Physics 139 Relativity
Relativity Notes 2001

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Notes to be found at
http://aether.lbl.gov/www/classes/p139/homework/homework.html

1 Radiation From Accelerated Charge

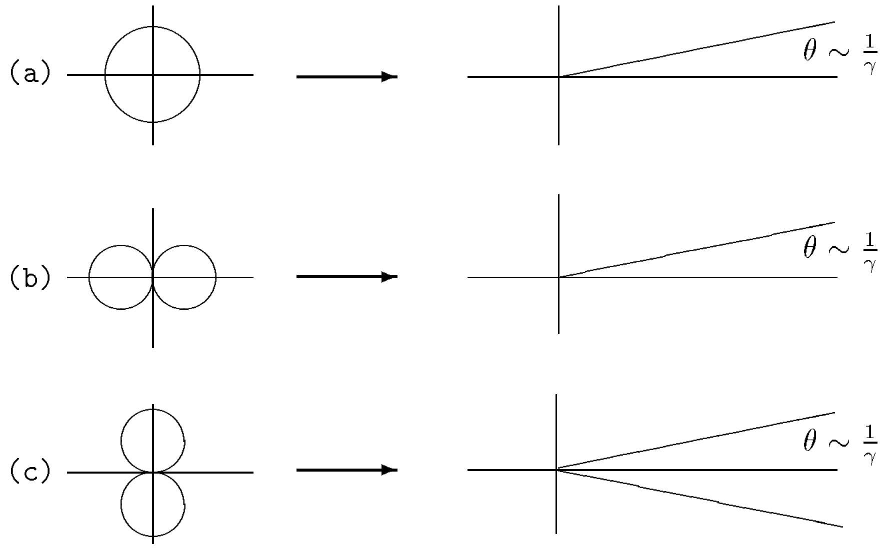
1.1 Introduction

You have learned about radiation from an accelerated charge in your classical electromagnetism course. We review this and treat it according to the prescriptions of Special Relativity to find the relativistically correct treatment.

Radiation from a relativistic accelerated charge is important in:
(1) particle and accelerator physics – at very high energies ($\gamma \gg 1$) radiation losses, e.g. synchrotron radiation, are a dominant factor in accelerator design and operation and radiative processes are a significant factor in particle interactions.
(2) astrophysics – the brightest sources from the greatest distances are usually relativistically beamed.
(3) Condensed matter physics and biophysics use relativistically beamed radiation as a significant tool. An example we will consider is the Advanced Light Source (ALS) at the Lawrence Berkeley Laboratory. Now free electron lasers are now a regular tool.

We will need to use relativistic transformations to determine the radiation and power emitted by a particle moving at relativistic speeds.

Lets look at the concept of relativistic beaming to get an idea before we go into the details which require a fair amount of mathematics.



Radiation from an accelerated relativistic particle can be greatly enhanced. Part of this effect is due to the aberration of angles and part due to the Doppler effect.

1.2 Doppler Effect

From time dilation we are used to the notion that a moving clock or system operating at frequency ν' in its rest frame will appear to be slower to a reference system.

$$\Delta t = \gamma \Delta t' \tag{1}$$

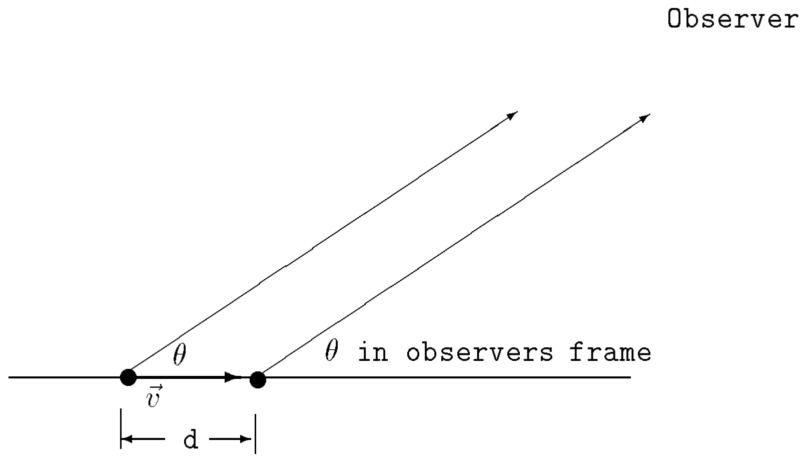
so that if the period in the rest frame is $\delta t' = 1/\nu'$, then

$$\nu = \nu'/\gamma \tag{2}$$

The factor leads to the relativistic transverse Doppler shift. The frequency shift one would observe for a clock or system moving transversely to the line of sight.

Thus the time between wave peaks (crests) or pulses is

$$\Delta t = \gamma \Delta t' = \frac{\gamma}{\nu'} \tag{3}$$



If the sources is moving at an angle θ to the observer's line of sight, then the difference in arrival times, Δt_A, of successive pulses or crests is

$$\begin{aligned} \Delta t_A &= \Delta t - \frac{d}{c} = \Delta t(1 - \frac{v}{c}cos\theta) \\ \frac{1}{\nu_{obs}} &= \frac{\gamma}{\nu'}(1 - \frac{v}{c}cos\theta_{obs}) \end{aligned} \tag{4}$$

which when inverted or multiplied by c yields:

$$\nu_{obs} = \frac{\nu'}{\gamma(1 - \frac{v}{c}cos\theta_{obs})} \qquad \lambda_{obs} = \gamma\lambda'(1 - \frac{v}{c}cos\theta) \tag{5}$$

1.3 Radiation by an Accelerated Charge Near Rest

In 1897 Larmor derived the formula for the radiation by an accelerated charged particle. He found for the power and angular distribution:

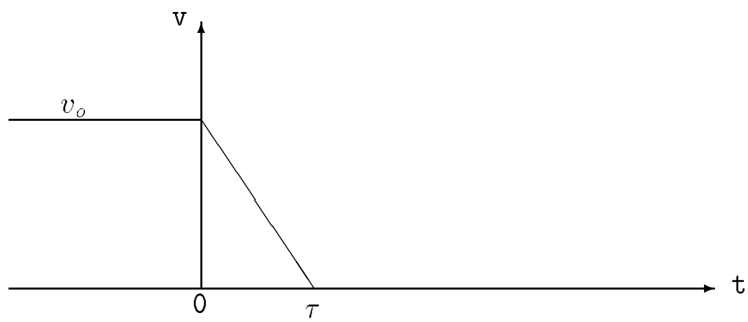
$$P = \frac{2q^2}{3c^3}\vec{a} \cdot \vec{a} \qquad \frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|a|^2 sin^2\Theta \tag{6}$$

where Θ is the angle to the direction of acceleration. It is our task to find the relativistically consistent and correct version of these formulae.

We can rederive the Larmor formula for your education. We consider the electric field to be a real physical entity that points radially back to a charge at rest. If we go into a moving frame, the electric field lines will continue to point radially back to the instantaneous position of the charge. The transformation of the electric field works out precisely that way. The Lorentz-Fitzgerald contraction along the direction of motion causes an increase by the factor γ of the transverse component of the field. Gauss's law continues to hold in that an integral over a closed surface, such as a sphere, gives the net charge within.

Now if a charge is diverted from uniform motion, then by our earlier arguments about causality, the electric field lines out at radius R can not be effected by that change from uniform motion until a time $t = R/c$ later. (In fact we expect that the electic field lines will change at the speed of light since light is an electromagnetic phenomenon.) Thus at a time t after a brief $\delta t = \tau$ disturbance (change from one state of uniform motion to another - also called acceleration) there is a critical radius $R = ct$. Inside of radius $R - c\tau$ the electric field lines point radially to the new instantaneous position of the charge and outside of radius $R + c\tau$ the electric field lines point radially to the virtual instanteous position of the undisturbed charge. The virtual instanteous position is where the charge would have been had it not been disturbed. There is a near discontinuity in the field lines where they must make a jaunt nearly perpendicular to radial. Nearly means that the angle between the field line and perpendicular to radial is of order $c\tau/vt$ where v is the velocity change due to the disturbance.

Consider: a charge moving with velocity $v \ll c$ abruptly, at time $t = 0$, is decelerated at a constant rate a until it comes to rest.



At $t = 0$, $x = 0$ and at $t = \tau$, $x = v_o\tau/2$.

Now consider fields at a time $t_f \gg \tau$. At a distance $r > ct_f$, the field will be that of a uniformly moving charge, emanating from the "virtual present position" (the point where the particle would have been, $x = v_0 t_f$, if it had continued unaccelerated. At a distance $r < c(t_f - \tau)$, the field will be that of a charge at rest with $x = v_o\tau/2$.

There is a transition region which is nearly a spherical shell ($v_o \ll c$) A particular field line L defines a cone of angle, θ, inside, which contains a certain flux. Its continuation L' defines another cone which must contain the same flux by reason of Gauss's law relating the field flux and the enclosed charge. Thus $\theta' = \theta$ and L' is parallel to L.

Consider the portion connecting these two regimes.

insert figure

The radial component of the electric field, E_r must be the same in the shell as just outside of it on either side (Gauss's law).

$$E_r = \frac{q}{r^2} = \frac{q}{ct_f r} = \frac{q}{(ct_f)^2} \tag{7}$$

By the geometry of the situation

$$\frac{E_\theta}{E_r} = \frac{v_o t_f sin\theta}{c\tau} \tag{8}$$

$$E_\theta = \frac{v_o t_f sin\theta}{c\tau} E_r = \frac{v_o t_f sin\theta}{c\tau}\frac{q}{(ct_f)^2} = \frac{q v_o sin\theta}{c^3 t_f \tau}. \tag{9}$$

Now $ct_f = r$, and $a = v_o/t_f$, so that

$$E_\theta = \frac{qasin\theta}{c^2 r} \tag{10}$$

The significance of this result is that $E_\theta \propto 1/r$ while $E_r \propto 1/r^2$. At a large distance the tangential electric field E_θ will dominate.

From our general knowledge of varying vacuum fields we know that there will be a component of $\vec{B}$ of strength equal to $\vec{E}$ and perpendicular both to $\vec{E}$ and $\vec{r}$.

The energy density (energy per unit volume) in the transition layer is

$$u = \frac{\text{Energy}}{\text{Volume}} = \frac{E_\theta^2}{8\pi} + \frac{B_\phi}{8\pi} = \frac{E_\theta^2}{4\pi} = \frac{q^2 a^2 sin^2\theta}{4\pi c^4 r^2} \tag{11}$$

The volume of the shell is it area ($4\pi r^2$) times its thickness ($c\tau$) and the average of $sin^2\theta = 2/3$,

$$\begin{aligned} < sin^2\theta > &= \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} sin^2\theta d(cos\theta)d\phi = \frac{1}{2}\int_{-1}^{1} sin^2\theta d(cos\theta) = \frac{1}{2}\int_{-1}^{1}(1-x^2)dx \\ &= \frac{1}{2}(x - \frac{1}{3}x^3)|_{-1}^{1} = \frac{2}{3} \end{aligned} \tag{12}$$

so that the energy in the transition layer is

$$E = \frac{2}{3}\frac{q^2 a^2 \tau}{c^3}.$$

The radiated power is then the energy per unit time:

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}. \tag{13}$$

which is precisely the formula derived by Larmor in 1897.

1.4 Radiation from Circular Orbit

Suppose that $\vec{a} \perp \vec{B}$, giving motion in a circle and that $\gamma \gg 1$:

insert figure: Reference frame S Laboratory and reference frame S′ which is the electron instantaneous rest frame

two column

The laboratory reference frame S has $\vec{B}$ perpendicular to the plane of the circular orbit ($B_x = B_y = 0$, and $B_z = B$ and $\vec{E} = 0$

$$F = e|\vec{v} \times \vec{B}| = \frac{\gamma m_o v^2}{r} \tag{14}$$

By transformation law $F' = \gamma^{-2,3} F$

column 2 Transform fields:

$$E' = \gamma \left(E - \beta B \right) \tag{15}$$

or more precisely

$$\begin{aligned} \vec{E}' &= \gamma \left(\vec{E} - \vec{\beta} \times \vec{B} \right) \\ &= \gamma \left(0 + \beta_x B_z (-\hat{e}_y) \right) \\ &= -\beta \gamma B \hat{e}_y \end{aligned} \tag{16}$$

Thus

$$\vec{F} = (-e)\vec{E}' = +\beta\gamma q B \hat{e}_y = m_o \vec{a} \tag{17}$$

Thus

$$a = \frac{\beta \gamma q B}{m_o} \tag{18}$$

The power emitted is

$$P = \frac{2}{3} \frac{q^2 a^2}{c^3} = \frac{2}{3} \frac{\beta^2 \gamma^2 q^4 B^2}{m_o^2 c^3} \tag{19}$$

This is the correct relativistic form! because P is energy per unit time and each is the 0-component of a 4-vector!!

To make a relativistic generalization we employ the concept of covariance.

The Poynting vector is the 0-0 component of the electromagnetic stress tensor and the radiated electromagnetic energy is the 0-component of a Lorentz 4-vector. Time is the 0-component of a Lorentz 4-vector. So the ratio energy per time is an invariant.

Can one find a Lorentz invariant that reduces to Larmor's formula as $\beta \to 0$? If so, it will be the correct relativistic formula! Is it unique? Yes, if we require it to involve only $\vec{\beta}$ and $d\vec{\beta}/dt$ and not higher powers.

How is one to construct it? Non-relativistically,

$$\frac{dv}{dt} = \frac{1}{m_o} \frac{dp}{dt} \tag{20}$$

So the Larmor power is

$$P = \frac{2}{3} \frac{e^2}{m_o^2 c^3} \left(\frac{d\vec{p}}{dt} \cdot \frac{d\vec{p}}{dt} \right) \tag{21}$$

To get an invariant, experience tells us to substitute $d\tau$ for dt, i.e. $d\vec{p}/dt \rightarrow d\vec{p}/d\tau$, and add the fourth component:

$$\frac{d\tilde{p}}{d\tau} \cdot \frac{d\tilde{p}}{d\tau} = \left(\frac{dE}{d\tau}\right)^2 - c^2 \left(\frac{d\vec{p}}{d\tau}\right)^2 \tag{22}$$

Notice that

$$EdE = c^2 pdp \tag{23}$$

So

$$\left(\frac{dE}{c}\right)^2 = \frac{(pc)^2}{E^2} (dp)^2 = \left(\frac{mvc}{mc^2}\right) (dp)^2 = \beta^2 (dp)^2 . \tag{24}$$

So that

$$\frac{1}{c^2} |\frac{d\tilde{p}}{d\tau}|^2 = |\frac{d\vec{p}}{d\tau}|^2 - \beta^2 \left(\frac{dp}{d\tau}\right)^2 \tag{25}$$

Giving

$$P = \frac{2}{3} \frac{e^2}{m_o^2 c^3} \left[|\frac{d\vec{p}}{d\tau}|^2 - \beta^2 \left(\frac{dp}{d\tau}\right)^2 \right] \tag{26}$$

It is possible to write this in many ways. One way is

$$P = \frac{2}{3} \frac{e^2}{c} \gamma^6 \left[|\dot{\vec{\beta}}|^2 - |\vec{\beta} \times \dot{\vec{\beta}}|^2 \right] \tag{27}$$

1.5 Power and Angular Distribution Summary

We can calculate these in a consistent way by using these formula as correct in the rest (primed) frame of the electron and transform the accelerations (forces), angles, frequencies, etc. into the laboratory frame. What we need is to show that powe is a Lorentz invariant $P = P'$ for any emitter that emits with front-back symmetry (zero net momentum) in its instantaneous rest frame. To do this we make use of the invariance of $\tilde{a} \cdot \tilde{u}$ which is zero for all systems.

$$\tilde{a} \cdot \tilde{u} = \frac{d\tilde{u}}{d\tau} \cdot \tilde{u} = \frac{1}{2} \frac{d}{d\tau} (u^\mu u_\mu) = \frac{1}{2} \frac{d}{d\tau} (c^2) = 0$$

This is a consequence invariance of the speed of light and four-vector velocity.

In the zero net radiation momentum (in instanteous rest frame) case $\tilde{a} \cdot \tilde{a} = \vec{a} \cdot \vec{a}$ since in the rest frame $a_0 = 0$. Thus the power can be evaluated in any frame can be found by computing the acceleration in that frame and squaring it.

$$\begin{aligned} P = \frac{2q^2}{3c^3} \vec{a}' \cdot \vec{a}' &= \frac{2q^2}{3c^3} \left(a_\perp'^2 + a_{||}'^2 \right) \\ &= \frac{2q^2}{3c^3} \gamma^4 \left(a_\perp^2 + \gamma^2 a_{||}^2 \right) \end{aligned} \tag{28}$$

where $a_\perp$ is the acceleration perpendicular to the motion of the charged particle and $a_{||}$ is the acceleration component parallel to the charge particle motion. In the last line we have made use of the transformation of accelerations $a'_{||} = \gamma^3 a_{||}$ and $a'_\perp = \gamma^2 a_\perp$ evaluated in the instantaneous rest frame (primed) of the electron. Note that there is a factor of γ difference in the transformation of accelerations perpendicular and parallel to the direction of motion. This translates into a difference between γ^4 and γ^6 in the perpendicular and parallel cases.

We get a similar expression for the angular distribution:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{a_\perp^2 + \gamma^2 a_{||}^2}{(1-\beta cos\theta)^4} sin^2\Theta' \tag{29}$$

We are making use of the conversion

$$\frac{dP}{d\Omega} = \frac{1}{\gamma^4(1-\beta cos\theta)^4} \frac{dP'}{d\Omega'}$$

Evaluation for perpendicular and parallel cases yields:

$$\begin{aligned} \frac{dP_\perp}{d\Omega} &= \frac{q^2 a_\perp^2}{4\pi c^3} \frac{1}{(1-\beta cos\theta)^4} \left[1 - \frac{sin^2\theta cos^2\phi}{\gamma^2(1-\beta cos\theta)^2}\right] \\ \rightarrow_{\gamma \gg 1} &\approx \frac{4q^2 a_\perp^2}{\pi c^3} \gamma^{\mathbf{8}} \frac{1 - 2\gamma^2\theta^2 cos 2\phi + \gamma^4\theta^4}{(1+\gamma^2\theta^2)^6} \end{aligned} \tag{30}$$

$$\frac{dP_{||}}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3} \frac{sin^2\theta}{(1-\beta cos\theta)^6} \rightarrow_{\gamma \gg 1} \approx \frac{4q^2 a_{||}^2}{\pi c^3} \gamma^{\mathbf{10}} \frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^6} \tag{31}$$

Note the large powers of γ 8 and 10 which shows the seriousness of the relativistic effects. Before we follow this up in detail, we review radiation near the rest frame of the emitting particle.

1.5.1 Case I: acceleration parallel to motion

Consider $\vec{\beta} \parallel \dot{\vec{\beta}}$; acceleration parallel to motion $\vec{\beta} \times \dot{\vec{\beta}} = 0$. Recalling that $\beta^2 = 1 - 1/\gamma^2$, then

$$\beta\dot{\beta} = \frac{\dot{\gamma}}{\gamma^3};$$

$$\dot{\beta} = \frac{\dot{\gamma}}{\beta\gamma^3}; \qquad \dot{\beta}^2 = \frac{\dot{\gamma}^2}{\beta^2\gamma^6}$$

$$P = \frac{2}{3}\frac{e^2}{c}\left(\frac{\dot{\gamma}}{\beta}\right)^2 \tag{32}$$

$$P = \frac{2}{3}\frac{e^2}{m_o^2 c^3}\left(1-\beta^2\right)\left(\frac{dp}{d\tau}\right)^2$$

$$
\begin{aligned}
&= \frac{2}{3}\frac{e^2}{m_o^2 c^3}\gamma^2 \frac{1}{\gamma^2}\left(\frac{dp}{dt}\right)^2 \qquad d\tau = \frac{dt}{\gamma} \\
&= \frac{2}{3}\frac{e^2}{m_o^2 c^3}(m_o c^2)^2 \left[\frac{d(\beta\gamma)}{dt}\right] \qquad p = \beta\gamma m_o c \\
&= \frac{2}{3}\frac{e^2}{c}\left(\frac{\dot{\gamma}}{\beta}\right)^2
\end{aligned} \tag{33}
$$

where the conversion makes use of the relations

$$
\begin{aligned}
(\beta\gamma) &= \sqrt{\gamma^2 - 1} \\
\frac{d(\beta\gamma)}{dt} &= \frac{1}{2}\frac{2\gamma\dot{\gamma}}{\sqrt{\gamma^2 - 1}} \\
&= \frac{\gamma\dot{\gamma}}{\beta\gamma} = \frac{\dot{\gamma}}{\beta}
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\frac{\dot{\gamma}}{\beta} &= \frac{c}{v}\frac{d}{dt}\left(\frac{E}{m_o c^2}\right) \\
&= \frac{1}{m_o c}\frac{dE}{v dt} = \frac{1}{m_o c}\frac{dE}{dx}
\end{aligned} \tag{35}
$$

$$
\frac{P}{dE/dt} \sim \frac{2}{3}\frac{e^2/m_o c^2}{m_o c^2}\frac{dE}{dx} \tag{36}
$$

So that the power radiated compared to the energy change per unit distance is

$$
P = \frac{2}{3}\frac{e^2}{m_o^2 c^3}\left(\frac{dE}{dx}\right)^2 \tag{37}
$$

Now we can compare the radiated power with the acceleration power

$$
\frac{P}{dE/dt} = \frac{2}{3}\frac{e^2/m_o c^2}{m_o c^2}\frac{dE/dx}{dE/dt}\frac{dE}{dx} \tag{38}
$$

Note $(dE/dx)/(dE/dt) \sim dt/(cdt)$ when $\beta \sim 1$. The ratio of powers, radiated to acceleration, is negligible unless energy gain in 2.8×10^{-13} cm is of order of the rest mass - i.e. for an electron ~ 0.511 MeV.

1.5.2 Case II: acceleration perpendicular to motion

Centripetal acceleration: $\dot{\vec{\beta}} \perp \vec{\beta}$. $\vec{a} = c\dot{\vec{\beta}}$ and $\vec{v} = c\vec{\beta}$.

insert figure / diagram to show vector directions

Then in the relation to find the rate of change of the energy-momentum four-vector

$$
\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2 - |\frac{d\vec{p}}{dt}|^2
$$

$E/dt \cong 0$; since $\vec{F} \perp \vec{v}$, so that no work is being done on the particle. Then

$$P = -\frac{2}{3}\frac{e^2}{m_o c^3}|\frac{d\vec{p}}{d\tau}|^2 \tag{39}$$

and

$$|\frac{d\vec{p}}{d\tau}| = \gamma\omega|\vec{p}| \tag{40}$$

where $\omega = \beta c/\rho$ is the orbital angular frequency of an orbit with radius ρ. One can derive this relationship

$$\frac{dp}{p} = d\theta = \frac{ds}{\rho} = \frac{vdt}{\rho} = \omega dt$$

$$\frac{dp}{dt} = \omega p$$

Thus

$$\omega = \frac{\beta c}{\rho}$$

Now we can move on to the power loss rate

$$d\tau = \frac{dt}{\gamma} \qquad p = \beta\gamma m_o c$$

$$\begin{aligned} P &= \frac{2}{3}\frac{e^2}{m_o^2 c^3}\gamma^2\omega^2|\vec{p}|^2 \\ &= \frac{2}{3}\frac{e^2}{m_o^2 c^3}\left(\frac{\beta c}{\rho}\right)\gamma^2\left(\beta\gamma mc\right)^2 \end{aligned} \tag{41}$$

$$P = \frac{2}{3}\frac{e^2 c}{\rho^2}\beta^4\gamma^4 \tag{42}$$

The energy gain for a particle per turn in an accelerator is

$$\delta E = 2\pi\rho\frac{P}{v} \tag{43}$$

The radiation loss is

$$\delta E = \frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4 \tag{44}$$

In practical units

$$\delta E/(1 \text{ MeV}) = 8.85 \times 10^{-2}\frac{(E/1 \text{ GeV})^4}{\rho/(1 \text{ meter})} \tag{45}$$

The power radiated by a bunch of electrons

$$\text{Power}/(1 \text{ watt}) = 10^6[\delta E/(1 \text{ MeV turn})][J/(1 \text{ amp})] \tag{46}$$

provided the radiation is incoherent.

Aside: How to get these practical unit relations:

$$\delta E = \frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4$$

Start by putting $\beta = 1$. If β is not very near to 1, then one gets negligible radiation power.

$$\gamma = \frac{E}{m_o c^2} = \frac{E(\text{in GeV})}{0.511\ \text{MeV}}$$

$$\frac{e^2}{\rho} = \frac{e^2}{m_o c^2}\frac{m_o c^2}{\rho} = r_o\frac{m_o c^2}{\rho} = (2.8\times10^{-13}\ \text{cm})\frac{0.511\ \text{MeV}}{\rho\ (\text{in cm})}$$

and the conversion from ρ in cm to m is $\rho_{cm} = 100\rho_m$. So that

$$\begin{aligned}\delta E &= \left[\frac{4\pi}{3}\frac{2.8\times10^{-13}}{100}\frac{0.511}{(5.11\times10^{-4})^4}\right]\frac{[E(\text{in GeV})]^4}{\rho\ (\text{in m})}\ \text{MeV} \\ &= \left[8.85\times10^{-2}\frac{[E(\text{in GeV})]^4}{\rho\ (\text{in m})}\right]\ \text{MeV} \\ &= 88.5\frac{[E(\text{in GeV})]^4}{\rho\ (\text{in m})}\ \text{keV}\end{aligned} \tag{47}$$

Giving the conversion used above.

$$\begin{aligned}\text{Power} &= \frac{\delta E}{\text{Electron turn}}\times\frac{\text{turn}}{\text{sec}}\times\text{Number of electrons} \\ &= \delta E\frac{V}{2\pi\rho}\frac{2\pi\rho J}{eV} = \frac{\delta E\times J}{e} \\ \text{Power (in kW)} &= 88.5[E(\text{in GeV})]^4 J\ (\text{in amps})/\text{R (in m)} \\ &= 26.5[E(\text{in GeV})]^3 B\ (\text{in teslas}) J\ (\text{in amps}) \\ \frac{\delta E\ (\text{in MeV})}{e} &= \delta V\ (\text{in MV})\end{aligned} \tag{48}$$

Now Some Numbers and History E.O. Lawrence invented the cyclotron and the first was built here at Berkeley. Later his colleague Edwin McMillen (and independently in the Soviet Union by V.I. Veksler) invented the idea of phase stability which made the synchrotron possible.

Synchrotron radiation was first observed in a laboratory in 1947. That laboratory was in Berkeley.

Early Synchrotrons:

First synchrotron was operated with 8 MeV electrons in 1946 by Goward and Barnes in Woolwich Arsenal, UK. In 1947 GE labs operated an electron synchrotron at 70 MeV. Soon after there were many operating.

Table 1: Parameters for Sample Accelerators

Accelerator	LBL	Cornell	LHC	ALS	FermiLab	SSC	Elo
Max. Energy (GeV)	0.3	10	45	1-2	1000	20,000	10^4
Particle	e	e	e	e	p-$\overline{p}$	p-p	p
B (Tesla)		0.33	0.135	1.248	4.4	6.6	7.7
Radius (m)	1	100	4249		1000	11.7 km	50 km
Bending R (m)				4.01		10.1 km	
Beam Current (ma)				400		73	100
Single Bunch (ma)				1.6			0.00167
E-gain/turn (MeV)	0.05	10.5	350	1		5.26	
E-loss/turn (MeV)	0.001	8.8		0.112	0.001		18
Synchrotron Power				45 kW		9.1 kW	1.8 MW
RF Power (kW)			16000	300	1600		61000
RF (MHz)		713.94	352	500	53.1	374.74	412
Harmonic		1800	31324	328	1113	103,680	146500
Beam lifetime (hrs)				14	4	~24	48
Fill time			30 min	2.1 min		40 min	4 hrs

An early synchrotron at Berkeley had a radius of about 1 meter and a maximum energy of about 0.3 GeV. The synchrotron radiation $\delta E_{max} \sim 1$ keV/turn could be noticed. The acceleration voltage was only a few keV/turn.

At big electron synchrotron was built at Cornell and operated at 10 GeV . The radius was about 100 meters. It encloses a football field. The magnetic field was $B = 3.3$ kG (0.33 Tesla). The accelerator voltage was about 10.5 MeV/turn and the synchrotron losses were $\delta E_{rad} \sim 8.8$ MeV/turn.

LBL Advanced Light Source is designed to provide synchrotron radiation as a tool for research.

1.6 Synchrotron Radiation Basics

Consider the non-relativistic case of a charged particle in a circular orbit caused by a magnetic field. That particle will radiate electromagnetic waves at a frequency given by the orbit frequency (or the Lamor frequency)

$$\omega_L = \frac{qB}{mc} \qquad \nu_L = \frac{qB}{2\pi mc}$$

due to the acceleration of bending in the magnetic field. As the particle's energy is increased relativistic effects will become important. For the same orbit the particle will both begin to radiate more energy and at more frequencies - which are at the orbit frequency and its harmonics. The peak power will be emitted at a frequency which is at $\approx \gamma^3$ times the orbit frequency. In the next sections we will understand this.

1.6.1 Synchrotron Emitted Power

To find the total emitted power we can use the Lamor (1897) formula

$$P_{emitted} = \frac{2}{3}\frac{q^2}{c^3}|\vec{a}_o|^2 = \frac{2}{3}\frac{q^2}{c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_{||}^2\right)$$

where $\vec{a}_o$ is the particle acceleration in its instantaneous rest frame and the right hand side of the equation uses the acceleration transform law from the particle rest frame.

$$\frac{dE}{dt} = q\vec{v}\cdot\vec{E}$$

and since $E = 0$ we have $\gamma =$ constant.

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(\gamma m_o \vec{v}) = q\vec{v}\times\vec{B}$$

With $\gamma =$ constant,

$$\gamma m_o \frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}$$

Thus

$$\frac{dv_{||}}{dt} = 0, \qquad \frac{dv_\perp}{dt} = \frac{q}{\gamma m_o}\vec{v}_\perp\times\vec{B}$$

We can conclude $|v_{||}| =$ constant and $|v_\perp| =$ constant. We have uniform circular motion of the projected motion on the normal plane. That is a simple helical motion around the uniform magnetic field.

The frequency of rotation or gyration is

$$\omega_B = \frac{qB}{\gamma m_o c}, \qquad \rightarrow \qquad a_\perp = \omega_B v_\perp$$

Note that the gyration (orbit) frequency is the Lamor frequency divided by γ.

We can now evaluate the transformation of the Lamor formula for the power radiated since we know $a_\perp = \omega_B v_\perp$ and $a_{||} = 0$

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^4\omega_B^2 v_\perp^2 = \frac{2}{3}\frac{q^2}{c^3}\gamma^4\left(\frac{qB}{\gamma m_o c}\right)^2 v_\perp^2 = \frac{2}{3}\frac{q^4 B^2}{m_o^2 c}\beta_\perp^2\gamma^2$$

$$= \frac{2}{3}r_o^2 c\beta_\perp^2\gamma^2 B^2$$

$$= 2\beta_\perp^2\gamma^2 c\sigma_T U_B = 2\beta^2\gamma^2 c\sigma_T U_B sin^2\alpha$$

where $r_o = e^2/m_e c^2$ is the classical radius of the electron, $\sigma_T = 8\pi r_o^2/3$ is the Thomson crossection, $U_B = B^2/8\pi$ is the energy density of the magnetic field, and α is the helix pitch angle (angle of the gyrating particle with respect the magnetic field lines). This is the relativistically correct form that we saw previously.

1.6.2 Synchrotron Radiation Frequency Spectrum

First we consider the frequency distribution of a monoenergetic distribution, i.e. we consider the radiation from a particle at an energy E corresponding to γ. When the particle's energy increases (as γ grows larger) the aberration of angles moves most of the radiated power into a cone of half angle $\Delta\theta \sim 1/\gamma$ in the instantaneous direction of motion of the particle. Thus an observer will see a pulse of radiation whenever the particle's instantaneous velocity sweeps past his direction. This will happen once per orbit. This pulse will be narrow both because the aberration of angles and because of the time dilation and Doppler effect. Since the relativistic particle is moving towards the receiver (observer), the received pulse is sharpened (compressed in time) by a factor of order γ^{-2}. The time compression goes at

$$\frac{dt}{d\tau} = 1 - \beta cos\theta \sim 1 - \beta + \Delta\theta^2/2 \rightarrow \gamma^{-2}$$

where the limit comes for $\gamma \gg 1$ since $\beta = \sqrt{1 - 1/\gamma^2} \rightarrow 1 - \gamma^{-2}/2$ and $\Delta\theta^2/2 \sim \gamma^{-2}/2$.

Thus the observer will see a pulse every orbit with width γ^{-3} of the pulse separation. Fourier theory tells us that the signal will appear at the orbit frequency and its harmonics and that the power will peak at a frequency which is near $\gamma^3\nu_L$ (where $\nu_B = \omega_L/2\pi = qB/m_ec$).

For a magnetic field $B = 10^{-5}$ Gauss, which is a typical value in the Galaxy and many powerful radio galaxies, $\nu_L = 28$ Hz. The electrons that produce emission at radio frequencies of a few GHZ therefore have Lorentz factors $\gamma \sim 10^3 - 10^4$. The spacing between successive harmonics is $\nu_B = \nu_L/\gamma$, for very high γ this spacing is so narrow as to negligible for all but the highest frequency resolution observations. In astrophysical sources, this is often blurred and smoothed by variations in the electron energy (a power law spectrum) and by variations in the magnetic field intensity and direction.

1.7 Astrophysical Synchrotron Radiation

1.7.1 Historical Note

Although nonthermal radiation had been observed from the Galaxy from the opening of radio astronomy in the pioneering work by Karl Jansky in 1933, there was no clear evidence of its origin. In 1950 Kiepenheuer suggested that Galactic nonthermal radio emission was synchrotron radiation and Alfvén and Herlofson proposed that non-thermal discrete sources were emitting synchrotron radiation. Kiepenheuer showed that the intensity of the nonthermal Galactic radio emission can be understood as the radiation from relativistic cosmic ray electrons that move in the general interstellar magnetic field. He found that a field of 10^{-6} Gauss (10^{-10} Tesla) and relativistic electrons of energy 10^9 eV would give about the observed intensity. The early 1950s saw the development of these ideas (e.g. Ginzburg et al. 1951 and following papers, see Ginzburg 1969) that synchrotron emission was the source of non-thermal "cosmic" radiation. This model was later supported by maps which showed that the sources of the non-thermal components were extended nebulae and external galaxies and by the discovery that the radiation was polarized as predicted by theory. The synchrotron theory is widely accepted and is the basis of interpretation of all data relating to nonthermal radio emission.

1.7.2 Context

Synchrotron radiation is a common phenonmen in astrophysics as there are almost always plasma and magnetic fields present and energetic electrons. Because of stochastic scattering processes, the energetic electrons tend to be isotropically distributed.

For an isotropic distribution of velocities one needs to average over all angles for a given speed β. If α is the pitch angle, the angle between the magnetic field direction and the particle velocity, then

$$< \beta_\perp^2 >= \frac{\beta}{4\pi} \int sin^2 \alpha d\Omega$$

Thus

$$P = \left(\frac{2}{3}\right)^2 r_o^2 c \beta^2 \gamma^2 B^2 = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 U_B$$

where $\sigma_T = 8\pi/3\ r_o^2$ is the Thomson cross section and $U_B = B^2/8\pi$ is the energy density in magnetic field.

Electrons of a given energy ($E = \gamma m_e c^2$) radiate over a wide spectral band, with the distribution peaking roughly at $\nu_c \approx 16.08(B_{\rm eff}/\mu{\rm G})(E/{\rm GeV})^2$ MHz, with a long low-power tail at higher frequencies, and most of the radiation in a 2:1 band from peak. The peak intensity is at $\nu_{max} = 0.29\nu_c = 4.6(B_{\rm eff}/\mu{\rm G})(E/{\rm GeV})^2$ MHz.

The radiation from a single electron is elliptically polarized with the electric vector maximum in the direction perpendicular to the projection of the magnetic field on the plane of the sky. Explicitly the total emissivity of a single electron via synchrotron radiation is the sum of parallel and perpendicular polarization

$$j(\nu) = \frac{\sqrt{3} e^3 B sin \alpha}{16\pi^2 \epsilon_0 c m_e} F(x) \tag{49}$$

where α is the electron direction pitch angle to the magnetic field B and $F(x) \equiv x \int_x^\infty K_{5/2}(\eta) d\eta$ is shown graphically in Figure **??**.

The quantity x is the dimensionless frequency defined as $x \equiv \omega/\omega_c = \nu/\nu_c$ where ω_c and ν_c are the critical synchrotron frequencies. An electron accelerated by a magnetic field B will radiate. For nonrelativistic electrons the radiation is simple and called cyclotron radiation and its emission frequency is simply the frequency of gyration of the electron in the magnetic field.

However, for extreme relativistic ($\gamma \gg 1$) electrons the frequency spectrum is much more complex and extends to many times the gyration frequency. This is given the name synchrotron radiation. The cyclotron (or gyration) frequency ω_B is

$$\omega_B = \frac{qB}{\gamma mc} \tag{50}$$

For the extreme relativistic case, aberration of angles cause the radiation from the electron to be bunched and appear as a narrow pulse confined to a time period much shorter than the gyration time. The net result is an emission spectrum characterized by a critical frequency

$$\omega_c \equiv \frac{3}{2}\gamma^2 \omega_B sin\alpha = \frac{3\gamma^2 qB}{2mc} sin\alpha \tag{51}$$

To understand the astrophysical radiation, one must consider that cosmic ray electrons are an ensemble of particles of different pitch angles α and energies E. It can generally be assumed that the directions are fairly isotropic so that integration over pitch angles is straightforward.

The next step is integration over electron energy spectrum to determine the total synchrotron radiation spectrum.

If the electrons' direction of motion is random with respect to the magnetic field, and the electrons' energy spectrum can be approximated as a power law: $dN/dE = N_0 E^{-p}$, then the luminosity is given by

$$I(\nu) = \frac{\sqrt{3}e^3}{8\pi mc^2} \left(\frac{3e}{4\pi m^3 c^5}\right)^{(p-1)/2} L N_0 B_{\rm eff}^{(p+1)/2} \nu^{-(p-1)/2} a(p), \tag{52}$$

where $a(p)$ is a weak function of the electron energy spectrum (see Longair, 1994, vol. 2, page 262 for a tabulation of $a(p)$), L is the length along the line of sight through the emitting volume, B is the magnetic field strength, and ν is the frequency.

At very low frequencies synchrotron self-absorption is very important as according to the principle of detailed balance, to every emission process there is a corresponding absorption process. At the lowest frequencies synchrotron self-absorption predicts an intensity that increases as $\propto \nu^{5/2}$.

The local energy spectrum of the electrons has been measured to be a power law to good approximation, for the energy intervals describing the peak of radio synchrotron emission (at GeV energies). The index of the power law appears to increase from about 2.7 to 3.3 over this energy range (Webber 1983, Nishimura et al 1991). Such an increase of the electron energy spectrum slope is expected, as the energy loss mechanisms for electrons increases with the square of the electron energy.

The synchrotron emission at frequency ν is dominated by cosmic ray electrons of energy $E \approx 3(\nu/\mathrm{GHz})^{1/2}$ GeV. The range of energies contributing to the radiation intensity at a given frequency depends on the electron energy spectrum: the steeper the electron distribution, the narrower the energy range (Longair 1994). For the case of most of the Galaxy, this range is of order 15 to 50. The observed steepening of the electrons' spectrum at GeV energies is used to model the radio emission spectrum at GHz frequencies (e.g. Banday & Wolfendale, 1990, Platania et al. 1998).

1.8 Free electron Lasers

The Free Electron laser (FEL) is a classical device that converts the kinetic energy of an electron beam into electromagnetic radiation by passing it through a transverse periodic magnetic field (called the "wiggler"). In contrast with conventional lasers, the radiation of the FEL is not constrained by the discrete energy levels that fix the wavelength of emission. The wavelength of FEL radiation depends mainly on the wavelength of the periodic magnetic field and the energy of the electron beam. High peak powers and its large range of operational wavelengths make it a laser of the future. A simple schematic representation of the FEL is given in the following figure.

A key feature is that the FEL is a true laser producing coherent radiation. Coherent radiation happens when the FEL is biased in the resonant condition. This leads to an effect where the electrons bunch more tightly so that they radiate as a single coherent bunch. For N electrons acting independently, the radiation is proportional to Ne^2. If the N electrons act coherently, as if a single particle, then the radiation is proportional to N^2e^2.

One could seed the laser with an electromagnetic wave for specific applications but to have a completely tunable laser, generally the FEL operates on the principle of a single-pass free electron laser operating the self-amplified spontaneous emission (SASE) mode. Electron motion through the undulator with alternating magnetic fields forces the electrons into a sinusoidal trajectory leading to electromagnetic radiation which recouples to the electron bunch causing laser action through SASE. The radiated power increases along the electron beam path leading to exponential increase in intensity. With high enough electron current and long enough undulator the power is saturated and energy oscillates between the electron and photon beam. If the resonant condition is met the energy exchange between the electron and photon beam leads to microbunching and coherent emission.

It should be noted that the FEL does not require any mirrors or resonating laser cavity structure. This is a great advantage at short wavelengths where, for example, mirrors and optics are technicaly difficult.

One can think about the FEL in steps: (1) What is the wavelength of light emitted by an electron traveling down the FEL magnet structure? Once can find this by using the synchrotron radiation formula or by transforming to the rest frame of the electron to find the frequency of oscillation by the magnets and then transforming the radiation to the lab by the Doppler formula. The approximate answer is

$$\lambda_\gamma = \frac{\lambda_{\text{magnetic structure}}}{2\gamma^2}$$

(2) What is the resonant condition? The undulator gives a resonance condition between the electron bunch and the electromagnetic wave, when one undulator period (travel length) λ_u gives a time difference between the electron bunch and electromagnetic wave corresponding to one period of the electromagnetic wave. In that situation the electrons are always going uphill against the electric field and thus adding power to the electromagnetic wave. That condition for very small transverse

movement of the electrons is that

$$\Delta t = \lambda_u/v - \lambda_u/c = \lambda_\gamma/c = \lambda_u/(2\gamma^2 c)$$

1.9 High Gain Free Electron Lasers

Motivation for high-gain FELs are as microwave sources for advanced accelerators and efficient sources of short wavelength radation. The basic physics is that a beam of electrons is injected along the axis of an undulator (a transverse, periodic (λ_0), magnetostatic field $B_o(z)$, N_o periods). The electrons are periodically deflected and as a result radiate synchrotron radiation. The primary features of synchrotron radiation are spontaneous emission which is incoherent: $I \sim N_e$, in a narrow cone: $\theta \sim 1/\gamma$, and narrow bandwidth:

$$\frac{dI}{d\omega d\Omega} \sim sinc^2\left(\pi N_0 \frac{\omega - \omega_s}{\omega_s}\right) \tag{53}$$

which peaks at $\omega = \omega_s = 2\pi/\lambda_s$ (and we will see that the resonant condition is at $\lambda_s = (1-\beta_{||})\lambda_o/\beta_{||}$.)

In the electron rest frame the wiggler field looks like N_0 period radiation field with wavelength

$$\lambda'_s = \lambda'_0 = \lambda_0/\gamma_{||}$$

where $\gamma_{||}^2 = 1/(1-\beta_{||}^2)$. Thus the electron oscillaes N_0 times. It produces a wavepacket of length $N_0\lambda'_0$ peaked at wavelength λ'_0. The spectrum of the radiation is the Fourier transform of a plane wave truncated after N_0 oscillations:

$$I(\omega) = sinc^2\left(\pi N_0 \frac{\Delta\omega}{\omega_s}\right) \tag{54}$$

In the laboratory frame, λ_s is the Doppler upshifted wavelength

$$\lambda_s = \frac{\lambda'_s}{2\gamma_{||}} \simeq \frac{\lambda_0}{2\gamma_{||}^2} \tag{55}$$

The exact solution is

$$\lambda_s = \frac{1-\beta_{||}}{\beta_{||}}\lambda_0$$

A free electron laser has tunability via change in electron energy or the undulator.

$$\frac{1}{\gamma^2} = 1 - \beta_{||}^2 - \beta_{\perp}^2 \simeq \frac{1}{\gamma_{||}^2} - \frac{a_0^2}{\gamma^2} \tag{56}$$

where a_0 is he dimensionaless vector potential of the undulator

$$a_0 = \frac{e\lambda_0 B_0}{2\pi m_0 c^2} \quad \text{helical}$$

$$
\begin{aligned}
&= 0.934 B_0 \lambda_0 \qquad \text{per Tesla cm} \\
&= \frac{e\lambda_0 B_0}{2\sqrt{2}\pi m_0 c^2} \qquad \text{planar} \\
&= 0.66 B_0 \lambda_0 \qquad \text{per Tesla cm}
\end{aligned} \tag{57}
$$

and

$$\gamma_{||}^2 = \frac{\gamma^2}{1 + a_0^2} \tag{58}$$

or

$$\lambda_s = \frac{\lambda_0}{2\gamma^2}\left(1 + a_0^2\right) \tag{59}$$

1.9.1 Stimulated Emission

Inject a laser beam with $\lambda \simeq \lambda_s$ along the axis of the undulator. The electrons move along curved path at $v_e{<}c$. Therefore $v_{||}{<}c$. Light moves down the4 axis at $v_z = c$. If an electron has the resonant energy $E_R = \gamma_R m_0 c^2$

$$\gamma_R^2 = \frac{\lambda_0}{2\lambda}\left(1 + a_0^2\right) \tag{60}$$

then the relative phase between transverse electron and radiation oscillations remains constant. Depending upon the phase the electron can give energy to the field and decellerate, $\dot{\gamma}{<}0$ (stimulated emission) or take energy from the field and accelerate, $\dot{\gamma}{>}0$.

An issue is that at the entrance of the undulator the electron phases are randomly distributed. For low gain, half of electrons will accelerate and half will decellerate. For low gain $< \gamma_0 >> \gamma_R$. This is what is observed for the first FEL, the Mdey laser in 1976 operated at 10.6 μm.

But if undulator is long enough and the current is high enough, then energy modulation will result in space modulation. There will be "self-bunching" and it will be around a "right" phase for gain. Most electrons will have the same phase and the intensity will be proportional to the number of electrons squared. $I \propto N_e^2$. This is collective instability of self-bunching and exponential gain.

1.9.2 Self-Consistent Theory

To fully describe FELs, we need a many particle, self-consistent theory that combines relativity for the electron mechanics and trajectories including the transverse current $J_\perp$, Maxwell's equations (or the special relativistic version), and an expression for the radiation field.

Wiggler Field

$$\vec{B}_0 = \vec{\nabla} \times \vec{A}_0$$

Radiation Field

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

Trajectory Equation

$$\frac{dp}{dt} = \frac{d}{dt}(\gamma m_0 v) = e\left(\vec{E} + \frac{1}{c}\vec{v} \times (\vec{B}_0 + \vec{B})\right)$$

Energy Equation

$$\frac{dE}{dt} == \frac{d}{dt}(\gamma m_0 c^2) = e\vec{E} \cdot \vec{v} = eEv_\perp$$

The total field on electrons from the vector potential $\vec{A}_{tot}$

$$\vec{A}_{tot} = \vec{A}_0 + \vec{A}$$

which is the total from the wiggler and radiation. $\vec{A}_0$ is periodic (spatially) either planar or helical

$$\vec{A}_0 = \frac{1}{\sqrt{2}}(\hat{e}e^{-ik_0 z} + c.c.)$$

for the helical field which leads to circularly polarized radiation:

$$\vec{A} = -\frac{i}{\sqrt{2}}\left[A\hat{e}e^{I(k_\parallel - \omega t)} - c.c.\right]$$

where $\omega = ck = c\sqrt{k_{||} + k_\perp^2}$ where $k_\perp$ allows for waveguides. Let $k_\perp = 0$. Then

$$\begin{aligned} \frac{d}{dt}(\gamma m_0 v_\perp) &= e\left[E + \frac{1}{c}(v \times B)_\perp\right] \\ &= -\frac{e}{c}\left[\frac{\partial A_{tot}}{\partial t} - (v \times \nabla \times A_{tot})_\perp\right] \\ &= -\frac{e}{c}\frac{d}{dt}A_{tot} \end{aligned} \tag{61}$$

$$\frac{d}{dt}(\gamma\beta_\perp) = -\frac{e}{mc^2}\frac{dA_{tot}}{dt}dt a_{tot} \tag{62}$$

For perfect on-axis injection $\beta_\perp(0) = 0$ and

$$\beta_\perp = -\frac{a_{tot}}{\gamma} \simeq -\frac{a_0}{\gamma}$$