Why Does $M \sim rv^2/G$?

This formula for approximating the masses of gravitating objects from their characteristic sizes, r, and characteristic velocities of their constituent parts, v, occurs repeatedly in this course. It is pretty obvious that the faster objects are moving around under the influence of gravity, the greater that force must be, and hence the greater the mass involved. Similarly, if the system is very large, it is likely to contain a lot of mass. However, why the particular formula

$$M \sim rv^2/G? \tag{1}$$

On the basis of simple dimensional arguments, we might argue that it is the only combination which returns a mass from the characteristic size and internal velocity of the object and the gravitational constant G (which has dimensions of [mass] [length]^3 [time]$^{-2}$). However, there are more direct physical reasons why this formula is valid in a wide variety of situations.

1. Circular Orbits

In the case of a spherically-symmetric system with material on circular orbits, this formula is exact: at radius r in such a system, the pull of gravity toward the centre of the system on a body of mass m is $F = GM(r)m/r^2$, where $M(r)$ is the total mass contained within radius r (Newton demonstrated that mass at larger radii in a spherically-symmetric system exerts no net force). To make the orbiting body follow a circular orbit at this radius, gravity must provide the centripetal acceleration, $a = v^2/r$. Applying Newton's Second Law ($F = ma$), we thus find that

$$M(r) = \frac{rv^2}{G} \tag{2}$$

exactly. So, for example, the flat rotation curves of spiral galaxies ($v \approx$ constant, independent of r) implies that $M(r) \propto r$ in these systems.

2. Escape Velocity

There is a natural limit to how quickly objects can move within a gravitating system. This limit is set by the maximum speed at which the objects can move while remaining bound to the system — the escape velocity. Any objects moving faster than this speed will rapidly escape from the system; unless such objects are being generated fast enough to replace them as they escape, we will not find many associated with the gravitating system. Objects on bound orbits have negative total (kinetic + potential) energies, while those on unbound orbits have positive total energies. Consider an object of mass m at radius r in the outskirts of a gravitating system, such that almost all the mass of the system, M, is at radii of less than r. In this location, we can approximate the gravitational force by a Keplerian field, for which the gravitational potential energy is $-GMm/r$. The escape speed, $v_{\rm esc}$, is thus given by

$$\frac{1}{2}mv_{\rm esc}^2 - \frac{GMm}{r} = 0. \tag{3}$$

Re-arranging this formula, we find

$$M = \frac{r v_{\text{esc}}^2}{2G} \tag{4}$$

Thus, if the characteristic speed v that we measure is the velocity of the fastest-moving object associated with the system (and assuming that this object is bound to the system), we once again find that equation (1) is obeyed to within a factor of two.

3. The Virial Theorem

In fact, the above examples are particular cases of an even more general result known as the virial theorem. Consider a system of N objects orbiting in a gravitational potential Φ. Further, assume that the system is in equilibrium, so that its over-all properties are not changing with time (for example, consider a globular cluster where the individual stars move around on their orbits, but the overall shape and size of the cluster does not change).

By Newton's Second Law, the force on object i is $\mathbf{F}_i = m_i \ddot{\mathbf{r}}_i$, and hence, taking a scalar product with r_i and summing over all particles,

$$\begin{aligned}
\sum_{i=1}^{N} \mathbf{F}_i.\mathbf{r}_i &= \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i.\mathbf{r}_i \\
&= \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i.\mathbf{r}_i\right) - \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2 \\
&= \frac{dK}{dt} - 2T,
\end{aligned} \tag{5}$$

where $K = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i.\mathbf{r}_i$, and $T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$ is the total kinetic energy of the system. Since the overall system is in equilibrium, its global properties such as K cannot change with time, and so $\frac{dK}{dt} \equiv 0$. Incidentally, the quantity on the left-hand side of this equation is called the Virial of Clausius.

If the system is spherically symmetric, with the coordinate origin at its centre, then the force of gravity on particle i is given by $\mathbf{F}_i = d\Phi/dr$ in the direction of r_i. So, for example, for a Keplerian potential with $\Phi \propto r^{-1}$, $\mathbf{F}_i.\mathbf{r}_i = -\Phi_i$, where Φ_i is the gravitational potential of object i. Summing over all objects, we thus find that

$$2T + V = 0, \tag{6}$$

where V is the kinetic energy of the entire system. The total kinetic energy of the system is $T = \frac{1}{2} M v_{\text{vt}}^2$, where v_{vt} is a weighted mean velocity of its constituents, and its total potential energy is $V = GM^2/r_{\text{vt}}$, where r_{vt} is a weighted mean radius of its constituent particles — its characteristic size. Subsituting these values into equation (6), we find once again that $M = r_{\text{vt}} v_{\text{vt}}^2 / G$, as in equation (1).