;rs} \equiv \left[A^k_{;r}\right]_{;s}$$

Going back to the definition of the covariant derivative $A^k_{;r}$ above gives

$$\frac{\partial}{\partial x^s}\left[A^k_{,r}\Gamma^k_{ar}A^a\right] + \Gamma^k_{as}\left[A^a_{,r} + \Gamma^a_{br}A^b\right] - \Gamma^a_{rs}\left[A^k_{,a} + \Gamma^k_{ba}A^b\right] \tag{607}$$

where the dummy index b is there to avoid repetitive use of index a in two of the terms. Multiplying out one has

$$A^k_{;rs} = A^k_{,rs} + \Gamma^k_{ar,s}A^a\Gamma^k_{ar}A^a_{,s} + \Gamma^k_{as}A^a_{,r} + \Gamma^k_{as}\Gamma^a_{br}A^b - \Gamma^a_{rs}A^k_{,a} + \Gamma^a_{rs}\Gamma^k_{ba}A^b \tag{608}$$

This seven term expression is the result of two sequential covariant differentiations of the vector $\vec{A}$ corresponding to the operations $D_{x^j}(D_{x^i}\vec{v})$ acting on vector $\vec{v}$. Now perform the same operations in reverse order and subtract. The reverse order of the covariant differentiation is found by exchanging the indices s and r.

$$A^k_{;sr} = A^k_{,sr} + \Gamma^k_{as,r}A^a + \Gamma^k_{as}A^a_{,r} + \Gamma^k_{ar}A^a_{,s} + \Gamma^k_{ar}\Gamma^a_{bs}A^b - \Gamma^a_{sr}A^k_{,a} + \Gamma^a_{sr}\Gamma^k_{ba}A^b \tag{609}$$

This is equation is subtracted from the previous and the first, third, fourth, sixth and seventh term in each cancels, leaving the difference between the second and fourth terms:

$$A^k_{;rs} - A^k_{;sr} = \Gamma^k_{ar,s}A^a - \Gamma^k_{as,r}A^a + \Gamma^k_{as}\Gamma^a_{br}A^b - \Gamma^k_{ar}\Gamma^a_{bs}A^b \tag{610}$$

The dummy indices a and b are now exchanged in the last two terms, and the vector component A^a removed from each term to give

$$A^k_{;rs} - A^k_{;sr} = \left[\Gamma^k_{ar,s} - \Gamma^k_{as,r} + \Gamma^k_{bs}\Gamma^b_{ar} - \Gamma^k_{ar}\Gamma^b_{as}\right]A^a \tag{611}$$

or one has

$$A^k_{;rs} - A^k_{;sr} = R^k_{ars}A^a \tag{612}$$

where

$$R^k_{ars} = \Gamma^k_{ar,s} - \Gamma^k_{as,r} + \Gamma^k_{bs}\Gamma^b_{ar} - \Gamma^k_{rb}\Gamma^b_{as} \tag{613}$$

This fourth-rank mixed tensor - R^k_{ars} - is the Riemann-Christoffel curvature tensor.

13.12.3 Ricci Tensor and Scalar Curvature

$$\text{Ricci} \qquad R_{ij} = R^k_{ijk}$$

$$\text{Scalar} \qquad R = R^i_i$$

We will contract the Riemann-Christoffel tensor in two steps. In the first of these, the contravariant index is contracted with the last of the covariant indices to give the Ricci tensor, which is defined as

$$\begin{aligned} R_{ij} &= R^k_{ijk} \\ R_{ij} &= \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^b_{ij}\Gamma^k_{kb} - \Gamma^b_{ik}\Gamma^k_{jb} \end{aligned} \tag{614}$$

It is easy to show that this tensor is symmetric

$$R_{ij} = R_{ji} \tag{615}$$

Other second-rank tensors can be found by contracting other indices; but the Ricci tensor is of special importance because of its symmetry and because of the unique properties of its derivatives. As a result it plays a major role in General Relativity. Since it is symmetric it has the same number of independent components as the metric tensor $g_{\mu\nu}$ and the stress energy tensor $T_{\mu\nu}$ and General Relativity provides a unique way to link them together.

14 General Relativity Principles

First we review Newtonian Gravitation to remind ourselves of what General Relativity must reduce to in the classical limit.

14.1 Newtonian Gravitation

We know very well Newton's three laws of motion and his law of Universal Gravitation. The most famous formula for the first is

$$\vec{F} = m_i \vec{a} \quad F_j = m_i a_j = m_i \frac{d^2 x_j}{dt^2} \qquad \tilde{F} = m_0 \tilde{a} \quad F^\alpha = m_0 a^\alpha = m_0 \frac{d^2 x^\alpha}{dt^2} \tag{616}$$

where m_i means inertial mass and m_0 means inertial rest mass.

Newton's Universal law of Gravitation is often expressed as

$$F_G = -\frac{G M_G m_G}{r^2} \quad \text{or} \quad \vec{F}_G = -m_G \nabla \Phi \quad F_G^j = \frac{\partial \Phi}{\partial x_j}$$

Where $\Phi = G M_G / r$ for a point mass or $\Phi = \int\int\int \rho_G / r \; dxdydz$ for a distributed density and the subscript G means the source charge of gravity. That is, m_G is the coupling for producing or being acted on (assuming action equals reaction) by gravitational forces.

14.1.1 Weak Equivalence Prinicple

The weak equivalence principle holds that the gravitational mass and inertial mass are exactly the same: $m_i = m_G$. This is well established by experiment to roughly the 10^{-14} level. We take it as correct here. Then we automatically get an equivalence of acceleration and gravity at a point as far as kinematics are concerned.

Continuing with our discussion of Newtonian gravity assuming the validity of the Weak Equivalence Principle we have for the equations of motion:

$$\frac{d^2 x^i}{dt^2} = -\delta^{ij} \frac{\partial \Phi}{\partial x^j} \tag{618}$$

This gives us a simple equation of motion based upon the first partial derivatives of the gravitational potential.

14.2 Tidal Forces Equations

Now it is both intuitive and well known that conventional gravitational fields, e.g. those from the Earth, moon, Sun etc. produce tidal forces. E.g. every thing dropped from rest (or near rest) near the Earth experience an acceleration directed toward the center of the Earth. Over small distances the directions are nearly parallel so everything in free fall (no other forces acting) appears to accelerate down at the same rate and direction. When viewed more globally, it is clear that the accelerations vary with the angle one is separated by around the circumference of the Earth.

We can write and equation for this for the general case by considering two particles one at location x^k and one at location $x^k + \Delta x^k$. The potential will be slightly different but we can expand Φ

$$\Phi(x^k + \Delta x^k) = \Phi(x^k) + \frac{\partial \Phi}{\partial x^k} \Delta x^k + ...$$

around the original point with position vector component x^k.

The equation of motion for the second particle is similar

$$\frac{d^2 x^i + \Delta x^i}{dt^2} = -\delta^{ij} \frac{\partial \Phi(x^k + \Delta x^k)}{\partial x^j} = -\delta^{ij} \left(\frac{\partial \Phi}{\partial x^j} + \frac{\partial^2 \Phi}{\partial x^j \partial x^k} \right)$$

Taking the difference between the original and displace by Δx^k test particles equations of motion, we have

$$\frac{d^2 \Delta x^i}{dt^2} = -\delta^{ij} \frac{\partial^2 \Phi}{\partial x^j \partial x^k} \tag{619}$$

Thus things that are separated by $\Delta \vec{x}$ accelerate together at a rate that is proportional to their separation and to the mixed second partial derivatives of the gravitational potential.

Consider one of Newton's spherical masses for an example.

$$\begin{aligned} \Phi &= -\frac{GM}{r} \\ \frac{d^2 x^i}{dt^2} &= -\frac{\partial \Phi}{\partial x^i} = \frac{GM}{r^2} \hat{n}_i \\ \frac{d^2 \Delta x^i}{dt^2} &= -\delta^{ij} \frac{\partial^2 \Phi}{\partial x^j \partial x^k} = -\left(\delta_{ij} - 3 n_i n_j\right) \frac{GM}{r^3} \end{aligned} \tag{620}$$

14.3 Geodesic Equation

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\frac{1}{2} g^{\alpha\sigma} \left[g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \right] \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \tag{621}$$

Consider the case of very 3-D low velocity, near the instanteous rest frame where $\tilde{U} = \gamma(1, \beta_x, \beta_y, \beta_z) c \simeq (v^0, 0, 0, 0) \approx (1, 0, 0, 0) c$

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu \simeq -\Gamma^\alpha_{00} (v^0)^2 \tag{622}$$

The first part of the equation shows that the geodesic equation is equivalent to the parallel transport of the 4-D velocity vector $\tilde{u}$.

Table 4: Gravitational Field Strengths for Astrophysical Objects

At Surface of Object	ϕ/c^2
proton	$\sim 10^{-39}$
Earth	$\sim 10^{-9}$
Sun	$\sim 10^{-6}$
White Dwarf	$\sim 10^{-4}$

14.4 Geometric Interpretation

Now we have come to understand inertia in curved space-time

Now we continue the Newtonian approximation by considering a static gravitational field in static coordinate system.

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\nu}(g_{\nu 0,0} + g_{0\nu,0} - g_{00,\nu}) = \frac{1}{2}g^{\alpha\nu}g_{00,\nu} \simeq -\frac{1}{2}g_{00,\nu} \tag{623}$$

The standard Newtonian equation of motion for a potential is

$$a^m = \frac{d^2x^m}{dt^2} = \frac{dv^m}{dt} = -\frac{\partial\phi}{\partial x^m} \tag{624}$$

A consistent solution is

$$g_{00,m} = 2\frac{\partial\phi}{\partial x^m} = (2\phi)_{,m} \tag{625}$$

$$g_{00} = 1 + 2\phi/c^2 \tag{626}$$

Note that for a point source or spherically distributed mass, M, within radius $R<r$ $\phi(r) = -GM/r$

$$g_{00} = 1 - \frac{2GM}{c^2r} \tag{627}$$

14.5 Weak Field Gravity

We are now in a position to understand gravity in the weak field limit through the Equivalence Principle and the idea that gravity is replaced by curved space-time. We can derive the weak field approximation determine its paramaterization from the Newtonian limit. We make use of the observation that Newton's law of gravity holds for situations in which the non-relativistic approximation is justified and thus valid. These are situations in which the space-time region is sufficiently small that it can be approximated by a coordinate system that is nearly flat.

We write the weak field approximation as

$$g_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} + h_{\mu\nu} \tag{628}$$

where $\eta_{\mu\nu}$ is the Minkowski metric (or later in the course this will be expanded to the Robertson-Walker metric which reduces to the Minkowski metric in the small space-time limit) and $h_{\mu\nu} << 1$ is a small perturbation to the metric.

Using the formula for determining the Christoffel symbol from metric tensor one has

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}\left(g_{\lambda\nu,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}\right) \simeq \frac{1}{2}\left(h_{\lambda\nu,\mu} + h_{\lambda\mu,\nu} - h_{\mu\nu,\lambda}\right) \quad (629)$$

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu} + h^{\nu}_{\alpha}h^{\alpha\nu} + \ldots \quad (630)$$

and thus

$$\Gamma^{\alpha}_{\mu\nu} = \eta^{\alpha\lambda}\Gamma_{\lambda\mu\nu} + Order(h^2) \quad (631)$$

The curvature tensor is

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + Order(h^2) \quad (632)$$

The Ricci tensor is

$$R_{\mu\nu} \simeq \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\nu\alpha,\mu} + Order(h^2) = \frac{1}{2}\left(-h_{\mu\nu,00} + \ldots\right) \quad (633)$$

The Ricci scalar curvature is

$$R = h_{\mu\mu,00} + h_{ij,ji} + Order(h^2) \quad (634)$$

A slowly moving particle has

$$\frac{dx^{\mu}}{d\tau} \sim (1,0,0,0) \quad (635)$$

so that the geodesic equation is

$$\frac{d^2x^i(\tau)}{d\tau^2} = -\Gamma^{i}_{00} \quad (636)$$

In a stationary (no time dependence) system the time derivatives are zero and therefore

$$-\Gamma_{i00} = \frac{1}{2}h_{00,i} \quad (637)$$

so that $-\frac{1}{2}h_{00}$ can be the gravitational potential.

Energy momentum conservation requires that the covariant derivative of the stress-energy tensor be zero:

$$D_{\mu}T_{\mu\nu} = 0. \quad (638)$$

$$-\Gamma^{i}_{00} \;=\; -\frac{\partial V(x)}{\partial x^i} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$$

$$
\begin{aligned}
-\frac{\partial \Gamma^i_{00}}{\partial x^i} &= 4\pi G_N T_{00} \\
\frac{\partial^2 h_{00}}{\partial (ct)^2} &= 8\pi G_N T_{00}
\end{aligned} \tag{639}
$$

Now this must be expressed in a way that is invariant under general coordinat transformations in order to deterimine the appropriate theory of gravity. Instead of using only one component of the stress-energy tensor $T_{\mu\nu}$ and certain partial derivatives of he connection fields Γ, we need a relation between covariant tensors. We reason from

$$R_{00} = -\frac{1}{2}\frac{\partial^2 h_{00}}{\partial (ct)^2}; \tag{640}$$

$$R = \frac{\partial^2 h_{ij}}{\partial x^i \partial x^j} + \vec{\partial}^2 (h_{00} - h_{ii}). \tag{641}$$

These equations give us a lead on the relationship between the tensors $T_{\mu\nu}$ and $R_{\mu\nu}$. The most general tensor relation of this thype would be

$$R_{\mu\nu} = AT_{\mu\nu} + Bg_{\mu\nu}T^\alpha_\alpha \tag{642}$$

where A and B are constants that have to be determined by comparing to the Newtonian limit. The trace of the energy-momentum (stress-energy) tensor is, in the non-relativisitic approximation

$$T^\alpha_\alpha = -T_{00} + T_{ii}. \tag{643}$$

so that the 00 component can be written as

$$R_{00} = -\frac{1}{2}\vec{\partial}^2 h_{00} = (A+B)T_{00} - BT_{ii} \tag{644}$$

In the newtonian limit the T_{ii} term (the pressure p) vanishes, not only because the pressure of the ordinary (non-relativistic) matter is very small but also because it averages out to zero as a source. In the stationary case

$$0 = \partial_\mu T_{\mu i} = \partial_j T_{ji} \tag{645}$$

$$\frac{d}{dx^1}\int T_{11} dx^2 dx^3 = -\int dx^2 dx^3 (\partial_2 T_{21} + \partial_3 T_{31}) = 0, \tag{646}$$

and therefore, if our source is surrounded by a vacuum

$$\int T_{11} dx^2 dx^3 = 0 \rightarrow \int d^3\vec{x}\ T_{11} = 0, \tag{647}$$

and similarly

$$\int d^3\vec{x}\ T_{22} = \int d^3\vec{x}\ T_{33} = 0. \tag{648}$$

This in turn implies

$$A + B = -4\pi G_N. \tag{649}$$

There is an additional piece of information in that the trace of $R_{\mu\nu}$ is $R = (A+4B)T^\alpha_\alpha$. Just at the general form of the second rank tensor for the stress energy included a trace component we msut make use of the Einstein curvature tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ to get

$$G_{\mu\nu} = AT_{\mu\nu} - \left(\frac{1}{2}A + B\right) T^\alpha_\alpha g_{\mu\nu} \tag{650}$$

Then

$$D_\mu G_{\mu\nu} = 0; \quad D_\mu T_{\mu\nu} = 0; \quad \text{therefore} \quad \left(\frac{1}{2}A + B\right) \partial_\nu T^\alpha_\alpha = 0. \tag{651}$$

For this to hold in general

$$B = -\frac{1}{2}A; \qquad A = -8\pi G_N, \tag{652}$$

The only tensor equation consistent with Newton's equation in a locally flat coordinate frame is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G_N T_{\mu\nu} \tag{653}$$

This is Einstein's General Theory of Relatitivity = gravitation.

Since both the left and right sides of the equation are symmetric under interchange of indices means that there are 10 equations. However, the conservation laws

$$D_\mu G_{\mu\nu} = 0; \quad D_\mu T_{\mu\nu} = 0 \tag{654}$$

give four equations that are automatically satisfied. This means that there are actually 6 non-trivial second-order non-linear partial differential equations. Ultimately one must determine the metric tensor $g_{\mu\nu}$ which has 10 independent components.

$$\begin{aligned} 8\pi G_N T^\mu_\mu &= R; \\ R_{\mu\nu} &= -8\pi G_N \left(T_{\mu\nu} - \frac{1}{2}T^\alpha_\alpha g_{\mu\nu}\right). \end{aligned} \tag{655}$$

therefore in regions of space-time where no matter is present one has

$$R_{\mu\nu} = 0 \tag{656}$$

but the complete Riemann tensor $R^\alpha_{\beta\gamma\delta}$ will not necessarily vanish.

The Weyl tensor is defined by subtracting from $R_{\alpha\beta\delta\gamma}$ in such a way that contraction on any pair of indices gives zero.

$$C_{\alpha\beta\delta\gamma} = R_{\alpha\beta\delta\gamma} + \frac{1}{2}\left[g_{\alpha\gamma}R_{\delta\beta} - \frac{1}{3}Rg_{\alpha\delta}g_{\beta\gamma} - \text{terms with } \gamma \text{ and } \delta \text{ exchanged}\right] \tag{657}$$

This construction is such that $C_{\alpha\beta\delta\gamma}$ has the same symmetry properties and

$$C^{\mu}_{\beta\mu\gamma} = 0. \tag{658}$$

If one counts the number of independent components at a given point $\tilde{x}$ that $R_{\alpha\beta\delta\gamma}$ has 20 degrees of freedom and $R_{\mu\nu}$ and $C_{\alpha\beta\delta\gamma}$ each have 10.

Though these eqations have been derived in a way that provided a unique match to Newtonian limit, if we allow a small deviation from Newton's laws, then another term is allowed. We have another conservation law (zero covariant derivative) for the metric tensor

$$D_{\mu} g_{\mu\nu} = 0, \tag{659}$$

Thus it is possible to modify General Relativity equation for gravity to

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu} \tag{660}$$

where Λ is the cosmological constant. This new term represents a renormalization

$$\delta T_{\mu\nu} \propto g_{\mu\nu} \tag{661}$$

14.6 Isometries

Tensor calculus is largely concerned with how quantities change under coordinate transformations. It is of particular interest when a quantity does not change, i.e. remains invariant, under coordinate transformations. For example, coordinate which leave a metric invariant are of importance since they contain information about the **symmetries** of the underlying Riemannian manifold. Just as in an ordinary Euclidean space, there are two sorts of transformations: **discrete** ones, like reflections, and **continuous** ones, like translations and rotations. In most applications, these latter types are the more important ones and they can in principle be obtained systematically by obtaining the so-called Killing vectors of the metric.

A metric g_{ab} is **from-invariant** or simply **invariant** under the transformation $x^a \to (x')^a$, if

$$g'_{ab}(\vec{y}) = g_{ab}(\vec{y}) \quad \text{for all coordinates} y^c, \tag{662}$$

that is, the transformed metric $g'_{asb}(\vec{x}')$ is the **same** function of its argument $\vec{x}'$ as the original metric $g_{ab}(\vec{x})$ is of its argument $\vec{x}$. Then a transformation leaving g_{ab} form-invariant is called an **isometry**. Since g_{ab} is a covariant tensor, it transforms according to the equation above, or equivalently (interchanging primes and unprimes)

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g'_{cd}(x'). \tag{663}$$

Then, using the equation from above, $x^a \to x'^a$ will be an isometry, if

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g_{cd}(x'). \tag{664}$$

Consider all quantities appearing in this equation to be functions of x using $x'^a = x'^a(x)$. In general, this condition is very complicated, but it may be greatly simplified, if we consider the special case of an **infinitesimal** coordinate transformation

$$x^a \to x'^a = x^a + \epsilon X^a(x) \tag{665}$$

where ϵ is small and arbitrary and X^a is a vector field. Differentiating gives

$$\frac{\partial x'^a}{\partial x^b} = \delta^a_b + \epsilon \partial_b X^a \tag{666}$$

Now substituting into the transformation equation and applying Taylor's theorem

$$\begin{aligned} g_{ab}(x) &= (\delta^c_a + \epsilon\partial_a X^c)(\delta^d_b + \epsilon\partial_b X^d) g_{cd}(x^e + \epsilon X^e) \\ &= (\delta^c_a + \epsilon\partial_a X^c)(\delta^d_b + \epsilon\partial_b X^d)[g_{cd}(x) + \epsilon X^e \partial_e g_{cd}(x) + ...] \\ &= g_{ab}(x) + \epsilon[g_{ad}\partial_d X^d + g_{bd}\partial_a X^d + X^e \partial_e g_{ab}] + O(\epsilon^2). \end{aligned} \tag{667}$$

Working to first order in ϵ and subtracting $g_{ab}(x)$ from each side, it follows that the quantity in the square brackets must vanish. This quantity is simply the Lie derivative of g_{ab} with respect to X, namely,

$$L_X g_{ab} = X^c \partial_e g_{ab} + g_{ad}\partial_b X^d + g_{bd}\partial_a X^d \tag{668}$$

Now we can replace ordinary derivative and so, the condition for an infinitesimal isometry becomes

$$L_X g_{ab} = X^c \nabla_e X_a + \nabla_a X_b = 0. \tag{669}$$

These equations are called **Killing's equations** and any solution of them is called a **Killing vector field** X^a. The metric is dragged into itself' by the vector field X^a.

Theorem: An infinitesimal isometry is generated by a Killing vector $X^a(x)$ **satisfying** $L_X g_{ab} = 0$.

It is sufficient to restrict attention to infinitesimal transformations because it is possible to build up any finite transformation with non-zero jacobian (i.e. a continuous transformation) by an integration process involving an infinite sequence of infinitesimal transformations.

15 The Schwarzschild Solution from Symmetry

15.1 Stationary Solutions

A metric will be stationary, if there exists a special coordinate system in which the metric is visibly time-independent, i.e.

$$\frac{g_{ab}}{\partial x^0} \equiv = 0, \tag{670}$$

where x^0 is a timelike coordinate. In an arbitrary coordinate system the metric will probably depend explicitly on all the coordinates; so we need to make the statement coordinate independent. Define a vector field

$$X^a \equiv \delta^a_0 \tag{671}$$

in the special coordinate system, then,

$$L_X g_{ab} = X^c g_{ab,c} + g_{ac} X^c_{,b} + g_{bc} X^c_{,a} \equiv \delta^c_0 g_{ab,c} = g_{ab,0} = 0 \tag{672}$$

$L_X g_{ab}$ is a tensor, or if it vanishes in one coordinate system, it vanishes in all coordinate systems. Hence, X^a is a **Killing vector field**. Conversely, a given **timelike** Killing vector field X^a, then there always exists a coordinate system which is **adapted** to the Killing vector field,, that in which the last equation holds, and then

$$0 = L_X g_{ab} \equiv g_{ab,0}, \tag{673}$$

and so the metric is stationary. This is a coordinate-independent definition.

A space-time is said to be stationary, if and only if, it admits a time like Killing vector field.

15.2 Hypersurface-orthogonal vector fields

To discuss static solutions in a coordinate-independent way, we need to introduce the concept of a hypersurface-orthogonal vector field. The equation of a **family** of hypersurfaces is given by

$$f(x^a) = \mu \tag{674}$$

where different members of the family correspond to different values of μ. Consider two neighboring points with coordinates x^a and $(x^a + dx^a)$, respectively, lying in one of the hypersurfaces, S.

$$\mu = f(x^a + dx^a) = f(x^a) + \frac{\partial f}{\partial x^a} dx^a \tag{675}$$

to first order. Thus

$$0 = \frac{\partial f}{\partial x^a} dx^a \tag{676}$$

evaluated at x^a. Define the **covariant vector field** n_a to the family of hypersurfaces by

$$n_a \equiv \frac{\partial f}{\partial x^a} dx^a \tag{677}$$

then becomes

$$n_a dx^a = g_{ab} n^a dx^b = 0$$

which tells us that n^a is orthogonal to the infinitesimal contravariant vector field dx^a. Since dx^a lies in S by construction, it follows that n^a is orthogonal to S and is therefore known as the **normal vector field** to S at x^a. Any other vector field X^a is said to be **hypersurface-orthogonal**, if it is everywhere orthogonal to the family of hypersurfaces, in which case it must be proportional to n^a everywhere, i.e.

$$X^a = \lambda(x) n^a$$

for some proportionality factor λ, which in general will vary from point to point.

16 Gravitational Lensing

The gravitational attraction of mass (stress-energy) deflects light. Because of this there can be multiple pathways for light to come from its source to an observer. It is possible for an intervening mass to produce multiple images of a distant source. When a mass concentration produces multiple images that is called strong *gravitational lensing*. When a mass (energy) concentration produces an image of a source that is distorted (magnified and sheared) that is generally called weak *gravitational lensing*. Gravitational lensing image can give information about the source, about the object acting as lens, and about the intervening large-scale geometry of the universe when source, lens, and observer are at cosmological distances from one another.

Realistic gravitational lenses may be clusters of distant galaxies withot any special symmetries. Light may propagate through them as well as around them. However, we first consider simple lensing by an concentrated spherical mass to illustrate the main features of lensing.

16.1 Lens Geometry and Image Position

The deflection angle α for a light ray passing by a mass M at an impact parameter $b >> R_s = 2GM/c^2$ is

$$\alpha = \frac{4GM}{c^2 b} = \frac{2R_s}{b}. \tag{678}$$

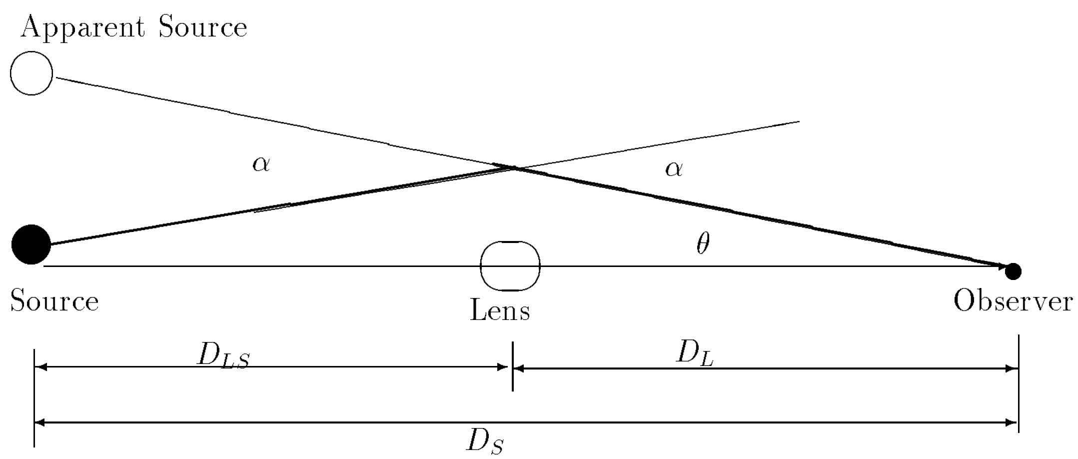
Proof of this is shown in lecture and is a homework problem.

For example and illustration, consider lensing by a stellar object, a galaxy, or a galactic cluster.

Object:	Planet	Star	Galaxy	Cluster
Mass	$0.01 M_\odot$	$M_\odot$	$10^{11} M_\odot$	$10^{13} M_\odot$
R_s	0.03 km	2.95 km	0.03 lyr (10^{11} km)	3.12 lyr
Distance		10 kpc	10^9 lyr	7.2×10^{13}
		3×10^4 lyr $\sim 10^{17}$ km	3×10^{22} km	
θ_E		$\sim 10^{-3\prime\prime}$	2.6×10^{-6} $0.5''$	$5''$

Table 5: Sample Parameters for Astrophysical Lenses

The following diagram explains the various quantities used in the the lensing formulae.



This figure shows the basic geometry of an astrophysical gravitational lens in the thin lens approximation. The Source is located a distance D_S from the observer and distance D_{LS} from the lens. Thus the distance from the lens to the source is

D_L. The heavy line from source to observer shows the approximate path of light bundle. The light bundle is bent through an angle α shown schematically (thin lens approximation) as a sharp bend at the distance of the lens. The apparent location extrapolating the incoming light bundle backwards appears at angle θ relative to line of sight to the lens. Not shown (I could not seem to draw it) is a straight line between the source and observer which is at an angle β to the line of sight to the lens.

The actual characteristic distance over which the bulk of the light bending occurs is of order of the Schwarzschild radius R_S which is typically much less than the other distances. Thus the then lens approximation will be fairly accurate. It is an excellent approximation that the light rays propagate as straight lines in flat space when far from the lens, and all the deflection occurs at the lens.

In realistic simulations, all the angles involved are very small, so that one can find transverse distances as angles times longitudinal distance:

$$b \simeq D_L\Theta \simeq D_L\beta + D_{LS}(\alpha + \beta - \theta) \qquad \theta D_S = \beta D_S + \alpha D_{LS} \tag{679}$$

This can be rewritten as

$$\theta = \beta + \frac{\theta_E^2}{\theta} \qquad \theta_E^2 \equiv 2R_S\left(\frac{D_{LS}}{D_S D_L}\right) \tag{680}$$

θ_E is called the Einstein angle.

The solutions to these equations gives the angular locations of two images

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \left(\beta^2 + 4\Theta_E^2\right)^{1/2}\right). \tag{681}$$

The arrangement of these two images as produced by a spherical mass is shown in the accompanying figure. There are two images on opposite sides of the position of the lens. One is at a position greater than the Einstein angle and one closer than the Einstein angle. If the source and lens are directly aligned with the observer ($\beta = 0$), then the images make a complete ring at the Einstein angle. In the limit of a small but finite-sized spherical lens, there is a third image behind the lens in addition to the two solutions.

The image locations are independent of the frequency of the light by the Equivalence Principle.

By measuring the angles $\theta_\pm$ between the position of the lens and the position of the images, the Einstein angle θ_E can be determined. If the distance to the lens is estimated, then the mas of the lens is determined.

Many sources have finite size and a defined shape. Because gravitational lensing is not true lensing but the power of the bending declines with impact parameter, the images not only are magnified but also sheared. The symmetry about the observer-lens axis implies that the light ray's value of azimuth, ϕ about that axis is unchanged by the deflection of the lens. the azimuthal angular width of the image $\Delta\phi$ is thus preserved. The polar width $\Delta\theta$ is changed by an amount that can be determined by differentiating the formula for $\theta_\pm$ to find

$$\Delta\theta_\pm = \frac{1}{2}\left(1 \pm \frac{\beta}{(\beta^2 + 4\theta_E^2)^{1/2}}\right)\Delta\beta \tag{682}$$

Thus the images of something like a galaxy will be elongated and distorted. The change in total brightness is called the magnification and is directly related to how much the image area as surface brightness is conserved.

$$\begin{aligned}
\frac{I_\pm}{I_*} &= \frac{\Delta\Omega_\pm}{\Delta\Omega_*} = \left|\frac{\theta_\pm\Delta\theta_\pm\Delta\phi}{\beta\Delta\beta\Delta\phi}\right| \\
&= \left|\left(\frac{\theta_\pm}{\beta}\right)\left(\frac{d\theta_\pm}{d\beta}\right)\right| = \frac{1}{4}\left(\frac{\beta}{(\beta^2+4\theta_E^2)^{1/2}} + \frac{(\beta^2+4\theta_E^2)^{1/2}}{\beta} \pm 2\right) \\
\frac{I_{total}}{I_*} &= \frac{I_+ + I_-}{I_*} = \frac{1}{2}\left(\frac{\beta}{(\beta^2+4\theta_E^2)^{1/2}} + \frac{(\beta^2+4\theta_E^2)^{1/2}}{\beta}\right)
\end{aligned} \tag{683}$$

The total intensity is always greater than unity. The gravitational lens always enhances total brightness for small angles to the lens. If the source is close to the observer-lens axis so that β is fairly small, this enhancement can be large. There are microlensing events that show a factor of ten increasse of star light. There are cases of distant galaxies being enhanced by significant amounts.

17 General Potential Distribution

All matter and energy, including dark matter, in the universe causes at least weak distortions in background images. The primary interest in astrophysical situations is weak gravitational lensing by clusters of galaxies. These are actually large gravitational potential wells that contain matter - roughly 10% baryons which are mostly in intracluster gas and plasma and some smaller part in galaxies - and some form of "Dark Matter" which clusters (clumps) to form the potential well. Individual sources behind a cluster of galaxies have their images distorted and magnified by the cluster gravitational lensing. Strong lensing produces "giant arcs", where background galaxies are magnified, stretched and curved. Weak lensing produces "arclets", in which galaxies are sheared by only a few percent. Although strong lensing provides information about the cluster core, it fails to disclose much about the mass distribution throughout the entire cluster. Fortunately, arclets can be found far from the core, sheared by the local gravitational potential; thus weak gravitational lensing can be used to determine the projected two-dimensional mass of the lens at large radii, thereby revealing dark matter concentrations contained in the cluster.

Gravitational lensing can be explained by Einstein's general theory of relativity. Galaxies, which are very massive objects, act on spacetime causing all four dimensions to become curved. This curved spacetime then influences particles in its vicinity, affecting their motion. Photons, for example, travel the path of least time. What appears to be the shortest path in flat spacetime is *not* the shortest path in curved spacetime. For this reason, photons passing a massive object take paths counterintuitive to Newton's classical law of gravity.

[width=.8clip=]curve.ps

Figure 1: Two-dimensional analogy for the curvature of spacetime.

The desire for photons to take the shortest path has significant consequences. As photons pass a massive object, they follow a curved path. This angular deflection gives an observer (along the line of sight) the impression that the light came from a different location (along a straight path) rather than its actual, original position. This is similar to the refraction of light rays passing through a prism. The result is that objects become distorted, and appear to be where they are not.

Some of the photons emitted by radiating bodies would not normally intersect an observer *if* they traveled along a straight path; however, because of a strong gravitational field, these photons can be "captured", curved around the lensing object and redirected to an observer. This enables more photons to be gathered and transported to a single location. The result are magnified objects with conserved surface brightness[4].

The next two subsections cover the theory of gravitational lensing, as well as a more simplified mathematical application for weak lensing.

Lensing by a Spherically Symmetric Mass Distribution[5]

The index of refraction[6] caused by curved spacetime is given by:

$$n = 1 - \frac{2\Phi}{c^2} \tag{684}$$

where Φ is the Newtonian gravitational potential of the lensing mass distribution, and c is the speed of light. It is assumed that $\Phi << c^2$, and $v << c$ (the velocity of the lensing mass perpendicular to the line of sight). The index of refraction, n, applies in any weak gravitational field.regardless of the configuration of the mass.

By applying Fermat's principle to (1), Euler's equation, and the assumption that individual lenses are thin, it can be shown that the deflection angle of a light ray passing through any gravitational field is given by:

$$\overrightarrow{\hat{\alpha}} = -\int \overrightarrow{\nabla}_{\perp} n \, dl = \frac{2}{c^2} \int \overrightarrow{\nabla}_{\perp} \Phi \, dl \tag{685}$$

[4]Measure of brightness per unit area: mags/arcsec^2.

[5](Narayan and Bartelmann 1997), (Bartelmann 2002)

[6]The ratio between the speed of light in a vacuum and the speed of light in that medium.

where dl is the infinitesimal coordinate along the light ray's path. The deflection angle is a two-component vector. It is assumed in gravitational lensing that the lenses are weak, and thus the angle of deflection is quite small. We can therefore use the Born approximation and integrate (2) along the line of sight of the incoming light ray, rather then the perturbed path.

For example, the Newtonian gravitational potential of a point mass is:

$$\Phi(r,l) = -\frac{GM}{(r^2+l^2)^{1/2}} \tag{686}$$

where G is the *universal gravitational constant*, and M is the point lens mass. the deflection angle for a circularly symmetric lens:

Narayan (1997) demonstrated that lensing by a singular isothermal sphere[7] is equivalent to lensing by the projected, circularly symmetric mass sheet. In reducing the general case of any lensing mass distribution to a circularly symmetric mass sheet lens, the deflection angle becomes:

$$\hat{\alpha}(r) = \frac{4GM(r)}{c^2 r} \tag{687}$$

where $M(r)$ is the mass enclosed within the radius r for a circularly symmetric lens (such as a point mass)[8].

There are three important distances when referring to the geometry of a gravitational lens: the distance between the observer and the lens, D_d, the distance between the lens and the source, D_{ds}, and the distance between the observer and the source, D_s (Figure 2 illustrates).

[width=.5clip=]fig204.ps

Figure 2: Geometry of a circularly symmetric gravitational lens system. $D_d\theta$ has been defined as ξ. Note that D_d, D_s, and D_{ds} are defined as angle $\times$ distance so that they are valid in curved spacetime; in general $D_{ds} \neq D_s - D_d$. (Diagram from of Narayan and Bartelmann 1997)

A light ray departing from the source, S, travels a distance D_{ds}, and is deflected by an angle $\hat{\alpha}$. When the ray arrives at the observer, it appears to be coming from point I, at an inclination of θ; in fact, the original source is at an angle β to the observer. These angles are related by:

$$D_s \overrightarrow{\beta}(\overrightarrow{\theta}) = D_s \overrightarrow{\theta} - D_{ds}\overrightarrow{\hat{\alpha}}(\overrightarrow{\theta}) \tag{688}$$

[7]A spherically symmetric mass distribution composed of particles in thermal and hydrostatic equilibrium, obeying the ideal gas equation of state.

[8]For a non-circular symmetric lens see Narayan and Bartelmann (1997)

where $\overrightarrow{\theta} \equiv (\theta_1, \theta_2)$, the azimuthal and polar angles.

The angle between the source and the image from the location of the observer, known as the *reduced deflection angle*, is:

$$\overrightarrow{\alpha} = \frac{D_{ds}}{D_s} \overrightarrow{\hat{\alpha}} \tag{689}$$

Substituting (6) into (5) and simplifying yields the *lens equation*:

$$\overrightarrow{\beta}(\overrightarrow{\theta}) = \overrightarrow{\theta} - \overrightarrow{\alpha}(\overrightarrow{\theta})$$

Now, defining a scalar potential such that it is the projected Newtonian potential of the lens, Φ, and has convenient derivation properties we have:

$$\psi(\overrightarrow{\theta}) = \frac{2D_{ds}}{c^2 D_d D_s} \int \Phi(D_d \overrightarrow{\theta}, l)\, dl \tag{690}$$

According to (2) and (6), the reduced deflection angle, $\overrightarrow{\alpha}$, is related to the gradient of the lensing potential, $\overrightarrow{\bigtriangledown}_\theta \psi$, with respect to the angle between the lens and the image, $\overrightarrow{\theta}$ by:

$$\overrightarrow{\bigtriangledown}_\theta \psi = \frac{2D_{ds}}{c^2 D_s} \int \overrightarrow{\bigtriangledown}_\perp \Phi(D_d \overrightarrow{\theta}, l)\, dl = \overrightarrow{\alpha} \tag{691}$$

It can then be shown that the Laplacian of the potential, $\bigtriangledown^2_\theta \psi$, which is proportional to the surface mass density of the lensing galaxy, is related to the convergence, κ, by (see Appendix A):

$$\bigtriangledown^2_\theta \psi = 2\frac{\Sigma(\overrightarrow{\theta})}{\Sigma_{critical}} \equiv 2\kappa(\overrightarrow{\theta}) \tag{692}$$

where the convergence has been defined as ratio of the surface mass density, $\Sigma(\overrightarrow{\theta})$, to the critical density[9], $\Sigma_{critical}$. Thus, the second derivative of the potential, which is the first derivative of the deflection angle, gives us the spread of the incoming light rays in the both the θ_1 and θ_2 directions. The convergence, κ, causes the symmetric expansion (or contraction) in a transmitted image. It is defined as an isotropic focusing of light rays and is given by:

$$\kappa \equiv \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial \theta_1^2} + \frac{\partial^2 \psi}{\partial \theta_2^2}\right)$$

The shear, which has two components, is a combination of the difference in the spread of the incoming light rays in the θ_1 and θ_2 directions, as well as the change

[9]The critical density is the density at which multiple images will be produced, that is when $\Sigma \geq \Sigma_{critical}$

in each direction with respect to the other direction. It is defined as the angular deformation:

$$\gamma_1(\overrightarrow{\theta}) \equiv \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_1^2} - \frac{\partial^2\psi}{\partial\theta_2^2}\right) \tag{693}$$

$$\gamma_2(\overrightarrow{\theta}) \equiv \frac{\partial^2\psi}{\partial\theta_1\partial\theta_2} = \frac{\partial^2\psi}{\partial\theta_2\partial\theta_1}$$

where the shear magnitude, $\gamma = (\gamma_1^2 + \gamma_2^2)^{1/2}$. The shear and convergence are responsible for the distortions seen in the transmitted images.

The Jacobian, which provides the lensing properties of the system, can be easily derived (see Appendix B), and is given by:

$$\mathcal{A} \equiv \frac{\partial\overrightarrow{\beta}(\overrightarrow{\theta})}{\partial\overrightarrow{\theta}} = \left(\delta_{ij} - \frac{\partial^2\psi(\overrightarrow{\theta})}{\partial\theta_i\partial\theta_j}\right) \tag{694}$$

$$= \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

Therefore, it is the second derivative of the lensing potential that provides the values for shear and convergence. Using the measured values for shear, the lensing potential, ψ, can be determined, and the mass of the lensing cluster can be reconstructed[10].

An important point in weak gravitational lensing is that galaxies project themselves in all sizes, ellipticities, and rotations. Due to this random nature, it is nearly impossible to determine the true orientation of a single, weakly sheared galaxy image. Because of this obstacle, an important assumption had to be made about the nature of galaxy formation: galaxies have no preferred direction when averaged over sufficient depth. This means that a sample of *unlensed* galaxies should have an average shear of zero; a weakly sheared field would have an average of a few percent. However, in order to account for statistical errors, a large number of galaxies must be measured and averaged in order to obtain an accurate determination of the local shear. Presently, the Hubble Deep and Medium-Deep Fields are two surveys with the potential for containing such large numbers of locally lensed galaxies. Other possible candidates are SUBARU, CTIO, NOAO, CFHT, and WHT[11].

[10]For more detail on the method of recovering the mass of a lensing cluster see Mellier & van Waerbeke (2001).

[11](Mellier & van Waerbeke (2001))

[width=1bb= 0 170 600 450, clip=]images/hdf.ps

Figure 3: The Hubble Deep Field (Photo courtesy of NASA and STScI). Deep fields, such as the Hubble Deep Field, are needed to reduce statistical errors in measuring shear by providing large numbers of weakly lensed galaxies.

APPENDIXES

Appendix A

Derivation of the Relation between the Laplacian of the Lensing Potential and the Convergence

Using the small angle approximation, the radius at closest approach is given by $D_d\theta$. Using this, we can obtain the mass of the cluster enclosed by integrating the product of the mass sheet in the lens plane and the distance from the center of the lens:

$$M(D_d\theta) = 2\pi \int_0^{D_d\theta} z\Sigma(z)\,dz = \pi(D_d\theta)^2\Sigma(D_d\theta)$$

By using the reduced deflection angle:

$$\overrightarrow{\alpha} = \frac{D_{ds}}{D_s}\overrightarrow{\hat{\alpha}}$$

and the deflection angle given in (4):

$$\hat{\alpha}(r) = \frac{4GM(r)}{c^2 r}$$

we obtain the relationship between the reduced deflection angle, surface mass density and distance from center:

$$\overrightarrow{\alpha}(\theta) = \frac{D_{ds}}{D_s}\frac{4GM(D_d\theta)}{c^2 D_d\theta} = \frac{D_{ds}}{D_s}\frac{4G\pi(D_d\theta)^2\Sigma(D_d\theta)}{c^2 D_d\theta} = \frac{4\pi G\Sigma(D_d\theta)}{c^2}\frac{D_d D_{ds}\theta}{D_s}$$

The critical surface mass density has been defined as:

$$\Sigma_{critical} = \frac{c^2}{4\pi G}\frac{D_s}{D_d D_{ds}} \tag{695}$$

The small angle approximation yields:

$$D_d\theta \simeq \theta \tag{696}$$

The lensing potential given in (7) is:

$$\psi(\overrightarrow{\theta}) = \frac{2D_{ds}}{c^2 D_d D_s}\int \Phi(D_d\overrightarrow{\theta}, l)\,dl \tag{697}$$

Laplace's equation states:

$$\bigtriangledown^2 \phi = 4\pi G\rho \tag{698}$$

And the integral of the density along a path, l, yields the surface mass density:

$$\int \rho(\overrightarrow{v}, l)\,dl = \Sigma(\overrightarrow{v}) \tag{699}$$

Using (12), (13), (14), (15), and (16), the Laplacian of the lensing potential can be represented as twice the ratio of the surface mass density and the critical surface mass density, which has been defined as twice the convergence:

$$\bigtriangledown_\theta^2\psi = \frac{2D_d D_{ds}}{c^2 D_s}\int \bigtriangledown_\perp^2\Phi(D_d\overrightarrow{\theta}, l)\,dl = \frac{2D_d D_{ds}}{c^2 D_s}4\pi G\Sigma(D_d\overrightarrow{\theta}) = 2\frac{\Sigma(\overrightarrow{\theta})}{\Sigma_{critical}} \equiv 2\kappa(\overrightarrow{\theta})$$

Appendix B

Derivation of the Jacobian Describing the Properties of the Lensing System

The angular separation between the lens and the image, as seen by the observer is characterized by two angles:

$$\overrightarrow{\theta} \equiv (\theta_1, \theta_2) \tag{700}$$

From Figure 2 and Equation (5), the lens equation is given by:

$$\overrightarrow{\beta}(\overrightarrow{\theta}) = \overrightarrow{\theta} - \overrightarrow{\alpha}(\overrightarrow{\theta}) \tag{701}$$

Substituting (17) into (18), the two components of β are:

$$\beta_1 = \theta_1 - \alpha_1(\overrightarrow{\theta}), \quad \beta_2 = \theta_2 - \alpha_2(\overrightarrow{\theta}) \tag{702}$$

The gradient of the lensing potential has been defined in (8) such that:

$$\overrightarrow{\nabla}_\theta \psi = \overrightarrow{\alpha} \tag{702}$$

Therefore, using (17), $\overrightarrow{\alpha}$ can be written in its components:

$$\alpha_1 = \partial\psi/\partial\theta_1, \quad \alpha_2 = \partial\psi/\partial\theta_2$$

The Jacobian is defined as:

$$\mathcal{A} \equiv \frac{\partial \overrightarrow{\beta}(\overrightarrow{\theta})}{\partial \overrightarrow{\theta}}$$

Substituting (17) and displaying in matrix form:

$$= \begin{pmatrix} \partial\beta_1/\partial\theta_1 & \partial\beta_1/\partial\theta_2 \\ \partial\beta_2/\partial\theta_1 & \partial\beta_2/\partial\theta_2 \end{pmatrix}$$

Substituting (19):

$$= \begin{pmatrix} 1 - \partial\alpha_1/\partial\theta_1 & -\partial\alpha_1/\partial\theta_2 \\ -\partial\alpha_2/\partial\theta_1 & 1 - \partial\alpha_2/\partial\theta_2 \end{pmatrix}$$

And finally, substituting (18) into the previous equation yields:

$$= \begin{pmatrix} 1 - \partial^2\psi/\partial\theta_1^2 & -\partial^2\psi/\partial\theta_1\partial\theta_2 \\ -\partial^2\psi/\partial\theta_1\partial\theta_2 & 1 - \partial^2\psi/\partial\theta_2^2 \end{pmatrix}$$

$$= \left(\delta_{ij} - \frac{\partial^2\psi(\overrightarrow{\theta})}{\partial\theta_i\partial\theta_j}\right)$$

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