Physics 139 Relativity

Relativity Notes 1999

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Notes to be found at
http://aether.lbl.gov/www/classes/homework/homework.html

4 Relativistic Dynamics of Particles

We will consider two approaches to reconciling Netonian mechanics with relativity:

(1) The phenomenological approach based upon experimental results (e.g. length contraction, time dilation, clock synchronization) and generalize Netonian mechanics to be consistent. We start with Newton's Principles (postulated three laws of mechanics) and generalize them to include Special Relativity.

(2) The second approach is to take literaly Minkowski 3+1-D space and use four-dimensional vectors and 4-D vector algebra in the way we are used to doing 3-D vector algebra. In the previous sections we have seen how velocity and force (and other important quantities) do not transform as 3-dimensional vectors in relativity. We can, however, generalize them to 4-D vectors successfully.

First let us consider the phenomenological generalization of Newtonian mechanics based upon experimental observations. This will provide a comparison and motivation for the 4-vector approach.

4.1 Generalize Newton's Laws

4.1.1 Newton's Laws

First and Second Laws are a definition of force and implicitly the law of conservation of mass. The Third law is the law of conservation of momentum.

1. Results of the Lorentz Transfomation
2. Generalized Conservation of Mass

   $\sum_i m_i =$ constant, not each m_i separately constant.
3. Conservation of Momentum

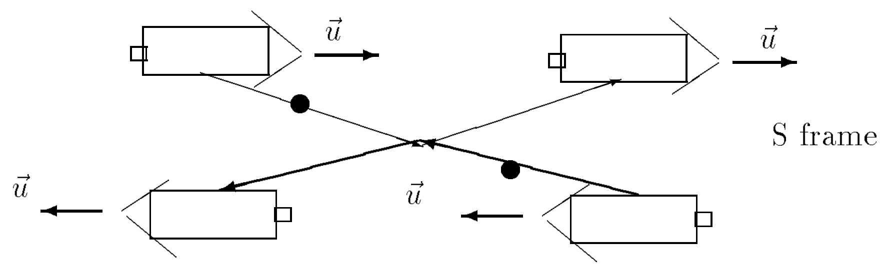
$$\sum m_i u_{xi} = \text{constant}$$
$$\sum m_i u_{yi} = \text{constant}$$
$$\sum m_i u_{zi} = \text{constant}$$

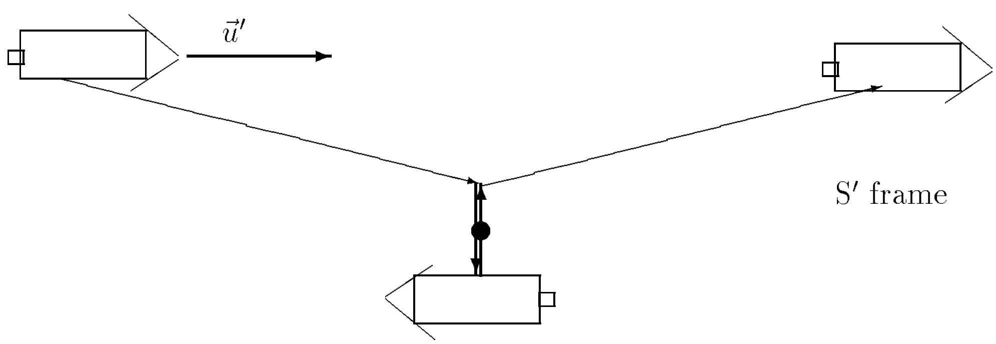
These postulates are as close as possible to those of Newton, but they produce different results.

4.2 The Mass of a Moving Particle

4.2.1 Space Billiards

Consider two identical space ships aligned to pass near each other with speed u in opposite directions at a distance d apart in frame S. At identical times ($t_i = -x_o/u = -d/(2u_y)$) in frame S each emits an identical steel ball with exactly opposite directions (in a direction perpendicular to their direction of motion in their rest frames) and with exactly equal transverse velocities u_y. The timing and positions are such that the two steel balls collide directly and elastically in the center (origin of frame S) between the ships and each ball rebounds to go back and be recaptured by the ship that emitted it at time $t_r = x_o/u = d/(2u_y)$. The location of the upper and lower ships is symmetrical by construction as is the collision.





Now consider the events in the frame S′ from the lower (in the picture) rest frame. That is, the inertial frame where the lower ship is at rest. By using the Lorentz transform it is easy to see that the events which were simultaneously in the observer O's frame S are no longer simultaneous in the frame S′. In fact in frame S′ the upper ship will emit its ball first, then the lower ship will emit its. The lower ship receives its ball back next and then the upper ship receives its ball back.

$$t' = \gamma\left(t + \frac{v}{c^2}x\right)$$

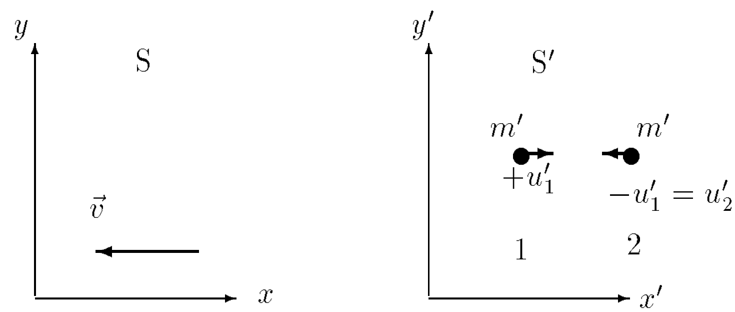
Spaceship	*event*	*Coordinates in S*	t' *in* S'	Δt
Upper	*emit*	$-x_o/u, -x_0$	$\gamma(-x_o/u - ux_o/c^2)$	
Lower	*emit*	$-x_o/u, +x_0$	$\gamma(-x_o/u + ux_o/c^2)$	$2\gamma v x_o/c^2$
Lower	*recapture*	$x_o/u, -x_0$	$\gamma(x_o/u - ux_o/c^2)$	
Upper	*recapture*	$x_o/u, +x_0$	$\gamma(x_o/u + ux_o/c^2)$	$2\gamma v x_o/c^2$

The balls still meet in the center by symmetry and because the Lorentz transformations leave transverse directions unchanged. The upper ball has a lower velocity and change in velocity upon scattering than the lower ball. If we want the y component of momentum conserved, $m_1 \Delta u_{y1} + m_2 \Delta u_{y2} = 0$ and since $\Delta u_{y1} < \Delta u_{y2}$, then the masses are not equal and $m_1 > m_2$. The more rapidly moving mass is greater, even though in frame S the two balls were identical and had identical mass by construction and symmetry.

This is exactly the case of transverse collisions discussed below.

4.2.2 Longitutional Collision

Consider the collision of two identical particles moving at each other with identical but oppositely directed velocities. These particles are aligned so that they will undergo an elastic collision and rebound in the opposite direction.



In S′ symmetry says that an elastic collision leads to a reversal of velocities of the colliding masses.

In S:

$$u_1 = \frac{u' + v}{1 + u'v/c^2}; \qquad u_2 = \frac{-u' + v}{1 - u'v/c^2}.$$

From the conservation of mass:

$$m_1 + m_2 = M = \text{constant}$$

From conservation of momentum

$$m_1 u_1 + m_2 u_2 = Mv = \text{constant}$$

By considering the motion at the instant of relative rest.

In the following algebra we show that $m = \gamma m_o$

$$m_1 \frac{u' + v}{1 + u'v/c^2} + m_2 \frac{-u' + v}{1 - u'v/c^2} = m_1 v + m_2 v$$

Subtracting the right hand side from both sides of the equation one has

$$m_1 \frac{u' + v - v - u'v^2/c^2}{1 + u'v/c^2} + m_2 \frac{-u' + v - v + u'v^2/c^2}{1 - u'v/c^2} = 0$$

$$m_1 \frac{u' - u'v^2/c^2}{1 + u'v/c^2} + m_2 \frac{-u' + u'v^2/c^2}{1 - u'v/c^2} = 0$$

$$\frac{m_1}{1 + u'v/c^2} = \frac{m_2}{1 - u'v/c^2}$$

$$\frac{m_1}{m_2} = \frac{1 + u'v/c^2}{1 - u'v/c^2}$$

The law of transformation of $\sqrt{1 - u^2/c^2}$ is

$$\sqrt{1 - u^2/c^2} = \frac{\sqrt{1 - (u')^2/c^2}\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2}$$

Multiplying the mass ratio equation by $\sqrt{1 - (u')^2/c^2}\sqrt{1 - v^2/c^2}$ divided by itself give

$$\frac{m_1}{m_2} = \frac{1 + u'v/c^2 \sqrt{1 - (u')^2/c^2}\sqrt{1 - v^2/c^2}}{1 - u'v/c^2 \sqrt{1 - (u')^2/c^2}\sqrt{1 - v^2/c^2}}$$

One can then group the first term with its reciprocal to find

$$\frac{m_1}{m_2} = \frac{\sqrt{1 - (u_2)^2/c^2}}{\sqrt{1 - (u_1)^2/c^2}}$$

Noting that $+u'_x = +u'$ goes with u_1 and $-u'_x = -u'$ goes with u_2. Thus

$$m_1 \sqrt{1 - (u_1)^2/c^2} = m_2 \sqrt{1 - (u_2)^2/c^2} = m_o$$

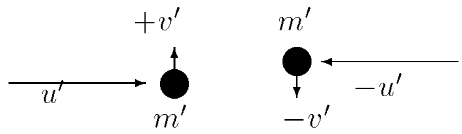
implying

$$m = \frac{m_o}{\sqrt{1 - u^2/c^2}} = \gamma m_o \tag{1}$$

4.2.3 Transverse Collision

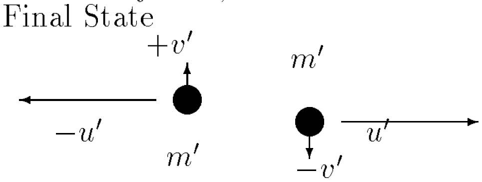
Consider a symmetric transverse collision

Inital State



This is a glancing collision with no change in the momenta in the x direction. In the S$'$ system, one sees a reversal of v''s on collision.

Final State



In S system:

$$v_1' = v'\frac{\sqrt{1 - v^2/c^2}}{1 + u'v/c^2}, \qquad v_2' = -v'\frac{\sqrt{1 - v^2/c^2}}{1 - u'v/c^2}$$

The change in velocity of mass 1 is

$$\Delta v_1 = v_{1f} - v_{1i} = -2v'\frac{\sqrt{1 - v^2/c^2}}{1 + u'v/c^2}.$$

The change in velocity of mass 2 is

$$\Delta v_2 = v_{2f} - v_{2i} = +2v'\frac{\sqrt{1 - v^2/c^2}}{1 - u'v/c^2}.$$

Note that $\Delta v_1 \neq \Delta v_2$, so to conserver momentum p_y total, $m_1 \neq m_2$.

$$-2m_1v'\frac{\sqrt{1 - v^2/c^2}}{1 + u'v/c^2} + 2m_2v'\frac{\sqrt{1 - v^2/c^2}}{1 - u'v/c^2} = 0$$

$$\frac{m_1}{m_2} = \frac{1 + u'v/c^2}{1 - u'v/c^2} = \frac{\sqrt{1 - u_2^2/c^2}}{\sqrt{1 - u_1^2/c^2}}$$

So that by factoring and substituting one has

$$m_1\sqrt{1 - u_1^2/c^2} = m_2\sqrt{1 - u_2^2/c^2}$$

So for this to always be true

$$m = \frac{m_o}{\sqrt{1-u^2/c^2}} = \gamma m_o$$

as derived for longitudinal case. We can generalize for any elastic collision with $\vec{p}_{total} = 0$ in S$'$ to get the same result.

4.2.4 "Internal" Mass

Consider an undeformed (unstrained) particle able to move freely. Let its mass be denoted by m_o when it is at rest.

In frame S$'$ during a symmetric longitudinal collission

$$M = \frac{m_o}{\sqrt{1-u_1^2/c^2}} + \frac{m_o}{\sqrt{1-u_2^2/c^2}}$$

$$= m_o\frac{1-u'v/c^2}{\sqrt{1-v^2/c^2}\sqrt{1-(u')^2/c^2}} + m_o\frac{1+u'v/c^2}{\sqrt{1-v^2/c^2}\sqrt{1-(u')^2/c^2}}$$

$$M = \frac{2m_o}{\sqrt{1-(u')^2/c^2}\sqrt{1-v^2/c^2}} = \gamma_{u'}\gamma_v\ 2m_o$$

At the instant of greatest deformation one might think $M = 2m_o/\sqrt{1-v^2/c^2}$ because at that instant each particle has velocity $\vec{v}$ in S; but that is incorrect! $M > 2m_o/\sqrt{1-v^2/c^2}$ because energy is added in deforming the particles.

4.2.5 Expressions for Force

We must choose a definition for force.

$$\vec{F} = \frac{d}{dt}(m\vec{u})$$

is more useful than $\vec{F} = m\frac{d\vec{u}}{dt}$, because then conservation of momentum implies that **action = reaction**.

$$m = \frac{m_o}{\sqrt{1-v^2/c^2}}; \qquad \frac{dm}{dt} = m_o\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right) = \frac{m_o}{(1-v^2/c^2)^{3/2}}\frac{u}{c^2}\frac{du}{dt}$$

So

$$\vec{F} = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{d\vec{u}}{dt} + \frac{m_o}{\left(\sqrt{1-u^2/c^2}\right)^3}\vec{u}\frac{u}{c^2}\frac{du}{dt}$$

So <u>force</u> is not parallel to <u>accelleration</u>!

We take components in S and S′:

$$F_x = m\dot{u}_x + \dot{m}u_x; \qquad F'_x = m'\dot{u}'_x + \dot{m}'u'_x$$

$$F_y = m\dot{u}_y + \dot{m}u_y; \qquad F'_y = m'\dot{u}'_y + \dot{m}'u'_y$$

$$F_z = m\dot{u}_z + \dot{m}u_z; \qquad F'_z = m'\dot{u}'_z + \dot{m}'u'_z$$

where $\dot{m}' = dm'/dt'$ and $\dot{u}'_x = du'_x/dt'$.

4.2.6 Transformation Equations for Force

The law of transformation of force is complicated by the fact that F_x, F_y, and F_z are **not** three of the four components of a relativistic four vector. Here are the relationships but we defer deriving them (see e.g. Rindler p. 91)

$$F'_x = \frac{F_x - \vec{F}\cdot\vec{u}v/c^2}{1 - u_x v/c^2} \tag{2}$$

$$F'_y = \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} F_y \tag{3}$$

$$F'_z = \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} F_z \tag{4}$$

4.2.7 Transformation Equations for Mass

$$m_o = m\sqrt{1 - u^2/c^2} = m'\sqrt{1 - (u')^2/c^2}$$

$$\sqrt{1 - u^2/c^2} = \frac{\sqrt{1 - (u')^2/c^2}\sqrt{1 - v^2/c^2}}{1 + u_x v/c^2}$$

So

$$m = m'\frac{1 + u_x v/c^2}{\sqrt{1 - v^2/c^2}}$$

Take the derivative with respect to time (d/dt)

$$\frac{dm}{dt} = \frac{1 + u_x v/c^2}{\sqrt{1 - v^2/c^2}}\frac{dm'}{dt'}\frac{dt'}{dt} + m'\frac{v/c^2}{\sqrt{1 - v^2/c^2}}\frac{du'_x}{dt'}\frac{dt'}{dt}$$

The inverse Lorentz transformation is

$$t = \frac{t' + x'v/c^2}{\sqrt{1 - v^2/c^2}}$$

so that

$$\frac{dt}{dt'} = \frac{1 + u'_x/c^2}{\sqrt{1 - v^2/c^2}}$$

Substituting for dt'/dt one finds

$$\frac{dm}{dt} = \frac{dm'}{dt'} + \frac{m'v}{c^2}\frac{1}{1 + u_x v/c^2}\frac{du'_x}{dt'} \tag{5}$$

4.3 Work and Kinetic Energy

We define differential work

$$dW \equiv \vec{F} \cdot \vec{dr}$$

and also define the differential kinetic energy

$$dE_{\text{kinetic}} \equiv dW = m\frac{d\vec{u}}{dt} \cdot \vec{dr} + \frac{dm}{dt}\vec{u} \cdot \vec{dr}$$

Now

$$\frac{d\vec{u}}{dt} \cdot \vec{dr} = d\vec{u} \cdot \frac{d\vec{r}}{dt} = d\vec{u} \cdot \vec{u} = \vec{u} \cdot d\vec{u}$$

$$\frac{1}{2}d\left(\vec{u} \cdot \vec{u}\right) = \vec{u} \cdot d\vec{u} = \frac{1}{2}d\left(u^2\right) = udu$$

So

$$dE_{\text{kinetic}} = mudu + u^2 dm$$

$$= \frac{m_o udu}{\sqrt{1 - u^2/c^2}} + \frac{m_o u^3/c^2 du}{\left(\sqrt{1 - u^2/c^2}\right)^3} = \frac{m_o udu}{\left(1 - u^2/c^2\right)^{3/2}}$$

$$E_{\text{kinetic}} = \int_{u=0}^{u} \frac{m_o udu}{\left(1 - u^2/c^2\right)^{3/2}} = \frac{m_o c^2}{\sqrt{1 - u^2/c^2}}\Big|_{u=0}^{u}$$

$$E_{\text{kinetic}} = \frac{m_o c^2}{\sqrt{1 - u^2/c^2}} - m_o c^2 \tag{6}$$

Notice that E_{kinetic} depends only on the final velocity squared and not on the way in which it is attained.

Of course for $v/c \ll 1$,

$$E_{\text{kinetic}} \rightarrow m_o c^2 \left[1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots - 1\right] = \frac{1}{2}mv^2\left[1 + O(\frac{v^2}{c^2})\right]$$

In an elastic collision

$$\Delta E_{\text{kinetic}}(1) = \Delta E_{\text{kinetic}}(2)$$

from the equality of action and reaction.

4.4 Relations Among Mass, Energy, and Momentum

4.4.1 Conservation of Energy

From the relation $dE_{\text{kinetic}} = m_o udu/\left(1-u^2/c^2\right)^{3/2}$, $dE = c^2dm$ and thus

$$\Delta E = c^2\Delta m$$

So conservation of mass implies conservation of energy. Association of mass with energy from $\Delta E = c^2\Delta m$, suppose

$$m = \frac{E}{c^2} \quad \text{and further} \quad E = mc^2 = \gamma m_o c^2$$

$$\vec{p} = \gamma m_o \vec{v}$$

4.4.2 Conservation of Momentum

With $dE = c^2dm$, we have $\Delta(\text{K.E.}) = \text{c}^2\left(\text{m}-\text{m}_\text{o}\right)$. So we postulate $E = mc^2$ and $m = E/c^2$ for any kind of E and m.

If energy E is being bodily transferred (transported) with velocity $\vec{u}$, the associated momentum is

$$\vec{G} = m\vec{u} = \frac{E}{c^2}\vec{u}.$$

But there are other ways to transfer energy, e.g. pushing at A and get out a B. so we have $\vec{g}$ defined as the density of momentum and $\vec{S}$ defined as energy flow, with $\vec{g} = \vec{S}/c^2$ for any mechanism of energy transfer.

Dimesions: $[G] = [Eu/c^2] = m/T$. $[g] = \text{m}/(\text{L}^2\text{T}; [\text{S}] = \text{m}/\text{T}^3] =$ Energy/(area * time).

5 Applications and Experimental Tests of Particle Dynamics

5.1 Mass of High Velocity Electrons

Bucher using β-rays and Hadka using cathode rays confirmed that $m = m_o/\sqrt{1-\beta^2}$ (where $\beta \equiv v/c$). In 1926 Gerlach put in the Handbuch der Physik

$$Ve = m_o c^2\left[\frac{1}{\sqrt{1-v^2/c^2}} - 1\right] = m_o c^2\left(\gamma - 1\right) \tag{7}$$

where V is the acceleration voltage.

All measurements really determine the ratio m/e; to get the relativistic result, they must assume e = constant. This results form requiring Maxwell's equations to be invariant under Lorentz transformation – Charge Conservation. We will revisit this issue later.

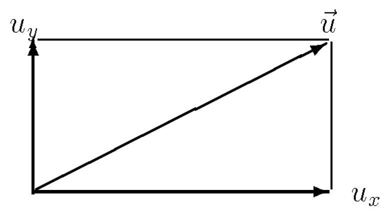
5.2 Relation Between Force and Acceleration

By our convention the vector force $\vec{F}$ is given by

$$\vec{F} = \frac{d}{dt}(m\vec{u}).$$

The force can be resolved into components parallel to $\vec{u}$ and $\vec{a} = d\vec{u}/dt$;

$$\vec{F} = m\frac{d\vec{u}}{dt} + \frac{dm}{dt}\vec{u} = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{d\vec{u}}{dt} + \frac{d}{dt}\left[\frac{m_o}{\sqrt{1-u^2/c^2}}\right]\vec{u}$$



$$\vec{u} = u_x\hat{i} + u_y\hat{j}; \qquad u^2 = u_x^2 + u_y^2$$

$$F_x = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{du_x}{dt} + \frac{d}{dt}\left[\frac{m_o}{\sqrt{1-u^2/c^2}}\right]u_x$$

$$F_y = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{du_y}{dt} + \frac{d}{dt}\left[\frac{m_o}{\sqrt{1-u^2/c^2}}\right]u_y$$

Consider an acceleration in the y direction:

$$\frac{du_x}{dt} = 0.$$

Then

$$\frac{du^2}{dt} = 2u_y\frac{du_y}{dt}$$

So that

$$F_x = \frac{m_o}{(1-u^2/c^2)^{3/2}}\frac{u_x u_y}{c^2}\frac{du_y}{dt}$$

$$F_y = \frac{m_o}{\sqrt{1-u^2/c^2}}\frac{du_y}{dt} + \frac{m_o}{(1-u^2/c^2)^{3/2}}\frac{u_y^2}{c^2}\frac{du_y}{dt}$$

$$= \frac{m_o}{(1-u^2/c^2)^{3/2}}\left[1 - u^2/c^2 + u_y^2/c^2\right]\frac{du_y}{dt}$$

$$= \frac{m_o}{(1-u^2/c^2)^{3/2}} \left[1 - u_x^2/c^2\right] \frac{du_y}{dt}$$

So the ratio of Fx/Fy is

$$\frac{F_x}{F_y} = \frac{u_x u_y}{c^2 - u_x^2}$$

We conclude that to accelerate in the y direction one must apply both F_y and $F_x = \frac{u_x u_y}{c^2 - u_x^2} F_y$. To accelerate in the x direction one must apply F_x and $F_y = \frac{u_x u_y}{c^2 - u_y^2} F_y$.

Under what conditions would the force $\vec{F}$ and acceleration $\vec{a}$ be in the same direction? If F_x and u_y are both zero, then F_y must be

$$F_y = \frac{m_o}{(1-u^2/c^2)^{1/2}} \frac{du_y}{dt}$$

.

Thus force and acceleration are parallel ($\vec{F} \parallel \vec{a}$) in y direction when perpendicular to the motion ($\perp \vec{u}$) and thus the velocity and acceleration are perpendicular ($\vec{a} \perp \vec{u}$). Then the "transverse mass" is $m_o/\sqrt{1-v^2/c^2}$.

If F_x and u_x are both zero:

$$F_y = \frac{m_o}{(1-u^2/c^2)^{1/2}} \frac{du_y}{dt} + = \frac{m_o}{(1-u^2/c^2)^{3/2}} \frac{u_y^2}{c^2} \frac{du_y}{dt}$$

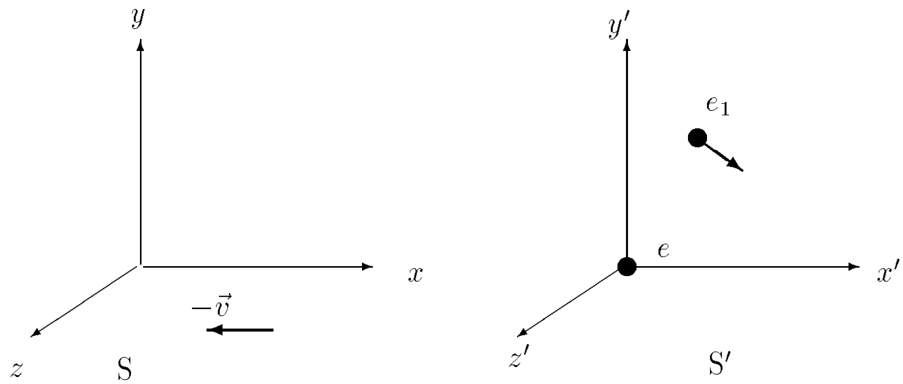
$$= \frac{m_o}{(1-u^2/c^2)^{3/2}} \frac{du_y}{dt}$$

Thus $\vec{F} \parallel \vec{a}$ in y direciton $\parallel \vec{u}$ ($\vec{a} \parallel \vec{u}$) is longitudional acceleration. Then the "longitudional mass" is $m_o/(1-u^2/c^2)^{3/2}$ and we conclude $m_{\text{transverse}} < m_{\text{longitudional}}$. The "longitudional mass" is bigger because the force must also do work to raise the energy. (A transverse force does no work.)

Do not be confused. The "true" mass is always $m_o/\sqrt{1-u^2/c^2}$, which is conserved in a closed system.

5.3 Force Exerted by a Moving Charge e on a charge e_1

Pick a coordinate system in which one charge is fixed. E. g. e fixed in S′ with e at the origin of S and e_1 located at x, y, z at time considered.



In frame S, $u_{e_x} = v$, $u_{e_y} = u_{e_z} = 0$.
The force on e_1 in S′ is electrostatic.

$$F'_x = \frac{ee_1}{(x'^2 + y'^2 + z'^2)^{3/2}} x'$$

$$F'_y = \frac{ee_1}{(x'^2 + y'^2 + z'^2)^{3/2}} y'$$

$$F'_z = \frac{ee_1}{(x'^2 + y'^2 + z'^2)^{3/2}} z'$$

Now transform to frame S in which e moves.

$$F_x = \frac{ee_1}{s^3} \left(1 - v^2/c^2\right) \{x + \frac{v}{c^2} (y u_y + z u_z)\}$$

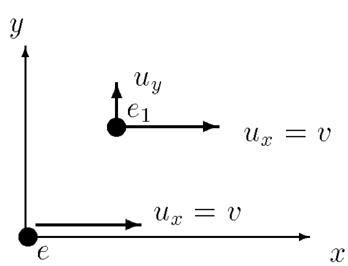
$$F_y = \frac{ee_1}{s^3} \left(1 - v^2/c^2\right) \left(1 - \frac{u_x v}{c^2}\right) y$$

$$F_z = \frac{ee_1}{s^3} \left(1 - v^2/c^2\right) \left(1 - \frac{u_x v}{c^2}\right) z$$

where $s^2 = x^2 + (1 - v^2/c^2)(y^2 + z^2)$.

This result is restricted to constant velocity v of e in x direction. If not constant, one must use the retarded potential.

Special Example: e fixed at origin, e_1 constrained to move in y direction.



The force on e_1 is:

$$F_x = \frac{ee_1}{s^3}\left(1 - v^2/c^2\right) y u_y$$

$$F_y = \frac{ee_1}{s^3}\left(1 - v^2/c^2\right)\left(1 - \frac{u_x v}{c^2}\right) y$$

$$\frac{F_x}{F_y} = \frac{u_x u_y}{c^2 - u_x^2},$$

which is just right to produce acceleration in the y direction only!

5.4 Nuclear Reactions

These were the first accurate tests of relations between mass and energy, also momentum. The experiments of Oliphant, Kinsey, and Rutherford (1933) and Bainbridge (1933) are cited.

$$Li^7 + H^1 \rightarrow Be^8 \rightarrow He^4 + He^4 + 17 \text{ MeV}$$

$$Li^6 + H^2 \rightarrow Be^8 \rightarrow He^4 + He^4 + 17 \text{ MeV}$$

Solar Energy: Around this time Hans Bethe realized that the primary source of solar energy is the reaction

$$4H^1 \rightarrow He^4 + 30 \text{ MeV}$$

Uranium Fission: Han and Sparissman found

$$U^{235} + n^1 \rightarrow F_1 + F_2 + \text{a few neutrons} + 200 \text{ MeV}$$

where F_1 and F_2 are the fission fragments which are randomly distributed but with average atomic number around 90 and 140 respectively.

"Monochromatic" Neutrons

$$Li^7 + H^1 \rightarrow Be^7 + n - 1.6 \text{ MeV}$$

is a reaction which absorbs energy.

5.5 Light Pressure

Let $\mathcal{E} \equiv$ energy density. Then $\mathcal{E}/c^2$ is the equivalent mass density of that energy. I.e. energy density has an effective inertia density. The momenutm density is then $\mathcal{E}c/c^2 = \mathcal{E}/c \equiv \pi$ and $\pi c = p =$ the change of momentum per unit area pe second if light is absorbed. Since $\pi c = \mathcal{E}$, then $p = \mathcal{E}$ in absorption.

For relfection $p = 2\mathcal{E}$.

In a cavity (as in hohlraum) $p = \mathcal{E}/3$.

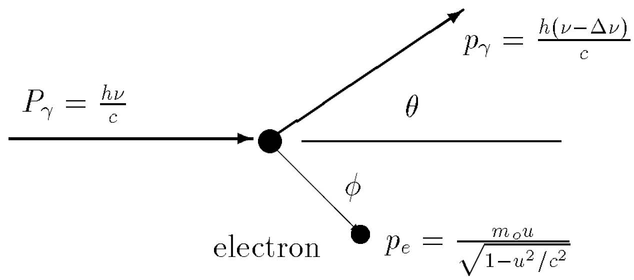
5.5.1 Mass, Energy, Momentum for Photons

$$E = h\nu; \qquad m = h\nu/c^2$$

$$G = h\nu/c; \qquad \vec{G} = \frac{h\nu}{c^2}\vec{c}$$

5.5.2 Compton Effect

Arthur Compton provided the relativistic treatment of a photon scattering on a free electron.



Conservation of energy yields:

$$h\Delta\nu = m_o c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]$$

Conservation of momentum in x-direction yields the equation:

$$\frac{h\nu}{c} = \frac{h(\nu - \Delta\nu}{c} cos\theta + \frac{m_o u}{\sqrt{1 - v^2/c^2}} cos\phi$$

Conservation of momentum in y-direction yields the equation:

$$\frac{h\nu + \Delta\nu}{c} sin\theta = \frac{m_o u}{\sqrt{1 - v^2/c^2}} sin\phi$$

The solution to these equations is:

$$\Delta\lambda = \frac{2h}{m_o c} sin^2 \left(\frac{\theta}{2}\right) \tag{8}$$

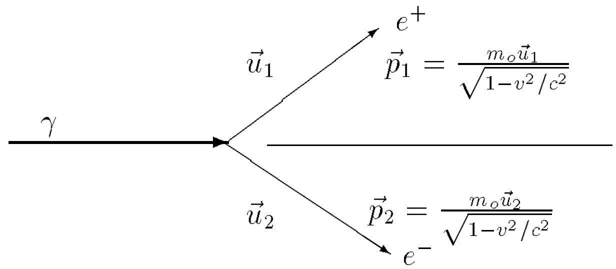
The importance to Quantum Mechanics was not only the direct evidence of photon scattering but the Compton effect is used to explain the "γ-ray microscope" interpretation of the Uncertainty Principle. The uncertainity in coordinate above is

$$\Delta x_{\min} \sim \frac{h}{mc}$$

To make it go to zero requires an infinite mass m and photon energy.

5.5.3 Pair Production

An energetic photon may have sufficient energy to convert into an electon-positron (e^-e^+) pair.



This process cannot occur in vacuum because it cannot conserve both energy and momentum. It must occur in the vicinity of a mass which can these by absorbing some of the E and $\vec{p}$.

5.5.4 Positron Annihilation

The reaction $e^+ + e^- \rightarrow \gamma$, that is, annihilation of a positron with an electron to a single energetic photon does not occur. However, the reaction $e^+ + e^- \rightarrow \gamma + \gamma$ is commonly observed.

5.6 Some Practical Examples of the Use of Invariants

In this section we show some practical examples of how to use invariants as derived either laboriously or through the use of 4-vectors of Minkowski space. This will motivate the next section in which we will learn the properties of vectors and tensors in 3+1-D space.

5.6.1 Mass, $\vec{\beta}$, γ

First consider the relation between mass, energy, and three momentum or equivalently the norm of the 4-momentum

$$\left(m_o c^2\right)^2 = E^2 - p^2 c^2 \tag{9}$$

Conservation of energy and momentum (4-momentum) for a collection of particles in a given frame leads to the invariant for the total mass M.

$$\left(Mc^2\right)^2 = \left(\sum_i E_i\right)^2 - \left(\sum_i p_i c\right)^2 \tag{10}$$

where M is the center-of-mass equivalent mass.

$$\vec{\beta} = \frac{\vec{p}}{E} \qquad \vec{\beta}_{CM} = \frac{\vec{p}_{CM}}{E_{CM}} \tag{11}$$

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}} = \frac{E}{m_o c^2} \qquad \gamma_{CM} = \frac{E_{CM}}{M_{CM}c^2} \tag{12}$$

5.6.2 Energy, momentum, and velocity of one particle in rest frame of another

The energy momentum, and velocity of one particle in rest frame of another can be calculated readily by making use of the concept of invariants. If one is given the 4-momenta of two particles in any frame the energy of particle two from the rest frame of particle one E_{21} can be found by a simple relationship. Let $\tilde{p}_1$ and $\tilde{p}_2$ are the momentum four vectors of particles one and two. The dot or inner product between the two 4-vectors is defined as

$$\tilde{p}_1 \cdot \tilde{p}_2 = E_1 E_2/c^2 - \vec{p}_1 \cdot \vec{p}_1$$

where $\vec{p}_1$ and $\vec{p}_1$ are the relativistic 3-momenta of particles one and two respectively. We can derive the relationship using the principle that this dot product should be invariant to frame of reference. Thus

$$\tilde{p}_1 \cdot \tilde{p}_2 = \tilde{p}_1' \cdot \tilde{p}_2'$$

Take the system S$'$ to be the frame in which particle one is at rest. Then $\tilde{p}_1' = (m_1 c, 0, 0, 0)$, where m_1 is the rest mass of particle one and thus

$$\tilde{p}_1 \cdot \tilde{p}_2 = \tilde{p}_1' \cdot \tilde{p}_2' = m_1 E_{21}$$

Dividing through by m_1 gives the relationship

$$E_{21} = \frac{\tilde{p}_1 \cdot \tilde{p}_2}{m_1} \tag{13}$$

Once we have the energy E_{21} of particle two in the rest frame of particle one, it is easy to find its three-momentum amplitude making use of the relationship between mass, energy, and three-momentum amplitude.

$$E_{21}^2 = |p_{21}c|^2 + (m_2c^2)^2 \quad \rightarrow \quad |p_{21}c|^2 = E_{21}^2 - (m_2c^2)^2$$

where m_2 is the rest mass of particle two.

$$|p_{12}|^2 = \frac{(\tilde{p}_1 \cdot \tilde{p}_2)^2}{(m_1 c)^2} - \frac{(m_2 c)^2}{c^2} = \frac{(\tilde{p}_1 \cdot \tilde{p}_2)^2 - m_1^2 m_2^2 c^4}{m_1^2 c^2} \tag{14}$$

And for the relative velocity we have

$$\beta_{21}^2 = \frac{|p_{21}|^2}{E_{21}^2} = \frac{(\tilde{p}_1 \cdot \tilde{p}_2)^2 - m_1^2 m_2^2 c^4}{(\tilde{p}_1 \cdot \tilde{p}_2)^2} = 1 - \frac{m_1^2 m_2^2 c^4}{(\tilde{p}_1 \cdot \tilde{p}_2)^2} = 1 - \left(\frac{m_2 c^2}{E_{21}}\right)^2 \tag{15}$$

5.6.3 Energy, momentum, and velocity of a particle in the center of momentum frame

Use the same formulae as above but for particle one use the center-of-momentum particle (fictious) as particle one.

POINT OF VIEW

In this chapter we consider relativisitic effects from different points of view. In essentially all the cases we have done before, we have assumed that we had a complete reference frame of meter sticks and clocks so that we could determine lengths and times at any place in space-time. This I refer to as the physicist's god-like view provided by his reference frame and ancillary tools. This concept of reference frames comes to us from Galileo and Newton.

Most mere mortals, such as astronomers and individuals, have more limited access to data about remote objects. In general, especially for astronomy, the observer either sits at a point in space-time and images light coming to his instrument – eye, telescope, camera, etc. – or sits at a point in space and observes the light arriving as a function of time.

The result of being limited to a single point of view, instead of the physicist's god-like plan view is to observe very different relativistic behavior than we have considered so far. One can observe cases of a moving clock running faster. Radio astronomers observe many objects moving superluminally (that is with velocities faster than light), and fast moving objects appear very differently than a resting object at the same place. Sometimes one can not see the front of an approaching object but can see the back.

We consider some of these effects in the following sections.

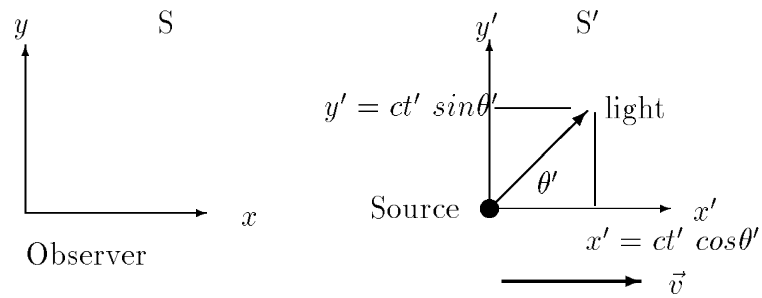
6 The Relativisitic Doppler Effect

From the point of view of a single observer confine to a location in space, a moving clock can run either faster or slower than an identical clock at rest with respect to the observer depending upon its velocity (direction and speed of motion). We consider the case of a clock that is a light source with a particular frequency and work out the relativistic Doppler shift. The frequency can be considered the beats of the clock.

We work the problem out by considering two different inertial frames and use the Lorentz transformations in order to determine what a single-place observer would see.

6.1 Ray Optics Approach

First, go to the frame S$'$ where the source is at rest and emits light at frequency $\nu' = \nu_o$. Now consider a pulse light going in the direction θ' relative to the x'-axis.



Now consider the frame S, where the source is moving in the x direction with velocity (speed) v, and consider the path of the light in this frame. We can use the Lorentz tranformations to calculate the location of a light pulse emitted at time $t' = 0$ and trace its path as a light ray.

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} = \frac{ct'cos\theta' + vt'}{\sqrt{1 - v^2/c^2}} = \frac{ct'\left(cos\theta' + v/c\right)}{\sqrt{1 - v^2/c^2}}$$

$$y = y' = ct'sin\theta'$$

By taking the ratio of y over x when can find $tan\theta$

$$tan\theta = \frac{y}{x} = \frac{sin\theta'}{cos\theta' + v/c}\sqrt{1 - v^2/c^2} = \frac{1}{\gamma}\frac{sin\theta'}{cos\theta' + v/c} \tag{16}$$

This is the full relativisitic aberration of light formula derived by ray optics argument. This is the same result as found using the Lorentz contraction and ether approach.

Now using the Lorentz transform for t and then t' we can derive a formula for the relative rate at which clocks appear to run.

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - v^2/c^2}} = \frac{t' + \frac{v}{c}t'cos\theta'}{\sqrt{1 - v^2/c^2}} = t'\frac{\left(1 + \frac{v}{c}cos\theta'\right)}{\sqrt{1 - v^2/c^2}} = \gamma t'\left(1 + \frac{v}{c}cos\theta'\right)$$

Similarly and symmetrically

$$t' = t\frac{\left(1 - \frac{v}{c}cos\theta\right)}{\sqrt{1 - v^2/c^2}} = \gamma t\left(1 - \frac{v}{c}cos\theta\right)$$

Taking the derivative of t with respect to t' and vice versa and inverting we find the relations

$$\frac{dt}{dt'} = \gamma\left(1 + \frac{v}{c}cos\theta'\right) = \left[\gamma\left(1 - \frac{v}{c}cos\theta\right)\right]^{-1} \tag{17}$$

Note that it matters whether one uses the angle θ or θ' because of the aberration of angles. The frequency of clock ticks would be:

$$\nu = \nu'\gamma\left(1 + \frac{v}{c}cos\theta'\right) = \nu'\left[\gamma\left(1 - \frac{v}{c}cos\theta\right)\right]^{-1} \tag{18}$$

6.2 Phase of Plane Wave Approach

Now we can calculate the direction and wavelength or frequency of light observed by considering the phase of a plane wave traveling in the same direction θ' in the frame S$'$ where the light source is at rest. Remember the relationship between wavelength λ, frequency ν, and the speed of light c $\lambda_o \nu_o = c$

$$\Phi = 2\pi \left[\nu_o t' - \frac{x' cos\theta' + y' sin\theta'}{\lambda_o} \right]$$

apply the Lorentz transforms expressing x', t' in terms of x and t and $y' = y$ to obtain:

$$\Phi = 2\pi \left[\nu_o \gamma \left(t - \frac{v}{c^2} x \right) - \frac{cos\theta'}{\lambda_o} \gamma \left(x - vt \right) - \frac{sin\theta'}{\lambda_o} y \right]$$

Now in the laboratory or observer rest frame coordinates

$$\Phi = 2\pi \left[\nu t - \frac{cos\theta}{\lambda} x - \frac{sin\theta}{\lambda} y \right] 2\pi \left[\nu \gamma \left(t' + \frac{v}{c^2} x' \right) - \frac{cos\theta}{\lambda} \gamma \left(x' + vt' \right) - \frac{sin\theta}{\lambda} y \right]$$

Since we realize that the phase must be the same in the two frames, we can compare the previous equations and obtain that the coefficients for t, x, and y must be the same. I.e. for t

$$\nu = \gamma \nu_o + \frac{cos\theta'}{\lambda_o} \gamma v = \gamma \nu_0 \left(1 + \frac{v}{c} cos\theta' \right)$$

Collecting the coefficients for t' yields

$$\nu_o = \gamma \nu \left(1 - \frac{v}{c} cos\theta \right)$$

These are the relativistic Doppler effect for frequency

$$\nu = \gamma \nu' \left(1 + \frac{v}{c} cos\theta' \right) = \nu' / \left[\gamma \left(1 - \frac{v}{c} cos\theta \right) \right] \tag{19}$$

These are the same equations we got for the ratio of clock running rates using the geometrical ray tracing.

We can also find aberration of angles, started by setting the coefficients for x and y equal from the two equations for the phase.

$$\frac{cos\theta}{\lambda} = \gamma \frac{cos\theta'}{\lambda_o} + \gamma \nu_o \frac{v}{c^2}$$

$$\frac{sin\theta}{\lambda} = \frac{sin\theta'}{\lambda_o}$$

where we make use of the relationship $\lambda' \nu' = \lambda_o \nu_o = c = \lambda \nu$. The ratio of these equations gives

$$tan\theta = \frac{sin\theta'}{\gamma \left(cos\theta' + v/c \right)}$$

is the same aberration from ray optics above. This is natural since one is geometrical (ray) optics and the other wave but rays propagate normal to wave fronts.

6.3 Special Cases

6.3.1 Doppler shift parallel to direction of observation

Consider the special case when the source is approaching or receding directly. That is to say that the velocity of the source is parallel to the line of sight. Then both versions of the formula yield the following relationship

$$\nu = \nu' \sqrt{\frac{1+\beta}{1-\beta}}$$

This is left as an exercise to the reader to show this and to show that the equation is exactly symmetrical on reversal of the frames

$$\nu' = \nu \sqrt{\frac{1+\beta'}{1-\beta'}} = \nu \sqrt{\frac{1-\beta}{1+\beta}}$$

6.3.2 Doppler shift perpendicular to direction of observation

The case of motion perpendicular to the direction of observation (in the frame observation frame). is quite simple:

$$\nu = \nu'/\gamma \qquad \nu' = \gamma\nu$$

This is called the transverse Doppler shift and is simply a result of time dilation as one would anticipate.

6.3.3 Fresnel's Velocity Dragging Coefficient

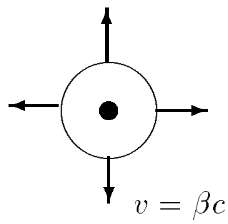
$$u = u' + vcos\theta(1 - u'^2/c^2) = c/n + \kappa vcos\theta$$

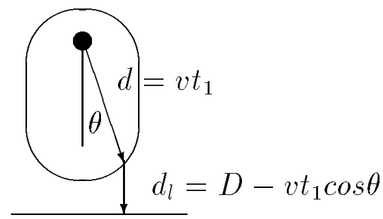
7 Superluminal

Radio astronomers routinely observe objects that they classify as superluminal. Operatinally this means that a radio astronomers uses his radio telescope (often an interferometer array) to make an image of an object at multiple times and the time rate of change of the angular diameter of the astronomical object times the estimated distance to the object gives a result that implies a velocity transverse to the line of sight which is greater than the speed of light, sometimes by up to five times.

There are a number of potential explanations for these observations but nearly all can be ruled out easily by companion observations.

Consider the following senario where the souce is at rest with respect to the observer (radio astronomer) and has sent out an relativistic expanding shell of light emitting matter.





A radio astronomy telescope images the incoming wavefront which means that it accepts photons which have arrived at the telescope at the same time. Hence we need to find the locus of points on the expanding wave front which have the same total travel time to the radio telescope. This means that the sum, t_{total}, of the time $t_1 = R/v$ taken for the point on the expanding sphere to reach the point at radius $R = vt_1$ where it emits the light plus the time $t_2 = (D - Rcos\theta)/c$ it takes light to travel from the point of emission to the radio telescope. Note that D is the distance from the original expanding source to the radio telescope.

$$t_{total} = R\left(\frac{1}{v} - \frac{cos\theta}{c}\right)$$

$$R = \frac{vt}{1 - \beta cos\theta}$$

note that for $\beta \ll 1$, this radius is $R \simeq vt(1 + \beta cos\theta)$.

Note also that this is an alternate definition of an ellipse with eccentricity $e = \beta$. Usually an ellipse is geometrically defined a the locus of points for which the sume of the distance from two points is a constant. However, a more general definition of a conic section is the locus of points whose distance between a point and a line, called the directrix (in this case the wavefront), is in a constant ratio e. In this case $e = v/c$. If e is less than 1, the resulting figure is an ellipse. If e is equal 1, the resulting figure is an parabola. If e is greater than 1, the resulting figure is a hyperbola. The eccentricity e of an ellipse varies between 0 and 1 and the value of e indicates the degree of departure from circularity. (Focus is at a distance of ae from the center and the directriz is at a distance a/e from the center of the ellipse.)

The apparent diameter set by the symmetric pair of such points is twice $Rsin\theta$.

$$\text{Diameter} = 2Rsin\theta = 2vt\frac{sin\theta}{1 - \beta cos\theta}$$

The velocity perpendicular to the line of sight is

$$v_\perp = \frac{vsin\theta}{1 - \beta cos\theta}$$

We can find the maximum apparent diameter (still assuming the expanding shell is opaque and emitting light) by taking the derivative of the diameter with respect to θ setting that to zero and finding the maximum apparent diameter at time t_o.

$$\frac{d\text{Diameter}}{d\theta} = 2vt\left(\frac{cos\theta}{1-\beta cos\theta} - \frac{\beta sin^2\theta}{(1-\beta cos\theta)^2}\right)$$

$$= \frac{2vt}{(1-\beta cos\theta)^2}\left(cos\theta - \beta\right)$$

The maximum clearly occurs at

$$cos\theta = \beta; \quad sin\theta = \sqrt{1-\beta^2}; \quad \theta = cos^{-1}\beta$$

At the maximum

$$R = \frac{2vt}{1-\beta cos\theta} = \frac{2vt}{1-\beta^2} = \gamma^2 vt$$

The diameter is then

$$\text{Diameter} = 2vt\frac{sin\theta}{1-\beta cos\theta} = 2vt\frac{\sqrt{1-\beta^2}}{1-\beta^2} = \frac{2vt}{\sqrt{1-\beta^2}} = 2\gamma vt$$

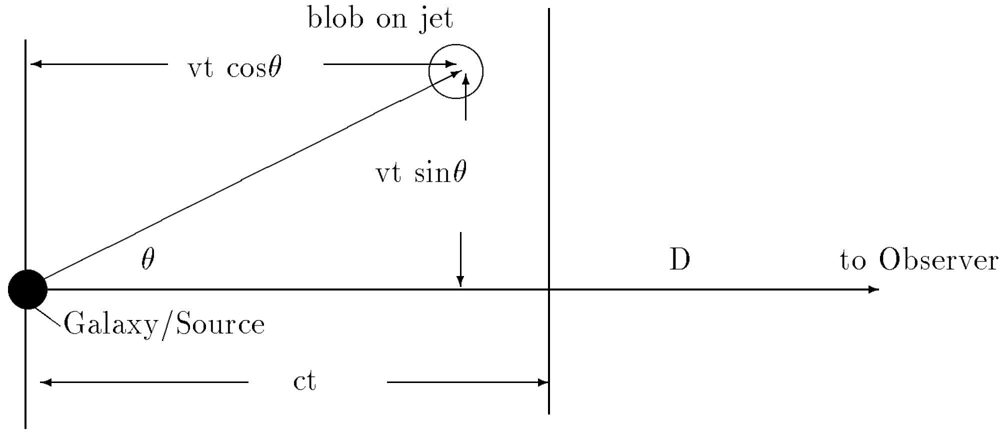
$$v_\perp = 2\gamma v$$

The subtended angle is $\simeq 2\gamma vt/D$ and the apparent velocity is γ times the expanding sphere velocity.

The most visible radio objects are double-lobe radio sources which have back-to-back relativitistic jets. In practice one generally only able to measure well relativisitic jet that is coming towards the observer because the Doppler effect both changes the observed temperature and intensity. The intensity of the portion coming towards the observer is typcally increased by the factor 8γ and the portion moving away decreased by the same factor. See the following exercise:

7.1 Superluminal Motion Exercise

Astronomers observe a large number of radio sources that move with apparent superluminal speed. That is the rate of change of angular separation between components times the distance to the radio souce gives a velocity well in excess of the speed of light ($v_{observed} = D \times d\alpha/dt$). Consider the following problem and diagram to help understand how an astronomer could measure apparent superluminal speed, if there is a relativistic beam coming from the source.



Neglect the source (host galaxy) motion relative to the observer and consider the motion of only a single blob on a radio jet. The blob moves at velocity v with respect to the galactic nucleus (and observer) beginning at time $t = 0$. Also assume that the blob and nucleus continuously emit radio waves so that they can be observed.

Consider the radio emission received as a function of time by the observing radio astronomer very far (distance D) away. Show that the observer sees the blob coincident with the galaxy source at time $t_0 = D/c$ corresponding to $t = 0$. Show also that the observer sees the blob with transverse displacement $vt\ sin\theta$ from the galactic nucleus at the time

$$t_r = t + (D - vt\ cos\theta)/c$$

Show that the elapsed time for the observer was

$$t_r - t_0 = t(1 - \beta cos\theta)$$

where $\beta = v/c$.

The apparent transverse velocity of the blob relative to the nucleus $v_{apparent-transverse}$ equals the transverse displacement divided by the time difference observed for the displacement to occur. Show that this leads to the formula:

$$\beta_{apparent-transverse} = \frac{\beta sin\theta}{1 - \beta cos\theta}$$

Plot this formula for the following values: $\beta = 0.5,\ 1$ (a special case) and $\gamma = 2,\ 3,\ 4,\ 5,\ 7,\ 16$.

Show that the maximum transverse velocity happens for $cos\theta = \beta$ (and thus $sin\theta = \sqrt{1 - \beta^2} = 1/\gamma$), as derived in class for an expanding spherical shell, and that the maximum apparent transverse velocity is

$$\beta_{apparent-transverse-max} = \beta/\sqrt{1 - \beta^2} = \gamma\beta$$

and that your graphs agree with this.

Note that for the critical angle and $\gamma >> 1$, the transverse speed is roughly $v_{apparent-transverse-max} \approx \gamma c$.

7.2 Too Rapid Time Variability

The minimum size for an astronomical object is often set by use of our earlier finding that no causal impulse can travel with a speed faster than the speed of light. Thus if an object is observed to vary its brightness very significantly in a given time period Δt, then it must be no larger than $d = \Delta t$ in extent. This is a good rule for non-relativistic objects. However, if the object, e.g. a jet, is moving towards the observer with relativistic speeds, then this can be compressed by a factor $\gamma_{\rm object}$.

This has been observed (R. A. Remillard, B. Grossan, H. V. Brandt, T. Ohashi, K. Hayashida, F. Makio, & Y Tanaka, Nature 1991 vol 350 p 589-592) in the rapid varibility of an energetic X-ray flare in the quasar PKS)Since we now the mass of the black hole from the limit of accretion efficiency, we know it's size. From the minimum (light) travel time across the source, we know the minimum variability time scale. The observed time is shorter, therefore, we must have relativistic beaming.

Another interesting example of variability, however, is the time dilation of supernova light curves. Nearby Type 1A supernova are observed to have a very standard brightness and time dependence of the light curve. (This can be made even a tighter standard by the correlation between the intensity and light curve width in time.) When observed at great distances, the light from a Type 1A supernova is observed to be reddened by an amount that is consistent with a Doppler frequency shift and the light curve time taken is stretched by the same amount predicted by the relativisitic Doppler shift formula. Most observed distant supernova have frequency shift factors ranging from 1.2 to 1.9. As we will see later this is evidence that the Universe is actually expanding and one can understand this stretching from a General Relativistic point of view also.

8 Appearance of Rapidly Moving Objects